RAIRO. INFORMATIQUE THÉORIQUE

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RAIRO. Informatique théorique, tome 13, nº 2 (1979), p. 107-133
<http://www.numdam.org/item?id=ITA_1979__13_2_107_0>

R.A.I.R.O. Informatique théorique/Theoretical Informatics
(vol. 13, n° 2, 1979, p. 107 à 133)

CATEGORICAL APPROACH TO NONLINEAR CONSTANT CONTINUOUS-TIME SYSTEMS (*)

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Communicated by W. Brauer

Abstract. — *The state-transition function of a constant, continuous-time system is shown to be a right action of a monoid on the state space. Using this fact, categorical approaches to reduction, reachability, observability and minimal realization, which were mainly developed for discrete-time automata and systems, can be also applied to continuous-time systems. In this way, some known results can be unified and several new results for different types of constant, continuous-time systems are obtained.*

Résumé. — *On montre que la fonction de transitions d'un système constant à temps continu est l'action d'un monoïde opérant à droite sur l'espace d'états. Grâce à quoi on peut appliquer aux systèmes à temps continu les approches catégorielles de la réduction, de la connexité, de l'observabilité et de la réalisation minimale qui avaient été développées surtout pour les automates et les systèmes à temps discret. De cette manière, on peut unifier quelques résultats connus et en obtenir de nouveaux pour différents types de systèmes constants à temps continu.*

INTRODUCTION

Continuous-time systems can be studied on two different levels of representation. On one hand, given by a differential equation and on the other hand given in state-transition form. While the first representation is used in most of the literature the general mathematical treatment of dynamical systems in [7], for example, is started with systems in state-transition form. For smooth systems it is shown that the state-transition function satisfies a differential equation leading to the first way of representation.

We claim that for problems of reachability, observability and minimal realization the state-transition form is much more adequate than the differential

(*) Received July 1977, revised september 1978.
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R.A.I.R.O. Informatique théorique/Theoretical Informatics, 0399-0540/1979/107/$ 4.00

equation form. In [10] and [11] for example, bilinear and nonlinear systems are studied with respect to these problems but the state-transition form is only used implicitely. Especially the fact that the state-transition function $\varphi : X \times \mathbf{R}^+ \times \Omega \to X$ can be regarded as a right action of the semigroup $\mathrm{R}^+ \times \Omega$ (Ω = space of input functions or controls) on the state space X can be used to solve the problems mentioned above in the same way as in the case of discrete-time automata where the state-transition function extended to the free monoid of the input alphabet is a right action. Using the concatenation of input functions as multiplication the space Ω and hence also $\mathbf{R}^+ \times \Omega$ becomes a semigroup. (This has already been used in [7], VI.1.) Actually, we will use a quotient monoid $\mathbf{R}^+ \underline{\times} \Omega$ of $\mathbf{R}^+ \times \Omega$, which is very similar to the semigroup $S(U)$ used in [12].

Starting with the basic definitions of dynamical systems in [7] we will show in section 1 that the state-transition function of constant, continuous-time systems becomes a right action of the monoid $\mathbf{R}^+ \underline{\times} \Omega$ on the state space such that these systems can be regarded as monoid automata. In section 2 we review the theory of monoid automata in closed categories from [3] and the concepts of minimal realization in [4]. Moreover, we give the explicit construction and proof for left and right adjoints of the forgetful functor $V : \mathbf{Ract}_M \to \mathbf{K}$ where $\mathbf{Ract}_M$ is the category of right actions of a monoid M in the closed base category $\mathbf{K}$. This result is most important in order to apply the general concepts of minimal realization to constant, continuous-time systems. These applications will be given in section 3. In particular, we will study the following types:

1. Constant, continuous-time systems (without additional structure, in the sense of [7]).
2. Bicontinuous systems (all basic sets are topological space and all functions are continuous in each variable).
3. Compactly generated systems (all basic sets are compactly generated Hausdorff spaces).
4. Semilinear systems ($\varphi(-, (t, w)) : X \to X$ is linear).
5. Bilinear systems (in the sense of [10] and [12]).
6. Semilinear bicontinuous systems (combination of 2 and 4).
7. Linear systems ($\varphi(-, t, -) : X \times \Omega \to X$ is linear).
8. Smooth systems (similar to [11] and [7]).

Let us point out, however, that this paper is only a first approach to study continuous-time systems with categorical methods. The basic concepts used from [4] are still not general enough to cover the case of smooth systems (*see* 3.8) explicitly.

ACKNOWLEDGEMENT

We thank W. Brauer for stimulating remarks and suggestions which led to a considerably improved exposition.

