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CYCLE AND PATH EMBEDDING ON 5-ARY N-CUBES

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Abstract. We study two topological properties of the 5-ary n-cube Q_n^5. Given two arbitrary distinct nodes x and y in Q_n^5, we prove that there exists an x-y path of every length ranging from $2n$ to 5^n-1, where $n \geq 2$. Based on this result, we prove that Q_n^5 is 5-edge-pancyclic by showing that every edge in Q_n^5 lies on a cycle of every length ranging from 5 to 5^n.

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1. Introduction

One of the most important problems in interconnection networks is the embedding problem. All algorithms developed for network topology A can be directly applied to topology B if topology A can be embedded into topology B. Linear arrays and rings are two fundamental networks for parallel and distributed computation. Numerous efficient algorithms based on linear arrays and rings for solving various algebraic and graph problems have been studied [1,8,10,12]. These applications motivated us to embed paths (linear arrays) and cycles (rings) into networks.

An interconnection network (*network* for short) is usually represented by a graph where vertices represent processors and edges represent communication links between processors. A *graph* $G = (V, E)$ is a pair comprised of the *vertex set* V and the *edge set* E, where V is a finite set and E is a subset of $\{(u, v) : (u, v)$ is an unordered pair of $V\}$. We sometimes use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively.

Keywords and phrases. Graph-theoretic interconnection networks, hypercubes, k-ary n-cubes, panconnectivity, edge-pancyclicity.

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A graph $G = (V, E)$ is said to be *panconnected* (respectively, *bipanconnected*) if for two arbitrary distinct nodes x and y of G, there exists an x-y path of every length l for $d(x, y) \leq l \leq |V| - 1$ (respectively, $2|(l - d(x, y)))$[1], where $d(x, y)$ is the *distance* of x and y (the length of a shortest path between x and y). On the other hand, a graph $G = (V, E)$ is *pancyclic* (respectively, *bipancyclic*) if G contains a cycle of every length (respectively, every even length) from three (respectively, four) to $|V|$. A bipancyclic graph G is further said to be *edge-bipancylic* if for any edge e of G, there exists a cycle C of every even length such that e is in C. Wang *et al.* [16] showed that Q_2^k is both bipanconnected and bipancyclic, and Q_n^k is Hamiltonian-connected when k is odd.

Many networks have been studied as attractive topologies for distributed and parallel systems, including mesh, torus (also called a wrap-around mesh), hypercube, and k-ary n-cube. In fact, the k-ary n-cube is an n-dimensional torus with each dimension of the same size k, and the hypercube is a k-ary n-cube with $k = 2$. Note that a mesh is a subgraph of a torus. Several parallel machines, both commercial and experimental, have been proposed based on k-ary n-cube, for example, Cray T3D and T3E (3D torus) [11], the Mosaic (k-ary n-cube) [13], and the iWarp (torus) [2]. In particular, the k-ary n-cube has been one of the most common interconnection networks for multi-processor systems [3,5,6,16].

Quite recently, Hsieh *et al.* [7] have investigated the panconnectivity and edge-pancyclicity of 3-ary n-cubes Q_n^3. In this paper, we further investigate the panconnectivity and edge-pancyclicity of the 5-ary n-cube Q_n^5 which is a more complicated topology than Q_n^3. Given two arbitrary distinct nodes x and y in Q_n^5, we prove that there exists an x-y path of length l ranging from $2n$ to $5^n - 1$, where $2n$ is the diameter of Q_n^5. Based on this result, we prove that every edge in Q_n^5 lies on a cycle of every length ranging from 5 to 5^n. The remainder of this paper is organized as follows. In the next section, some basic definitions and notations are introduced. Our main results are presented in Sections 3 and 4. Finally, some concluding remarks are presented in Section 5.

