RAIRO-Theor. Inf. Appl. **41** (2007) 447–462
DOI: 10.1051/ita:2007028

RECURSIVE COALGEBRAS OF FINITARY FUNCTORS *

Jiří Adámek[1], Dominik Lücke[2] and Stefan Milius[1]

Abstract. For finitary set functors preserving inverse images, recursive coalgebras A of Paul Taylor are proved to be precisely those for which the system described by A always halts in finitely many steps.

Mathematics Subject Classification. 18A25, 08C05, 68R65.

1. Introduction

For finitary endofunctors H of the category of sets we study *recursive coalgebras*. A coalgebra for H is recursive if it admits a unique homomorphism into every algebra for H. This concept stems from the work of Osius [11] (see also Montague [10]) on coalgebras for the power-set functor. For an arbitrary endofunctor the notion of a recursive coalgebra appears in the monograph of Taylor [13] under the name "coalgebra obeying the recursion scheme", and the name recursive coalgebra stems from a recent paper of Capretta *et al.* [6]. It was proved by Taylor that whenever a set functor H preserves inverse images, then recursive coalgebras are precisely the well-founded ones. In the present paper we prove that if H is, moreover, finitary, then recursive coalgebras are precisely those having the *halting property* which means that the corresponding systems halt in finitely many steps no matter what the initial state is and what input is processed.

Recall that a coalgebra is a set A of states together with a function $\alpha : A \longrightarrow HA$ assigning to every state a the collection $\alpha(a)$ of all observations about a. For example, if $H = H_\Sigma$ is the polynomial functor of a signature Σ, then a coalgebra can be understood as a deterministic system given by a set A of states

Keywords and phrases. Recursive coalgebra, coalgebra, definition by recursivity.

* *The first author acknowledges the support of the Grant MSM 6840770014 of the Ministry of Education of Czech Republic.*

[1] Technical University of Braunschweig, Institute of Theoretical Computer Science, Braunschweig, Germany; {adamek, milius}@iti.cs.tu-bs.de

[2] Department of Computer Science, University of Bremen, PO Box 330440, 28334 Bremen, Germany; luecke@tzi.de

and by a dynamics

$$\alpha : A \longrightarrow H_\Sigma A = \coprod_{n \in \mathbb{N}} \coprod_{\sigma \in \Sigma_n} A^n$$

assigning to every state an expression of the form $\sigma(a_0, \dots, a_{n-1})$ for some n-ary symbol σ. The states with $n = 0$ are the *halting states* of the system, the states with $n > 0$ react to an n-ary input, they have the output σ, and $a_0, \dots, a_{n-1}$ are the successor states. The initial algebra I_Σ can be described as the algebra of all finite Σ-*trees* (*i.e.*, trees labeled by Σ so that an n-ary label implies that the node has n children). The systems with a homomorphism into I_Σ are precisely those which always halt in finitely many steps; this is called the *halting property* of the system. Thus, recursive coalgebras are precisely the systems having the halting property.

P. Taylor also mentioned in [13] that every recursive coalgebra $\alpha : A \longrightarrow HA$ satisfies an inductive principle called *parametric recursivity* in [6] which states that for the endofunctor $H(-) \times A$ the coalgebra $\langle \alpha, id_A \rangle : A \longrightarrow HA \times A$ is recursive. Explicitly: for every morphism $e : HX \times A \longrightarrow X$ there exists a unique morphism $e^\dagger : A \longrightarrow X$ such that the square

$$\begin{tikzcd} A \arrow[r, "\langle \alpha, id_A \rangle"] \arrow[d, "e^\dagger"'] & HA \times A \arrow[d, "He^\dagger \times id_A"] \\ X & HX \times A \arrow[l, "e"] \end{tikzcd} \tag{1.1}$$

commutes. This is the dual concept of the concept of a *completely iterative algebra* of [9].

We believe that in addition to their theoretical importance our results have many interesting applications which we illustrate with several examples. In particular, in functional programming one often uses the universal property of an initial algebra to provide a semantics of a recursive program. Recursive coalgebras extend that universal property beyond the initial algebra (considered as a coalgebra). So this provides a larger set of tools for semantics of functional programs. For example, divide-and-conquer algorithms like Quicksort can easily be formulated using recursive coalgebras. Furthermore, our characterization of recursive coalgebras gives a necessary and sufficient condition which is easy to check in order to establish recursivity in concrete examples. Finally, parametric recursivity yields an extended universal property of recursive coalgebras that is useful for the semantics of programs where the calling parameter is used not only in the base case of the recursion. This happens frequently, for example in primitive recursion.

The above results hold for every finitary endofunctor H which preserves inverse images or satisfies $H\emptyset = \emptyset$. In case H is a nontrivial, connected functor, we prove that, conversely, if for every coalgebra the equivalence

a homomorphism into the initial algebra exists $\iff$ recursive

is valid, it follows that H preserves inverse images or satisfies $H\emptyset = \emptyset$.

Preservation of inverse images is a relatively weak assumption on H: it is weaker than the (often used, see *e.g.* [12]) assumption that H preserves weak pullbacks. We provide a complete description of finitary functors preserving inverse images in Section 2 based on the concept of regular equations well known in universal algebra; our characterization appears to be new. We also present simple functors which fail to preserve inverse images but have the above equivalence property. The results of this article were announced at the workshop CALCO-jnr 2005, see [4].