2. Preliminaries

Two vertices u and v are *adjacent* if $(u, v) \in E(G)$. A *subgraph* of $G = (V, E)$ is a graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. Given a set $V' \subseteq V$, the *subgraph of* $G = (V, E)$ *induced by* V' is the graph $G' = (V', E')$, where $E' = \{(u, v) \in E : \ u, v \in V'\}$. A *path* $P[v_0, v_k] = \langle v_0, v_1, ..., v_k \rangle$ in a graph G is a sequence of distinct vertices such that any two consecutive vertices are adjacent, and we call v_0 and v_k the *end-vertices* of the path. A cycle is a path $P[v_0, v_k]$ with $v_0 = v_k$ and $k \geq 3$. A path with end-vertices u and v is said to be a *u-v path*. The *length* of a path is the number of edges contained in the path. The *distance* between u and v, denoted by $d_G(u, v)$, is the length of a shortest path $P[u, v]$. A path may contain another path as a *subpath*, denoted by $\langle v_0, v_1, ..., v_i, P[v_i, v_j], v_j, v_{j+1}, ..., v_k \rangle$, where

[1] The notation $d|a$ means that $a = kd$ for some integer k.

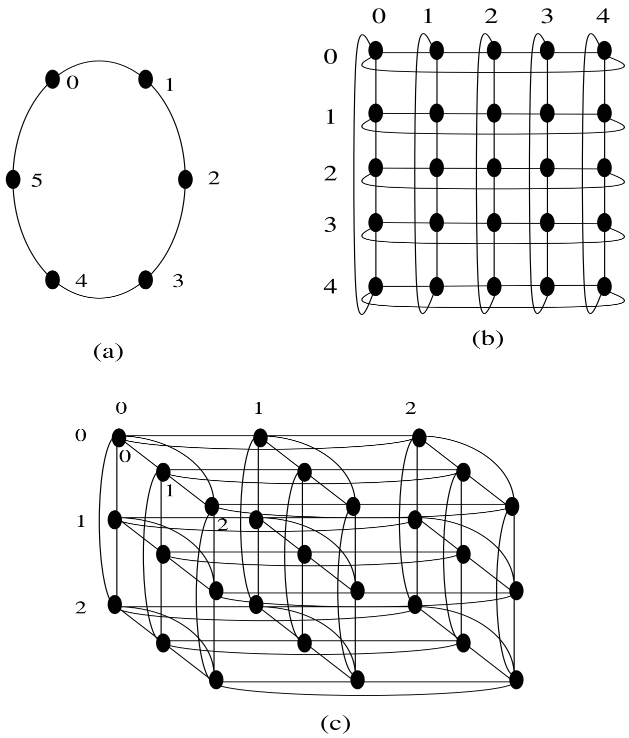


FIGURE 1. (a) Q_1^6, (b) Q_2^5, and (c) Q_3^3.

$P[v_i, v_j] = \langle v_i, v_{i+1}, ..., v_{j-1}, v_j\rangle$. A cycle (respectively, path) in G is called a *Hamiltonian cycle* (respectively, *Hamiltonian path*) if it contains every vertex of G exactly once. A graph G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and *Hamiltonian-connected* if there exists a Hamiltonian path between every two distinct vertices of G.

A graph $G = (V, E)$ is *pancyclic*, if G contains any cycle of length l satisfying $3 \leq l \leq |V|$, that is, any cycle of length l can be embedded into G with dilation one. Furthermore, G is *edge-pancyclic* if every edge of G lies on a cycle of every length from 3 to $|V|$. G is *L-edge-pancyclic* if every edge of G lies on a cycle of every length from L to $|V|$. The *k-ary n-cube* Q_n^k ($k \geq 2$ and $n \geq 1$) has $N = k^n$ nodes each of the form $x = x_n x_{n-1} \dots x_1$, where $0 \leq x_i < k$ for all $1 \leq i \leq n$. Two nodes $x = x_n x_{n-1} \dots x_1$ and $y = y_n y_{n-1} \dots y_1$ in Q_n^k are adjacent if and only if there exists an integer j, $1 \leq j \leq n$, such that $x_j = y_j \pm 1 \pmod{k}$ and $x_l = y_l$ for every $l \in \{1, 2, ..., n\} - \{j\}$. For clarity of presentation, we omit writing "(mod k)" in similar expressions for the remainder of the paper. Note that each node has degree $2n$ when $k \geq 3$, and n when $k = 2$. Obviously, Q_1^k is a cycle of length k, Q_n^2 is an n-dimensional hypercube, and Q_2^k is a $k \times k$ wrap-around mesh. In this paper, we pay our attention on $k = 5$. Figure 1 illustrates Q_1^6, Q_2^5, and Q_3^3.

The ith position, from the right to the left, of the n-bit string $x_n x_{n-1} \dots x_1$ is called the *i-dimension*. We can *partition Q_n^5 along the i-dimension* by regarding the graph comprised of 5 disjoint copies, $Q_{n-1}^5[0]$, $Q_{n-1}^5[1]$, $Q_{n-1}^5[2]$, $Q_{n-1}^5[3]$, and $Q_{n-1}^5[4]$,[2] where $Q_{n-1}^5[j]$ is the subgraph of Q_n^5 induced by $\{x \in V(Q_n^5)|$ the ith bit x_i of $x = x_n x_{n-1} \dots x_1$ is fixed by $j \in \{0, 1, 2, 3, 4\}\}$. We call each $Q_{n-1}^5[j]$ a *subcube* of Q_n^5. Note that $Q_{n-1}^5[j]$ is isomorphic to a 5-ary $(n-1)$-cube. Clearly, there are exactly 5^{n-1} edges which form a perfect matching between $Q_{n-1}^5[j]$ and $Q_{n-1}^5[j+1]$. We call $Q_{n-1}^5[j]$ and $Q_{n-1}^5[j+1]$ *adjacent subcubes*, and call the edges between two adjacent subcubes *bridges*.

3. Panconnectivity of 5-ary n-cubes

In this section, we investigate the panconnectivity of 5-ary n-cubes. We first provide some previously known properties which are useful in our method.

Lemma 1 [3]**.** *The diameter of Q_n^k equals $\lfloor \frac{k}{2} \rfloor n$.*

Lemma 2 [16]**.** *The following two statements hold:*
(1) Q_2^k is bipanconnected.
(2) Q_n^k is Hamiltonian-connected when k is odd.

Lemma 3. *For any two distinct nodes $x, y \in V(Q_2^5)$ and any integer l with $4 \leq l \leq 24$, Q_2^5 contains an x-y path of length l.*

Proof. We attempt to construct an x-y path of every length l for $4 \leq l \leq 24$. Due to the structure property of Q_2^5, we need to consider the following five cases.

Case 1. $x = 00$ and $y = 01$.

- **Case 1.1:** l is odd. We have $d(x, y) = 1$ and since Q_2^k is bipanconnected by Lemma 2(1), all desired x-y paths can be obtained.
- **Case 1.2:** l is even. A desired path of every even length can be obtained from a Hamiltonian path $\langle 00, 04, 14, 24, 34, 44, 43, 42, 41, 40, 30, 31, 32, 33, 23, 22, 21, 20, 10, 11, 12, 13, 03, 02, 01\rangle$ by subtracting length 2 successively till length of 4, using a routing strategy illustrated in Figures 2a–2f.

Case 2. $x = 00$ and $y = 02$.

- **Case 2.1:** l is odd. A desired path of every odd length can be obtained from a path $\langle 00, 04, 14, 24, 34, 44, 43, 42, 41, 40, 30, 31, 32, 33, 23, 22, 21, 20, 10, 11, 12, 13, 03, 02\rangle$ by subtracting length 2 successively till length of 5, using a routing strategy illustrated in Figures 3a–3f.
- **Case 2.2:** l is even. We have $d(x, y) = 2$ and since Q_2^k is bipanconnected by Lemma 2(1), all desired x-y paths can be obtained.

[2] Since the index i is not necessary to be specified, we omit it from the notation.

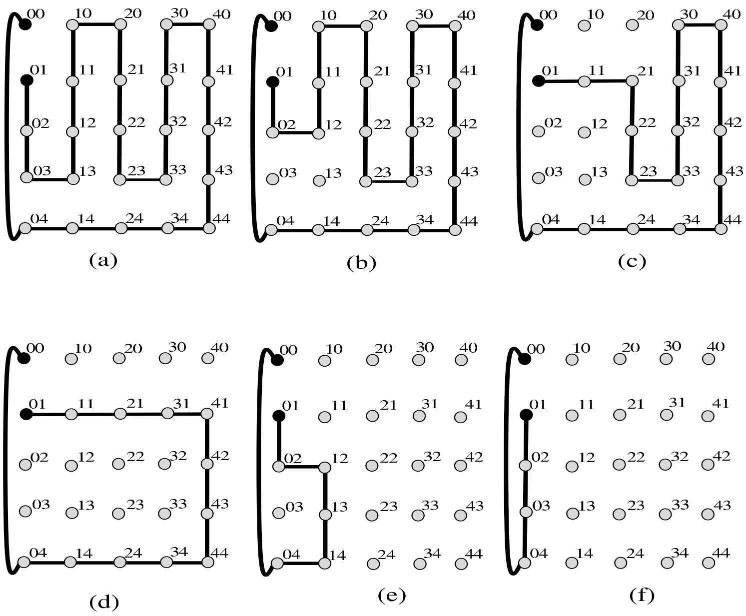


FIGURE 2. An illustration of case 1 in the proof of Lemma 3.