2. Preservation of inverse images

Assumption 2.1. Throughout this section H denotes a finitary endofunctor of Set.

Notation 2.2. For every signature $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ we define the *polynomial endofunctor* $H_\Sigma : X \longmapsto \coprod_{n\in\mathbb{N}} \coprod_{\sigma\in\Sigma_n} X^n$.

Remark 2.3. Recall that an endofunctor H of Set is finitary, if it fulfils one of the equivalent conditions:

(i) H preserves directed colimits;
(ii) every element of HX, where X is an arbitrary set, lies in the image of Hm for some finite subset $m : M \hookrightarrow X$;
(iii) H is a quotient of some polynomial functor.

See [3]. For example, the passage (ii) $\implies$ (iii) is provided by the Yoneda Lemma: given H satisfying (ii), let Σ be the signature with $\Sigma_n = H(n)$ for all $n \in \mathbb{N}$. By the Yoneda Lemma each element of Σ_n corresponds to precisely one natural transformation $(-)^n \longrightarrow H$. These natural transformations for every n and every element of Σ_n give a natural transformation $\epsilon : H_\Sigma \longrightarrow H$, and it is easy to see that each component ϵ_X is surjective.

Definition 2.4. We call a functor F a *quotient* of a functor H, if there is a natural transformation $\varepsilon : H \longrightarrow F$ with surjective components. In case $H = H_\Sigma$, we call (Σ, ε), a *presentation* of F.

Example 2.5. The finite-power-set functor $\mathcal{P}_{fin} : X \longmapsto \{A \subseteq X \mid A \text{ finite}\}$ is finitary. It has a presentation with Σ having a unique n-ary symbol σ_n for every $n \in \mathbb{N}$, and $\varepsilon_X(\sigma_n(x_0, \dots, x_{n-1})) = \{x_0, \dots, x_{n-1}\}$.

Remark 2.6. Every finitary functor has a presentation (Σ, ε). And ε is completely described by the kernel pairs of each component ε_X, where X is a finite set, which are written in the form of equations

$$\sigma(x_0, \dots, x_{n-1}) = \varrho(y_0, \dots, y_{k-1}), \tag{2.1}$$

where σ and ϱ are operation symbols from Σ and where $x_0, \ldots, x_{n-1}$ and $y_0, \ldots, y_{k-1}$ are variables from X, see [3], III.3.3. The above equations are called *ε-equations*. Notice that for every ε-equation the function $\varepsilon_X : H_\Sigma X \longrightarrow HX$ merges both sides (which are elements of $H_\Sigma X$).

Definition 2.7. A presentation is called *regular* provided that every ε-equation has the same set of variables on both sides; more precisely: $\{x_0, \ldots, x_{n-1}\} = \{y_0, \ldots, y_{k-1}\}$ in the equations (2.1) above.

Remark 2.8. Recall that an *inverse image* of a subobject $m : B_0 \hookrightarrow B$ under a morphism $f : A \longrightarrow B$ is simply a pullback of f along m

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, hook, "n"'] & B_0 \arrow[d, hook, "m"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{2.2}$$

A functor preserving such pullbacks is said to *preserve inverse images*.

Polynomial functors H_Σ and the functor $\mathcal{P}_{fin}$ preserve inverse images. Moreover, products, coproducts and composites of functors preserving inverse images also preserve them.

Examples 2.9.

(i) The functor $(-)^3_2$, which is the subfunctor of $X \longmapsto X \times X \times X$ given by all triples (x_1, x_2, x_3) which do not have pairwise distinct components, does not preserve weak pullbacks, see [1], but it of course preserves inverse images.

(ii) Let R be the functor defined on objects by $RX = \{(x, y) \in X \times X \mid x \neq y\} + \{d\}$ and on morphisms $f : X \longrightarrow X'$ by

$$Rf(d) = d \qquad \text{and} \qquad Rf(x,y) = \begin{cases} d & \text{if } f(x) = f(y) \\ (f(x), f(y)) & \text{else.} \end{cases}$$

This functor does not preserve inverse images, consider *e.g.*

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & \{0\} \arrow[d, hook] \\ \{0,1\} \arrow[r, "\mathrm{const}_1"'] & \{0,1\} \end{tikzcd}$$

(for the elements $(0,1) \in R\{0,1\}$ and $d \in R\{0\}$ there is no suitable element of $R\emptyset$).

Theorem 2.10. *A finitary endofunctor H of* Set *preserves inverse images iff it has a regular presentation.*

Proof. (1) Let H preserve inverse images. Recall from [3], VII.2.5, that a presentation $\varepsilon : H_\Sigma \longrightarrow H$ is *minimal* provided that no n-ary operation of Σ can be substituted by an operation of arity $k < n$. More precisely, that means that for every n-ary $\sigma \in \Sigma$ the element

$$\hat{\sigma} = \varepsilon_n(\sigma(0, 1, \ldots, n-1)) \in Hn \quad (\text{where } n = \{0, 1, \ldots, n-1\})$$

does not lie in the image of Hr for any function $r : k \longrightarrow n$ with $k < n$. Every finitary functor obviously has a minimal presentation: every operation σ with $\hat{\sigma} \in Hr([Hk])$ can be substituted by a k-ary operation.