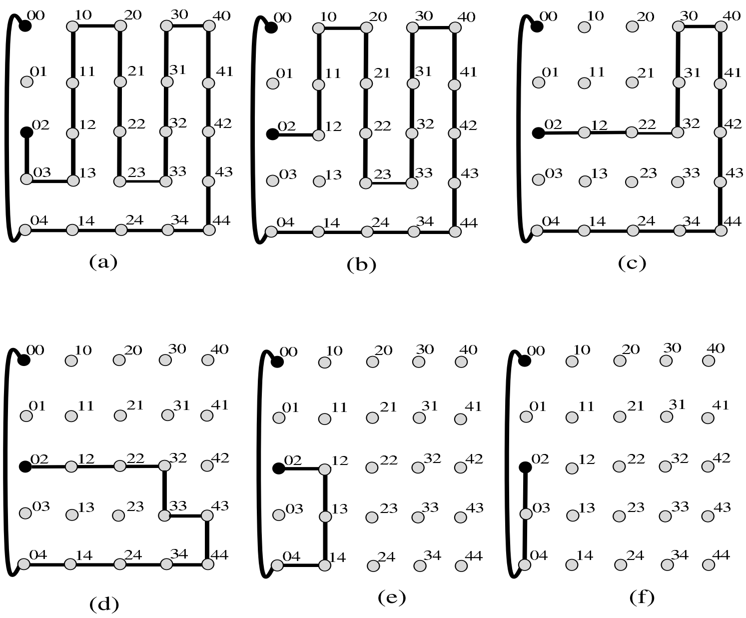


FIGURE 3. An illustration of case 2 in the proof of Lemma 3.

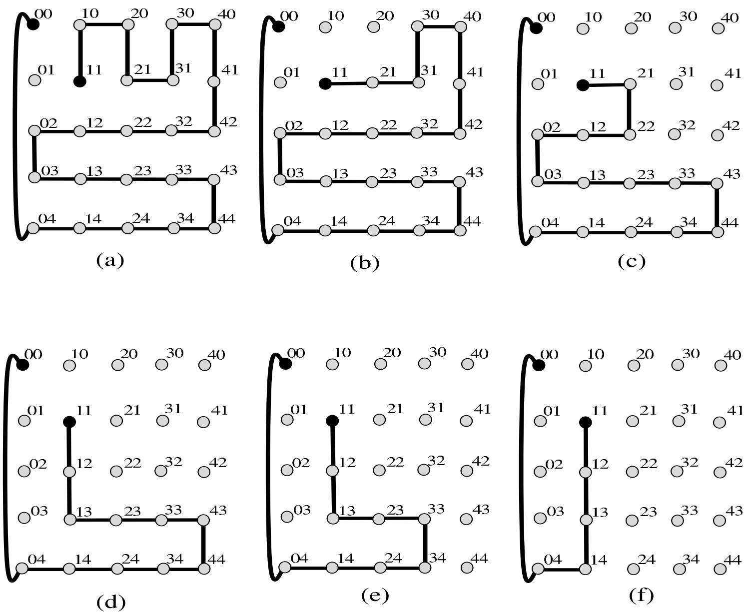


FIGURE 4. An illustration of case 3 in the proof of Lemma 3.

Case 3. $x = 00$ and $y = 11$. The proof is similar to Case 2 (see Fig. 4).

Case 4. $x = 00$ and $y = 12$. The proof is similar to Case 1 (see Fig. 5).

Case 5. $x = 00$ and $y = 22$. The proof is similar to Case 2 (see Fig. 6). □

For convenience, we use the notation "$u \mapsto v$" to mean that (u, v) is a bridge between two adjacent subcubes.