We prove that every minimal presentation of H is regular. In fact, let

$$\sigma(x_0, \ldots, x_{n-1}) = \varrho(y_0, \ldots, y_{k-1})$$

be an ε-equation. We derive a contradiction from the assumption, that

$$x_{i_0} \notin \{y_0, \ldots, y_{k-1}\}$$

for some i_0. By symmetry, this proves the regularity. Let

$$B = \{\, i \in n \mid x_i \notin \{y_0, \ldots, y_{k-1}\} \,\} \neq \emptyset.$$

Consider the n-tuple $(x_0, \ldots, x_{n-1})$ as a function $x : n \longrightarrow X$, and denote by $\bar{x} : n - B \longrightarrow \bar{X}$ its domain-codomain restriction, where $\bar{X} = X - \{x_i \mid i \in B\}$. For the inclusion map $v : \bar{X} \longrightarrow X$ form the inverse image

$$\begin{tikzcd} n - B \arrow[r, "\bar{x}"] \arrow[d, hook, "w"'] & \bar{X} \arrow[d, hook, "v"] \\ n \arrow[r, "x"'] & X \end{tikzcd}$$

The element $\sigma(0, 1, \ldots, n-1)$ of $H_\Sigma(n)$ is mapped by ε_n to $\hat{\sigma}$ and the element $\varrho(y_0, \ldots, y_{k-1})$ of $H_\Sigma \bar{X}$ is mapped by $\varepsilon_{\bar{X}}$ to

$$\varepsilon_{\bar{X}}(\varrho(y_0, \ldots, y_{k-1})) = \varepsilon_X(\sigma(x_0, \ldots, x_{n-1})) \in HX.$$

Thus in the pullback

$$\begin{tikzcd} H(n - B) \arrow[r, "H\bar{x}"] \arrow[d, hook, "Hw"'] & H\bar{X} \arrow[d, hook, "Hv"] \\ Hn \arrow[r, "Hx"'] & HX \end{tikzcd}$$

the elements $\hat{\sigma}$ and $\varepsilon_{\bar{X}}(\varrho(y_0, \ldots, y_{k-1}))$ are mapped by Hx and Hv, respectively, to the same element of HX. This implies that $\hat{\sigma}$ lies in the image of Hw, in contradiction to the minimality of the presentation ε.

(2) Let H have a regular presentation. Suppose we have an inverse image

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, hook, "w"'] & Y_0 \arrow[d, hook, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where v and w are inclusion maps, and two elements

$$\begin{array}{rcll} a & = & \varepsilon_X(\sigma(x_0, \dots, x_{n-1})) & \in HX \\ b & = & \varepsilon_{Y_0}(\varrho(y_0, \dots, y_{k-1})) & \in HY_0 \end{array}$$

with $Hf(a) = Hv(b)$. Then

$$\sigma(f(x_0), \dots, f(x_{n-1})) \quad = \quad \varrho(y_0, \dots, y_{k-1})$$

is an ε-equation because

$$\begin{array}{rcl} \varepsilon_Y(\sigma(f(x_0), \dots, f(x_{n-1}))) & = & \varepsilon_Y \cdot H_\Sigma f(\sigma(x_0, \dots, x_{n-1})) \\ & = & Hf(\varepsilon_X(\sigma(x_0, \dots, x_{n-1}))) \\ & = & Hf(a) \\ & = & Hv(b) \\ & = & Hv(\varepsilon_{Y_0}(\varrho(y_0, \dots, y_{k-1}))) \\ & = & \varepsilon_Y(\varrho(y_0, \dots, y_{k-1})). \end{array}$$

Consequently, $\{f(x_i) \mid 0 \le i \le n-1\} = \{y_j \mid 0 \le j \le k-1\} \subseteq Y_0$. Therefore, the subset $X_0 = f^{-1}(Y_0)$ contains all the variables of $\sigma(x_0, \dots, x_{n-1})$. Thus the element $a_0 = \varepsilon_{X_0}(\sigma(x_0, \dots, x_{n-1}))$ of HX_0 fulfils $Hw(a_0) = a$ and $Hf_0(a_0) = b$. $\square$

3. Recursive coalgebras

Notation 3.1. Throughout this section H denotes a finitary endofunctor of Set. Recall from [5] that H has a terminal coalgebra

$$\tau : \quad T \longrightarrow HT$$

and an initial algebra

$$\varphi : HI \longrightarrow I.$$

We consider I as a coalgebra *via* φ^{-1}. (Recall that φ is invertible due to Lambek's Lemma, see [8].) We denote by $u : I \longrightarrow T$ the unique coalgebra homomorphism.

Example 3.2. For a polynomial functor H_Σ we can describe a terminal coalgebra T_Σ as the coalgebra of all Σ-trees and an initial algebra I_Σ as the algebra of all finite Σ-trees. A coalgebra $\alpha : A \longrightarrow H_\Sigma A$ yields the unique homomorphism $h : A \longrightarrow T_\Sigma$ assigning to every state the tree unfolding.

Definition 3.3. We say that a H_Σ-coalgebra A has the *halting property*, if every tree in the image of the unique homomorphism $h : A \longrightarrow T_\Sigma$ is finite.

Example 3.4 (Ex. 3.2 continued)**.** If a system A has the halting property, then it halts after finitely many steps (no matter what the initial state is and what input string comes), and vice versa. This property becomes trivial if Σ has no constant symbols: then $I = \emptyset$ and only the empty coalgebra has the halting property.