Theorem 1. *For any two distinct nodes $x, y \in V(Q_n^5)$, where $n \geq 2$, and any integer l with $2n \leq l \leq 5^n - 1$, there exists an x-y path of length l.*

Proof. We prove this theorem by induction on n. By Lemma 3, the base result holds for $n = 2$. Suppose that the result holds for the 5-ary $(n-1)$-cube. We now consider Q_n^5, where $n \geq 3$. We partition Q_n^5 along the dimension i into five subcubes $Q_{n-1}^5[0]$, $Q_{n-1}^5[1]$, $Q_{n-1}^5[2]$, $Q_{n-1}^5[3]$, and $Q_{n-1}^5[4]$. We will attempt to construct an x-y path of every length l with $2n \leq l \leq 5^n - 1$. There are the following two scenarios.

Case 1: $2n \leq l \leq 5^{n-1} - 1$. Without loss of generality, we assume that x is in $Q_{n-1}^5[0]$. Due to the structure of Q_n^5, we only consider that y is in $Q_{n-1}^5[j]$, where j=0, 1, or 2. Let u_1 be a node in $Q_{n-1}^5[1]$ such that $x \mapsto u_1$ if y is in $Q_{n-1}^5[1]$. Let u_2 be a node in $Q_{n-1}^5[2]$ such that $x \mapsto u_1 \mapsto u_2$ if y is in $Q_{n-1}^5[2]$, where u_1 is a node in $Q_{n-1}^5[1]$. Let $u_0 = x$ if y is in $Q_{n-1}^5[0]$. If $u_1 = y$ or $u_2 = y$, then we can partition Q_n^5 along another dimension $i'(\neq i)$ such that x and y are in the same subcube. By the induction hypothesis,

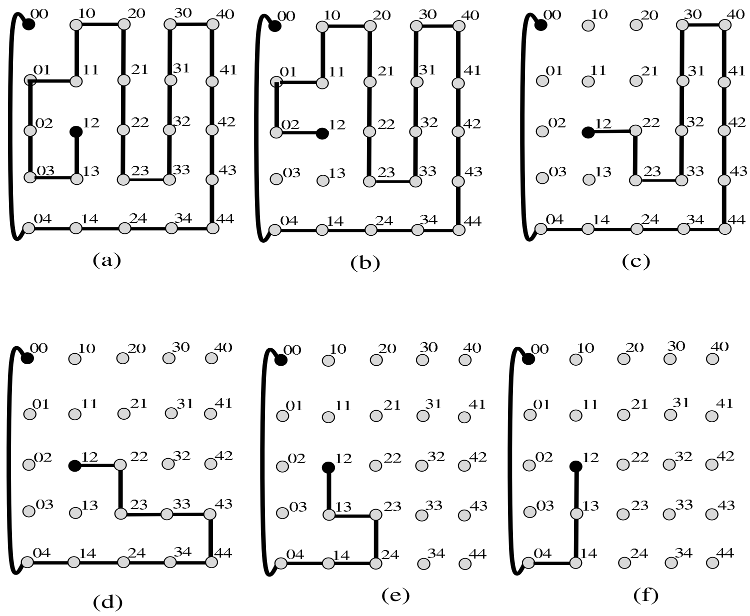


FIGURE 5. An illustration of case 4 in the proof of Lemma 3.