Definition 3.5 (see [6, 13])**.** A coalgebra (A, α) is called *recursive* if for every algebra (X, e) there exists a unique coalgebra-to-algebra morphism $e^\dagger : A \longrightarrow X$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "e^\dagger"'] & HA \arrow[d, "He^\dagger"] \\ X & HX \arrow[l, "e"] \end{tikzcd}$$

A coalgebra (A, α) is called *parametrically recursive* if for every morphism $e : HX \times A \longrightarrow X$ there exists a unique morphism $e^\dagger : X \longrightarrow A$ such that the diagram (1.1) commutes.

Remarks 3.6.
(i) It is obvious that the implications

parametrically recursive $\Longrightarrow$ recursive $\Longrightarrow$ has a homomorphism into I

hold for all endofunctors H: for the first one, turn every algebra $e : HX \longrightarrow X$ into a morphism

$$HX \times A \xrightarrow{\mathsf{outl}} HX \xrightarrow{e} X.$$

(ii) The converse implications need not hold. In fact, for the functor R of 2.9(ii) both fail. Observe that here $I = T = 1$, thus every coalgebra has a homomorphism into I. However, the coalgebra $A = \{0, 1\}$ with

$$\alpha(0) = (0, 1) \quad \text{and} \quad \alpha(1) = d$$

is not recursive. In fact, let

$$e : RX \longrightarrow X$$

be any algebra which contains an element $x \in X$ such that $e(x, y) = e(y, x) = x$ for $x \neq y = e(d)$. Then any candidate of $e^\dagger : A \longrightarrow X$ must satisfy $e^\dagger(1) = y$. But, there are two possible choices $e^\dagger(0) = y$ and $e^\dagger(0) = x$.

And the recursive coalgebra $B = \{0, 1\}$ with

$$\beta(0) = \beta(1) = (0, 1)$$

is not parametrically recursive. In fact, recursivity is easily seen: for every algebra $e : RX \longrightarrow X$ the only candidate of $e^\dagger : B \longrightarrow X$ sends both 0 and 1 to $y = e(d)$. But consider any morphism $e : RX \times \{0,1\} \longrightarrow X$ such that RX contains more than one pair $(x_0, x_1), x_0 \neq x_1$, with $e((x_0, x_1), i) = x_i$ for $i = 0, 1$. Each such pair yields $e^\dagger : B \longrightarrow X$ by $e^\dagger(i) = x_i$. Thus, B is not parametrically recursive.

Remark 3.7. In the definition of recursive coalgebras the uniqueness of the morphism cannot be lifted. In fact, a coalgebra with a (not necessarily unique) homomorphism into every algebra is precisely a coalgebra with a homomorphism into I. So the non-recursive coalgebra A of Remark 3.6(ii) has, for every algebra $e : RX \longrightarrow X$, a coalgebra-to-algebra morphism, *e.g.*, the constant function with value $e(d)$. Notice that our result of Theorem 3.17 shows that the uniqueness can be lifted whenever H preserves inverse images.

Remark 3.8. The equivalence of the conditions (i), (iii) and (iv) in the following theorem can be deduced from results of Taylor [13], see Proposition 6.3.9, Theorem 6.3.13, Corollary 6.3.6 and Exercise 6.24. We present a (short) full proof for the sake of completeness:

Theorem 3.9. *For every Σ-coalgebra A the following conditions are equivalent:*

(i) *A is recursive;*
(ii) *A has the halting property;*
(iii) *a coalgebra homomorphism from A to I_Σ exists; and*
(iv) *A is parametrically recursive.*

Proof. The equivalence of (iii) and (ii) is obvious from the fact that the unique coalgebra homomorphism $A \longrightarrow T_\Sigma$ assigns to every state the tree-unfolding. And A has the halting property iff the unique homomorphism from A to T_Σ factors through $u : I_\Sigma \longrightarrow T_\Sigma$.

It remains to prove (iii) $\Rightarrow$ (iv). Given $e : H_\Sigma X \times A \longrightarrow X$, we are to prove that there exists precisely one $e^\dagger : A \longrightarrow X$ equal to $e \cdot (H_\Sigma e^\dagger \times id_A) \cdot \langle \alpha, id_A \rangle$. We start with a homomorphism

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & H_\Sigma A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle H_\Sigma h} \\ I_\Sigma & \xrightarrow[\varphi_\Sigma^{-1}]{} & H_\Sigma I_\Sigma \end{array}$$

Then $A = \bigcup_{i \in \mathbb{N}} A_i$ where A_0 are the halting states,

$$A_0 \quad = \quad \{a \in A \mid \alpha(a) \in \Sigma_0\}$$

and given A_i then

$$A_{i+1} \quad = \quad A_i \cup \{a \in A \mid \alpha(a) \in H_\Sigma A_i\}.$$

In fact, since h is a homomorphism, it is easy to prove by induction on i that A_i is the inverse image of the set of all Σ-trees of depth $\leq i$ under h, therefore, every element of A lies in some A_i.