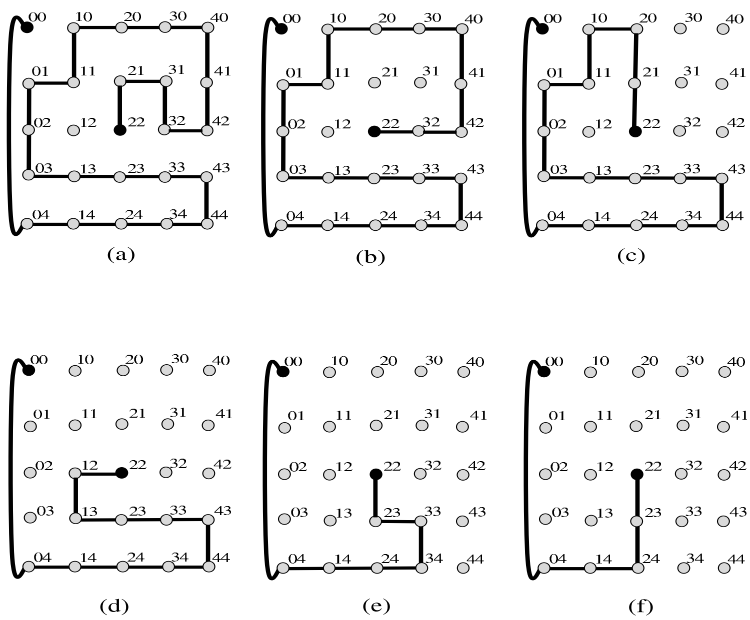


FIGURE 6. An illustration of case 5 in the proof of Lemma 3.

$Q^5_{n-1}[j]$ contains a path $P_j[u_j, y]$ of length l_j with $2(n-1) \leq l_j \leq 5^{n-1}-1$. An x-y path of length l_1 with $2(n-1)+1 \leq l_1 \leq 5^{n-1}$ can be constructed by $\langle x, u_1, P_1[u_1, y], y\rangle$ if y is in $Q^5_{n-1}[1]$, and an x-y path of length l_2 with $2n \leq l_2 \leq 5^{n-1}+1$ can be constructed by $\langle x, u_1, u_2, P_2[u_2, y], y\rangle$ if y is in $Q^5_{n-1}[2]$. Therefore, any path of the specified length can be constructed.

Case 2: $5^{n-1} \leq l \leq 5^n - 1$. Without loss of generality, we assume that x is in $Q^5_{n-1}[0]$. Due to the structure of Q^5_n, we only consider y to be in $Q^5_{n-1}[0]$, $Q^5_{n-1}[1]$, or $Q^5_{n-1}[2]$. No matter which subcube y is in, we can construct an x-y path by using a routing strategy illustrated in Figure 7. Note that P_0 is an x-y path in Figure 7a and an x-v path in Figure 7b and c. By the induction hypothesis, $Q^5_{n-1}[k]$ contains a path P_k of length l_k with $2(n-1) \leq l_k \leq 5^{n-1}-1$, where $0 \leq k \leq 4$. Thus, an x-y path of length $l = l_0 + l_1 + l_2 + l_3 + l_4 + 4$ with $10n - 6 \leq 5^{n-1} \leq l \leq 5^n - 1$ can be constructed, where $n \geq 3$.

The proof is completed from the above two cases. □

Remarks 1. Note that when $d(x, y) = 1$, there is no x-y path of length 2 in Q^5_n. Therefore, the range of length l in the statement of Theorem 1 cannot start with $d(x, y)$.

4. Edge-pancyclicity of 5-ary n-cubes

In this section, we investigate the edge-pancyclicity of 5-ary n-cubes according to the results obtained in Section 3.

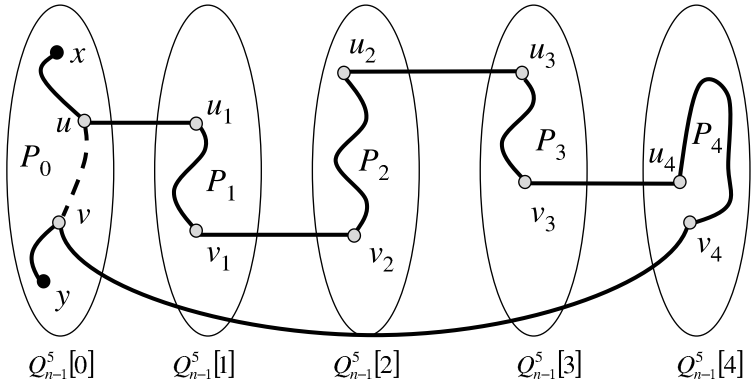
Lemma 4. *For any edge $(x, y) \in E(Q^5_2)$ and any integer l with $5 \leq l \leq 25$, there exists a cycle C of length l such that (x, y) is in C.*

Proof. Given two adjacent nodes x and y in Q^5_2, without loss of generality, let $x = 00$ and $y = 01$, there exists an x-y path $P[x, y]$ of every length l for $4 \leq l \leq 24$ by the Case 1 of proof in Lemma 3. Then $P[x, y] + (x, y)$ forms a cycle of every length l for $5 \leq l \leq 25$. Therefore, the result holds. □

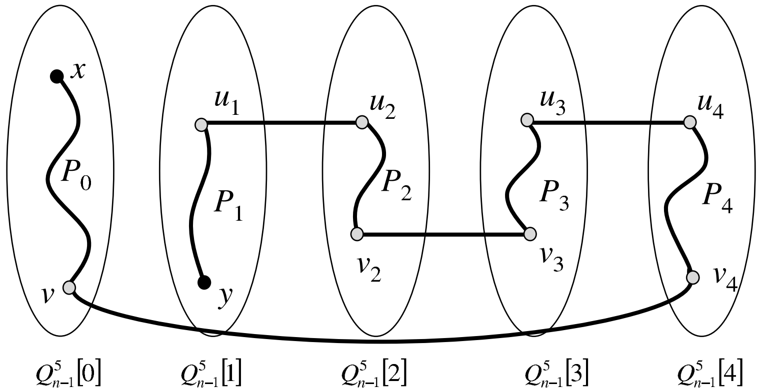
Theorem 2. *For any edge $(x, y) \in E(Q^5_n)$, where $n \geq 1$, and any integer l with $5 \leq l \leq 5^n$, there exists a cycle C of length l such that (x, y) is in C. That is, Q^5_n is 5-edge-pancyclic.*

Proof. We prove this theorem by induction on n. The result clearly holds for $n = 1$ because Q^5_1 is a cycle of length 5, and also holds for $n = 2$ by Lemma 4. Suppose that the result holds for 5-ary $(n-1)$-cubes. We now consider Q^5_n. We partition Q^5_n along the dimension i into five subcubes $Q^5_{n-1}[0]$, $Q^5_{n-1}[1]$, $Q^5_{n-1}[2]$, $Q^5_{n-1}[3]$, and $Q^5_{n-1}[4]$. Without loss of generality, we assume that (x, y) is in $Q^5_{n-1}[0]$.

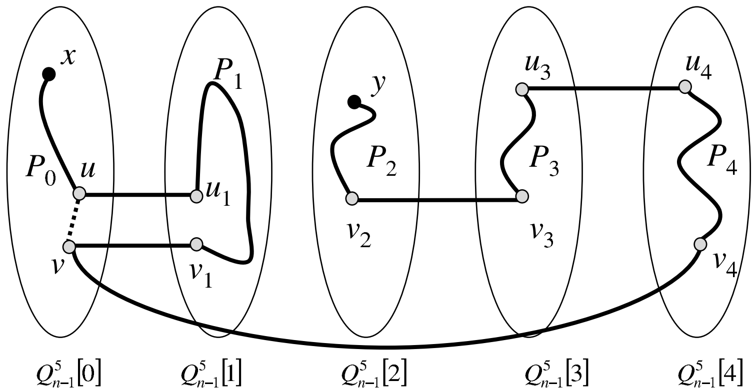
Case 1 $5 \leq l \leq 5^{n-1}$ (see Fig. 8a). By the induction hypothesis, $Q^5_{n-1}[0]$ contains a cycle C_0 of length l such that (x, y) is in C_0.



(a)



(b)



(c)

FIGURE 7. An illustration of the proof of Theorem 1.