The morphism $e^\dagger$ is uniquely determined

(a) on A_0, since if $\alpha(a) = \sigma \in \Sigma_0$, then $e^\dagger(a) = e(H_\Sigma e^\dagger(\sigma), a) = e(\sigma, a)$;

(b) on A_{i+1} whenever it is uniquely determined on A_i since if $\alpha(a) = \sigma(a_0, \ldots, a_{n-1})$ for some $\sigma \in \Sigma_n$ and $a_t \in A_i$ with $0 \leq t < n$, then

$$e^\dagger(a) = e(H_\Sigma e^\dagger(\sigma(a_0, \ldots, a_{n-1})), a) = e(\sigma(e^\dagger(a_0), \ldots, e^\dagger(a_{n-1})), a).$$

Therefore, A is parametrically recursive. □

Example 3.10. The functor

$$HX = X + 1$$

has unary algebras with a constant as H-algebras, and partial unary algebras as H-coalgebras. The coalgebra $\mathbb{N}$ of natural numbers with the partial operation $n \longmapsto n - 1$ (defined iff $n > 0$) obviously has the halting property. Consequently, it is parametrically recursive. Thus every function

$$e = [u, v] : HX \times \mathbb{N} = X \times \mathbb{N} + \mathbb{N} \longrightarrow X$$

(with $u : X \times \mathbb{N} \longrightarrow X$ and $v : \mathbb{N} \longrightarrow X$) defines a unique sequence

$$e^\dagger : \mathbb{N} \longrightarrow X, \quad e^\dagger(n) = x_n$$

in X such that the diagram (1.1) commutes, which means that $x_0 = v(0)$ and $x_{n+1} = u(x_n, n + 1)$. For example, the factorial function is then given by the choice $X = \mathbb{N}$; $u(n, m) = n \cdot m$ and $v(0) = 1$.

Example 3.11. For the functor H given by

$$HX \quad = \quad X \times X + 1$$

H-algebras are the algebras on one binary operation and one constant. Coalgebras are deterministic systems with a binary input and with halting states (expressed by the inverse image of the right hand summand 1 under the dynamics $\alpha : A \longrightarrow A \times A + 1$).

The coalgebra $\mathbb{N}$ of natural numbers with halting states 0 and 1 and dynamics $\alpha : n \longmapsto (n-1, n-2)$ for $n \geq 2$ obviously has the halting property. Consequently, $\mathbb{N}$ is parametrically recursive.

To define the Fibonacci sequence, consider the morphism

$$e : H\mathbb{N} \times \mathbb{N} = \mathbb{N}^3 + \mathbb{N} \longrightarrow \mathbb{N}$$

given by

$$(i, j, k) \longmapsto i + j \quad \text{and} \quad n \longmapsto \begin{cases} a_0 & n = 0 \\ a_1 & n = 1 \\ 0 & n \geq 2. \end{cases}$$

We know that there is a unique sequence $e^\dagger$ such that the diagram (1.1) commutes, which means $x_0 = a_0$, $x_1 = a_1$ and $x_{n+2} = x_{n+1} + x_n$.

Example 3.12 (Quicksort, see [6])**.** Let A be any linearly ordered set (of data elements). Then Quicksort is usually given in terms of the following recursive definition

$$\begin{array}{rrcl} \mathsf{q}_{\mathsf{sort}}: & A^* & \longrightarrow & A^* \\ & \varepsilon & \longmapsto & \varepsilon \\ & a \cdot w & \longmapsto & \mathsf{q}_{\mathsf{sort}}(w_{\leq a}) \star (a \cdot \mathsf{q}_{\mathsf{sort}}(w_{>a})), \end{array}$$

where A^* is the set of all lists on A, ε is the empty list, $\star$ is the concatenation of lists and $w_{\leq a}$ and $w_{>a}$ denote the lists of those elements of w which are less than or equal, or greater than a, respectively. Now consider the functor $HX = A \times X \times X + 1$, where $1 = \{\bullet\}$, and form the coalgebra

$$\begin{array}{rrcl} \mathsf{q}_{\mathsf{split}}: & A^* & \longrightarrow & A \times A^* \times A^* + 1 \\ & \varepsilon & \longmapsto & \bullet \\ & a \cdot w & \longmapsto & (a, w_{\leq a}, w_{>a}). \end{array}$$

This coalgebra obviously has the halting property. Thus, for the H-algebra

$$\begin{array}{rrcl} \mathsf{q}_{\mathsf{merge}}: & A \times A^* \times A^* + 1 & \longrightarrow & A^* \\ & \bullet & \longmapsto & \varepsilon \\ & (a, w, v) & \longmapsto & w \star (av) \end{array}$$

there exists a unique function $\mathsf{q}_{\mathsf{sort}}$ on A^* such that

$$\mathsf{q}_{\mathsf{sort}} = \mathsf{q}_{\mathsf{merge}} \cdot H(\mathsf{q}_{\mathsf{sort}}) \cdot \mathsf{q}_{\mathsf{split}}.$$

Notice that the last equation reflects the idea that Quicksort is a "divide-and-conquer"-algorithm. The coalgebra structure $\mathsf{q}_{\mathsf{split}}$ divides a list into two parts $w_{\leq a}$ and $w_{>a}$, then $H(\mathsf{q}_{\mathsf{sort}})$ sorts these two smaller lists, and finally in the "combine"-step (or "conquer"-step) the algebra structure $\mathsf{q}_{\mathsf{merge}}$ merges the two sorted parts to obtain the desired whole sorted list.