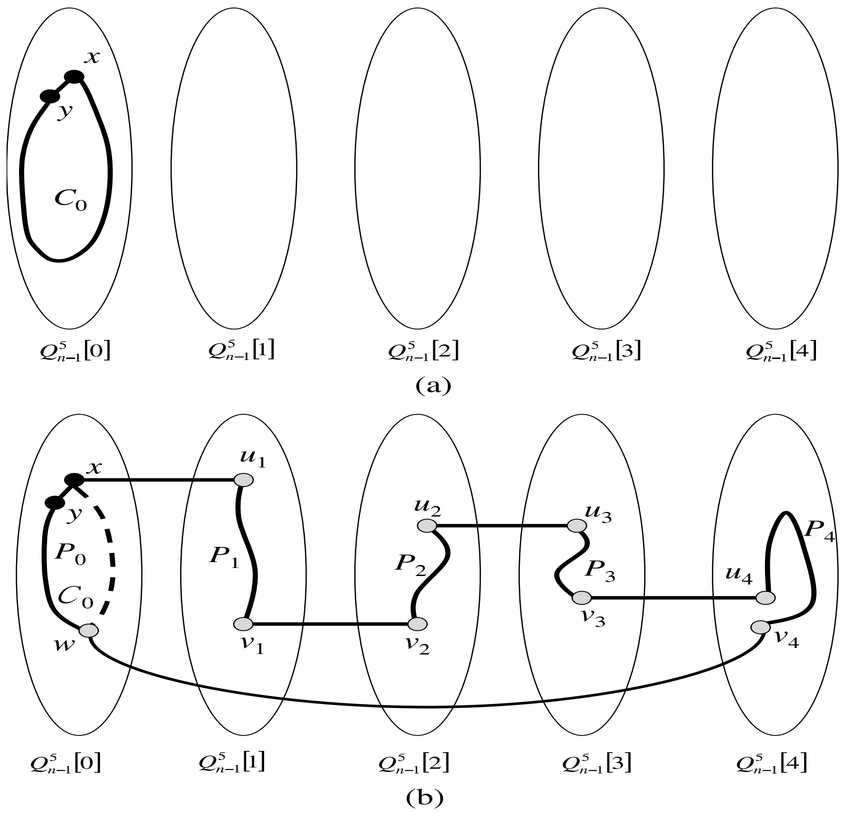


FIGURE 8. An illustration of the proof of Theorem 2.

Case 2 $5^{n-1}+1 \leq l \leq 5^n$ (see Fig. 8b). By the induction hypothesis, $Q^5_{n-1}[0]$ contains a cycle C_0 of length 5^{n-1} such that (x, y) is in C_0. Clearly, we can select a path $P_0[x, w] = \langle x, y, ..., w\rangle$ from C_0 whose length l_0 satisfies $5^{n-1} - 8n + 4 \leq l_0 \leq 5^{n-1} - 1$. We can represent $P_0[x, w]$ as $\langle x, y, P'_0[y, w], w\rangle$. Furthermore, we can select two distinct nodes u_i and v_i in $Q^5_{n-1}[i]$ for $i \in \{1, 2, 3, 4\}$ such that $x \mapsto u_1$, $v_1 \mapsto v_2$, $u_2 \mapsto u_3$, $v_3 \mapsto u_4$, and $v_4 \mapsto w$. By Theorem 1, $Q^5_{n-1}[i]$ contains a path $P_i[u_i, v_i]$ of length l_i with $2(n-1) \leq l_i \leq 5^{n-1} - 1$ for $i \in \{1, 2, 3, 4\}$.

Therefore, $\langle x, y, P'_0[y, w], w, v_4, P_4[v_4, u_4], u_4, v_3, P_3[v_3, u_3], u_3, u_2, P_2[u_2, v_2], v_2, v_1, P_1[v_1, u_1], u_1, x\rangle$ forms a cycle of length $l(= l_0+l_1+l_2+l_3+l_4+5)$ such that (x, y) is in the cycle and $5^{n-1} + 1 \leq l \leq 5^n$.

By combining cases 1–2, we complete the proof. □

Remarks 2. Since Q^5_1 has no cycle of length $l = 3, 4$ and Q^5_2 has no cycle of length 3, we can conclude that Q^5_n for $n \geq 1$ is only 5-edge-pancyclic.

A graph G is *L-node-pancyclic* if every node of G lies on a cycle of every length from L to $|V(G)|$ for some constant L. The following result follows directly from Theorem 2.

Corollary 1. *The 5-ary n-cube* Q_n^5 *is 5-node-pancyclic.*

5. Concluding remarks

In this paper, we have focused on topology embedding, where a 5-ary n-cube Q_n^5 acts as the host graph and paths (cycles) represent the guest graphs. Given two arbitrary distinct nodes x and y, we prove that Q_n^5 can embed an x-y path of length l ranging from $2n$ to $5^n - 1$, where $2n$ is the diameter of Q_n^5. Based on this result, we also prove that every edge in Q_n^5 lies on a cycle of every length ranging from 5 to 5^n.

Paths (linear arrays) and cycles (rings) are two fundamental networks for parallel and distributed computation, and are suitable for designing simple algorithms with low communication costs. Our results show that algorithms designed for paths (cycles) can also be executed well on Q_n^5. A future work is to extend our result to the k-ary n-cube for a general k.

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