Similarly, functions defined by parametrical recursivity, see Diagram (1.1), can be understood as "divide-and-conquer"-algorithms, where the "combine"-step is allowed to access the original parameter additionally. For instance, in our current example the "divide"-step $\langle \mathsf{q}_{\mathsf{split}}, id_{A^*} \rangle$ produces the pair consisting of $(a, w_{\leq a}, w_{>a})$ and the original parameter $a \cdot w$, and the "combine"-step which is given by an algebra $HX \times A^* \longrightarrow X$ will by the commutativity of (1.1) get $a \cdot w$ as its right-hand input.

Definition 3.13. Let $\varepsilon : H_\Sigma \longrightarrow H$ be a presentation of a set functor H. A coalgebra $\alpha : A \longrightarrow HA$ for H is said to be *presented* by a H_Σ-coalgebra $\bar{\alpha} : A \longrightarrow H_\Sigma A$ if $\alpha = \varepsilon_A \cdot \bar{\alpha}$. If some presentation of A has the halting property (w.r.t. H_Σ), we say that the H-coalgebra A *has the halting property.*

Observation 3.14. Let $\varepsilon : H_\Sigma \longrightarrow H$ be a presentation, and $\alpha : A \longrightarrow HA$ be a coalgebra. Choose any $m : HA \longrightarrow H_\Sigma A$ with $\varepsilon_A \cdot m = id_{HA}$ and consider

A as a Σ-coalgebra *via* $\bar{\alpha} = m \cdot \alpha$. Clearly, this is a presentation of A, and A is a (parametrically) recursive coalgebra for H if it is (parametrically) recursive for H_Σ. In fact, given $e : HX \times A \longrightarrow X$, then morphisms $f = e^\dagger$ for H are precisely the morphisms $f = \bar{e}^\dagger$ for H_Σ, where $\bar{e} = e \cdot (\varepsilon_X \times id_A)$:

$$\begin{tikzcd}
A \arrow[r, "\langle \bar{\alpha}, id_A \rangle"] \arrow[d, "f"'] \arrow[rr, bend left, "\langle \alpha, id_A \rangle"] & H_\Sigma A \times A \arrow[r, "\varepsilon_A \times id_A"] \arrow[d, "H_\Sigma f \times id_A"] & HA \times A \arrow[d, "Hf \times id_A"] \\
X & H_\Sigma X \times A \arrow[l, "\bar{e}"] \arrow[r, "\varepsilon_X \times id_A"'] & HX \times A \arrow[ll, bend left, "e"]
\end{tikzcd}$$

In fact, the outer square of this diagram commutes iff the left-hand inner square does since all other parts trivially commute.

Remarks 3.15.

(i) For every presentation $\varepsilon : H_\Sigma \longrightarrow H$ we have the initial H-algebra I as a quotient of the initial Σ-algebra I_Σ *via* the unique Σ-algebra homomorphism

$$i : I_\Sigma \longrightarrow I\,,$$

where I is considered as the Σ-algebra

$$H_\Sigma I \xrightarrow{\varepsilon_I} HI \xrightarrow{\varphi} I\,.$$

In fact, I can be considered as the quotient of the Σ-algebra I_Σ modulo the congruence generated by ε-equations, see Remark 2.6.

(ii) Let $\alpha : A \longrightarrow HA$ be a coalgebra with a presentation $\bar{\alpha} : A \longrightarrow H_\Sigma A$. Every homomorphism $f : A \longrightarrow I_\Sigma$ of H_Σ-coalgebras defines a homomorphism $i \cdot f : A \longrightarrow I$ of H-coalgebras. In fact, from the equations $i \cdot \varphi_\Sigma = (\varphi \cdot \varepsilon_I) \cdot H_\Sigma i$ and $\varphi_\Sigma^{-1} \cdot f = H_\Sigma f \cdot \bar{\alpha}$ we easily derive $\varphi^{-1} \cdot f = Hf \cdot \alpha$.

(iii) We also have the terminal H-coalgebra T as a quotient of the terminal Σ-coalgebra T_Σ *via* the unique H-coalgebra homomorphism

$$j : T_\Sigma \longrightarrow T\,,$$

where T_Σ is considered as the H-coalgebra

$$T_\Sigma \xrightarrow{\tau_\Sigma} H_\Sigma T_\Sigma \xrightarrow{\varepsilon_{T_\Sigma}} HT_\Sigma\,.$$

In fact, as proved in [2], T can be considered as the quotient of the Σ-coalgebra T_Σ modulo infinite application of ε-equations.

Finally, for every functor H we have the unique coalgebra homomorphism

$$u : I \longrightarrow T.$$

In case $H = H_\Sigma$ we denote it by

$$u_\Sigma : I_\Sigma \longrightarrow T_\Sigma;$$

this is the inclusion map (of all finite Σ-trees into all Σ-trees).

Lemma 3.16. *If H is a finitary functor preserving inverse images, then a regular presentation leads to a pullback*

$$\begin{tikzcd} I_\Sigma \arrow[r, "u_\Sigma"] \arrow[d, "i"'] & T_\Sigma \arrow[d, "j"] \\ I \arrow[r, "u"'] & T \end{tikzcd}$$

Proof. It is quite easy to show that $j \cdot u_\Sigma$ and $u \cdot i$ are both H-coalgebra homomorphisms, and since T is the terminal H-coalgebra, we obtain that they are equal. Given Σ-trees $s \in I_\Sigma$ and $t \in T_\Sigma$ with $u(i(s)) = j(t)$, it is our task to show that $t \in I_\Sigma$—it then follows that the above square is a weak pullback, and since u_Σ is a monomorphism, it is a pullback. The proof is an easy induction on the depth n of the finite tree s: we prove that t and s have the same depth. The equality $u(i(s)) = j(t)$ implies, due to the regularity of the presentation, that we can obtain t from s by applying ε-equations on (subtrees of) nodes of s. Since s is finite, it is sufficient to consider one ε-equation applied to one node of s.

Case $n = 0$: the regularity of the presentation tells us that since s is a nullary symbol, every ε-equation with s on one side has a constant symbol on the other side. Thus, t is a nullary symbol.

Induction step: if the node of s to which the given ε-equation is applied is not the root, use the induction hypothesis. And if it is the root, then we consider the form

$$\sigma(x_0, \dots, x_{m-1}) \quad = \quad \varrho(y_0, \dots, y_{k-1})$$

of the ε-equation used, see Remark 2.6: it follows that

$$s \quad = \quad \sigma(s_0, \dots, s_{m-1})$$

for trees $s_0, \dots, s_{m-1}$, and since the variables $y_0, \dots, y_{k-1}$ form the same set as $x_0, \dots, x_{m-1}$, we conclude that t has the root labeled by ϱ and has the same set of children as s, thus, t has the same depth as s. □

Theorem 3.17. *Let H be a finitary endofunctor of* Set *preserving inverse images. Then for every H-coalgebra A the following conditions are equivalent:*

(i) *A is recursive;*
(ii) *A has the halting property;*
(iii) *a coalgebra homomorphism from A to I exists; and*
(iv) *A is parametrically recursive.*

Proof. (i) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii) Let $h : A \longrightarrow I$ be a homomorphism. For the coalgebra $\alpha : A \longrightarrow HA$ choose a presentation by putting $\bar{\alpha} = m \cdot \alpha : A \longrightarrow H_\Sigma A$ where $m : HA \longrightarrow H_\Sigma A$ is a morphism with $\varepsilon_A \cdot m = id_{HA}$. Let $k : A \longrightarrow T_\Sigma$ be the unique Σ-coalgebra homomorphism from $(A, m\cdot\alpha)$ to (T_Σ, τ_Σ). Then $u \cdot h$ and $j \cdot k$ are both H-coalgebra homomorphisms from (A, α) to (T, τ), in fact, for $j \cdot k$ consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[rrr, bend left, "\alpha"] \arrow[d, "k"'] & HA \arrow[r, "m"] & H_\Sigma A \arrow[r, "\varepsilon_A"] \arrow[d, "H_\Sigma k"] & HA \arrow[d, "Hk"] \\
T_\Sigma \arrow[rr, "\tau_\Sigma"'] \arrow[d, "j"'] & & H_\Sigma T_\Sigma \arrow[r, "\varepsilon_{T_\Sigma}"'] & HT_\Sigma \arrow[d, "Hj"] \\
T \arrow[rrr, "\tau"'] & & & HT
\end{tikzcd}$$

Due to the pullback in Lemma 3.16 we obtain the unique morphism

$$l : A \longrightarrow I_\Sigma \quad \text{with} \quad h = i \cdot l \quad \text{and} \quad k = u_\Sigma \cdot l.$$

Then l is a Σ-coalgebra homomorphism because $H_\Sigma u_\Sigma$ is a monomorphism and in the diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "l"] \arrow[dd, bend right=60, "k"'] & HA \arrow[r, "m"] & H_\Sigma A \arrow[d, "H_\Sigma l"'] \arrow[dd, bend left=60, "H_\Sigma k"] \\
I_\Sigma \arrow[rr, "\varphi_\Sigma^{-1}"'] \arrow[d, "u_\Sigma"] & & H_\Sigma I_\Sigma \arrow[d, "H_\Sigma u_\Sigma"'] \\
T_\Sigma \arrow[rr, "\tau_\Sigma"'] & & H_\Sigma T_\Sigma
\end{tikzcd}$$

the outside square and all inner parts except the upper one commute. Thus, the H_Σ-coalgebra A has the halting property by Theorem 3.9.

(ii) $\Rightarrow$ (iv) Let $\alpha : A \longrightarrow HA$ have the halting property, and choose some presentation $\bar{\alpha} : A \longrightarrow H_\Sigma A$ having the halting property, too (see Def. 3.13). By Theorem 3.9, A is parametrically recursive for H_Σ. Finally, the same argument as in Observation 3.14 shows that A is parametrically recursive for H.

(iv) $\Rightarrow$ (i) is trivial. $\square$

Example 3.18. A $\mathcal{P}_{fin}$-coalgebra is a finitely branching graph A: the structure map $\alpha : A \longrightarrow \mathcal{P}_{fin} A$ assigns to every node the set of all neighbor nodes. Such a graph is recursive iff it has no infinite paths.

Example 3.19. Finitely branching labelled transition systems are coalgebras for the functor $\mathcal{P}_{fin}(\Sigma \times -)$, where Σ is the set of all actions. Recursivity means that every development ends in finitely many steps in a state without transitions.

Remark 3.20. Recall from [14] that a set functor H is *connected* (*i.e.*, is not a coproduct of proper subfunctors) iff $H1 \cong 1$. We call H *trivial* if $HA \cong 1$ for all sets $A \neq \emptyset$.

Theorem 3.21. *For a nontrivial, connected endofunctor H the following conditions are equivalent:*

(i) *every coalgebra, for which a homomorphism into I exists, is recursive;*
(ii) $H\emptyset = \emptyset$.

Proof. It is obvious that (ii) $\Rightarrow$ (i) since $I = \emptyset$, thus, only the empty coalgebra has a homomorphism into I. Conversely, suppose $H\emptyset \neq \emptyset$, then we construct a non-recursive coalgebra. This is sufficient because every coalgebra has a homomorphism into I: since H is connected, $T = 1$, and since $H\emptyset \neq \emptyset$, we have $I \neq \emptyset$. However, there always exists a monomorphism $u : I \hookrightarrow T$, thus,

$$I \cong T.$$

By Lemma 4.3 in [7], since H is nontrivial, there exists a set A such that

$$\operatorname{card} HA \geq \operatorname{card} A > 1.$$

Choose $e : HA \longrightarrow A$ and $\alpha : A \longrightarrow HA$ with $e \cdot \alpha = id_A$. Then the coalgebra (A, α) is not recursive: for the algebra (A, e) one candidate of $e^\dagger$ is id_A:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "id_A"'] & HA \arrow[d, "Hid_A"] \\ A & HA \arrow[l, "e"] \end{tikzcd}$$

Another candidate is obtained by choosing an element $d \in H\emptyset$: for every set X the empty map $r_X : \emptyset \longrightarrow X$ yields an element $d_X = Hr_X(d)$ such that

$$Hf(d_X) = d_Y \quad \text{for all functions} \quad f : X \longrightarrow Y.$$

Consequently, the constant function $c : A \longrightarrow A$ of value $e(d_A)$ also makes the square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "c"'] & HA \arrow[d, "Hc"] \\ A & HA \arrow[l, "e"] \end{tikzcd}$$

commute: in fact, since c factorizes through $A \longrightarrow 1$, it follows that Hc factorizes through $H(A \longrightarrow 1)$, thus, since H is connected, Hc is the constant function of value d_A. And $c \neq id_A$ because $\operatorname{card} A > 1$. $\square$

Example 3.22. There exists a functor not preserving inverse images, but having the property that for all coalgebras the equivalences

$$\text{a homomorphism into } I \text{ exists} \iff \text{recursive} \iff \text{parametrically recursive}$$

hold. Change the value of R, see Example 2.9(ii), in the empty set to the value $\emptyset$. The only coalgebra having a homomorphism into $I = \emptyset$ is the empty one.

4. Conclusions

We study coalgebras for finitary set functors H, making use of the presentation of such functors as (precisely all) quotients of polynomial functors H_Σ modulo ε-equations. We proved that the condition of H preserving inverse images, useful in various parts of coalgebra theory, is equivalent to the fact that ε-equations have the same set of variables on both sides.

Our main result is a characterization of recursive H-coalgebras as studied by Taylor [13] and recently by Capretta *et al.* [6]; those are coalgebras with a unique morphism into every algebra. We prove that recursive coalgebras are precisely those describing systems with the "halting property", *i.e.*, such that when started in any fixed state, the system halts in finitely many steps. This holds for finitary set functors preserving inverse images.

References

[1] P. Aczel and N. Mendler, *A Final Coalgebra Theorem*, Proceedings Category Theory and Computer Science, edited by D.H. Pitt *et al. Lect. Notes Comput. Sci.* (1989) 357–365.
[2] J. Adámek and S. Milius, Terminal coalgebras and free iterative theories. *Inform. Comput.* **204** (2006) 1139–1172.
[3] J. Adámek and V. Trnková, *Automata and Algebras in Categories*. Kluwer Academic Publishers (1990).
[4] J. Adámek, D. Lücke and S. Milius, *Recursive coalgebras of finitary functors*, in CALCO-jnr 2005 CALCO Young Researchers Workshop Selected Papers, edited by P. Mosses, J. Power and M. Seisenberger, Report Series, University of Swansea, 1–14.
[5] M. Barr, Terminal coalgebras in well-founded set theory. *Theoret. Comput. Sci.* **114** (1993) 299–315.
[6] V. Capretta, T. Uustalu and V. Vene, Recursive coalgebras from comonads. *Inform. Comput.* **204** (2006) 437–468.
[7] V. Koubek, Set functors. *Comment. Math. Univ. Carolin.* **12** (1971) 175–195.

[8] J. Lambek, A fixpoint theorem for complete categories. *Math. Z.* **103** (1968) 151–161.
[9] S. Milius, Completely iterative algebras and completely iterative monads. *Inform. Comput.* **196** (2005) 1–41.
[10] R. Montague, Well-founded relations; generalizations of principles of induction and recursion (abstract). *Bull. Amer. Math. Soc.* **61** (1955) 442.
[11] G. Osius, Categorical set theory: a characterization of the category of sets. *J. Pure Appl. Algebra* **4** (1974) 79–119.
[12] J. Rutten, Universal coalgebra, a theory of systems. *Theoret. Comput. Sci.* **249** (2000) 3–80.
[13] P. Taylor, *Practical Foundations of Mathematics.* Cambridge University Press (1999).
[14] V. Trnková, On a descriptive classification of set-functors I. *Comment. Math. Univ. Carolin.* **12** (1971) 143–174.

Communicated by Z. Esik.
Received June 14, 2005. Accepted February 9, 2007.