1. FROM CONTINUOUS-TIME SYSTEMS TO MONOID-AUTOMATA

Following [7] definitions (1.1), (1.2) and (1.3) we use the following notation:

1.1. Definition

A *continuous-time dynamical system* is a construct

$$\Sigma=(X, \mathbf{R}, U, \Omega, Y, \varphi, \eta)$$

where X (state set), U (input values), and Y (output values) are sets, $\mathbf{R}$ the real numbers, Ω (input functions) is a nonempty subset of all functions from $\mathbf{R}$ to U, φ (state-transition function) a (partial) function

$$\varphi : \quad X \times \mathbf{R} \times \mathbf{R} \times \Omega \to X$$

and η (readout map) a function

$$\eta : \quad X \times \mathbf{R} \to Y$$

satisfying the following axioms:

(*a*) Ω is closed under concatenation of inputs, i. e. for $w, w' \in \Omega$ and $t_1 < t_2 < t_3$ there is an $w'' \in \Omega$ such that we have for the restrictions of the functions

$$w''\big|_{(t_1, t_2]} = w\big|_{(t_1, t_2]} \qquad \text{and} \qquad w''\big|_{(t_2, t_3]} = w'\big|_{(t_2, t_3]}.$$

(*b*) (Direction of time). φ is defined for all

$$(x, \tau, t, w) \in X \times \mathbf{R} \times \mathbf{R} \times \Omega \qquad \text{with} \quad t \geqq \tau.$$

(*c*) (Consistency). $\varphi(x, t, t, w)=x$ for all $x \in X$, $t \in \mathbf{R}$, $w \in \Omega$.

(*d*) (Composition property). For any $t_1 < t_2 < t_3$ we have

$$\varphi(x, t_1, t_3, w)=\varphi(\varphi(x, t_1, t_2, w), t_2, t_3, w).$$

(*e*) (Causality). If $w, w' \in \Omega$ with $w\big|_{(\tau, t]} = w'\big|_{(\tau, t]}$ then

$$\varphi(x, \tau, t, w)=\varphi(x, \tau, t, w').$$

The only difference to [7] is that we have not mentioned the output function explicitly. But we use the same output function namely that given by

$$\eta(\varphi(x, \tau, t, w), t) \qquad \text{for} \quad t \geqq \tau.$$

1.2 Definition

A continuous-time dynamical system is *constant* iff:

(*a*) Ω is closed under the *shift operator* $Z^\tau : w \mapsto w'$ defined by $w'(t) = w(t-\tau)$ for $\tau, t \in \mathbf{R}$.

(*b*) $\varphi(x, \tau, t, w) = \varphi(x, \tau + s, t + s, Z^s w)$ for all $s \in \mathbf{R}$.

(*c*) The map $\eta(., t) : X \to Y$ is independent of t.

In the following we will assume that all our systems are constant. Then we will make the following notational simplifications:

Using 1.2 (*b*) we can assume that the initial time τ is 0 and by 1.1 (*b*) φ becomes a total function $\varphi : X \times \mathbf{R}^+ \times \Omega \to X$ where the initial time $\tau = 0$ is omitted and $t \in \mathbf{R}^+ = [0, \infty)$.

Using 1.2 (*c*), η becomes a function $\eta : X \to Y$. Causality 1.1 (*e*) means that φ can be regarded as a function $\varphi : X \times (\mathbf{R}^+ \underline{\times} \Omega) \to X$ where:

1.3 $\mathbf{R}^+ \underline{\times} \Omega$ denotes the quotient space of $\mathbf{R}^+ \times \Omega$ by the equivalence relation

$$(t, w,) \sim (t', w') \quad \Leftrightarrow \quad t = t' \quad \text{and} \quad w\,|_{(0,t]} = w'\,|_{(0,t]}.$$

Using 1.2 (*b*) and initial time $\tau = 0$ the composition property 1.1 (*a*) can be written as

$$\varphi(x, t_3 - t_1, Z^{-t_1} w) = \varphi(\varphi(x, t_2 - t_1, Z^{-t_1} w), t_3 - t_2, Z^{-t_2} w)$$

for all $t_1 < t_2 < t_3$ and $w \in \Omega$.

Using $t'_3 = t_3 - t_2$ and $t'_2 = t_2 - t_1$ and $w' = Z^{-t_1} w$ we obtain

$$\varphi(x, t'_3 + t'_2, w') = \varphi(\varphi(x, t'_2, w'), t'_3, Z^{-t'_2} w').$$

That is in other notation

$$\varphi(x, t_1 + t_2, w) = \varphi(\varphi(x, t_1, w), t_2, Z^{-t_1} w)$$

for all $t_1, t_2 > o$ and all $w \in \Omega$.

But this is also equivalent to

1.4 $$\varphi(x, t_1 + t_2, w_1 \star_{t_1} w_2) = \varphi(\varphi(x, t_1, w_1), t_2, w_2)$$

for all $t_1, t_2 > 0$ and all $w_1, w_2 \in \Omega$ taking $w = w_1 \star_{t_1} w_2$ with

1.5
$$w_1 \star_{t_1} w_2(t) = \begin{cases} w_1(t) & \text{for } 0 < t \leqq t_1, \\ w_2(t-t_1) & \text{for } t_1 < t \leqq t_1 + t_2. \end{cases}$$

In fact, we have $w\,|_{(0,\, t_1]} = w_1$ and $Z^{-t_1} w(t) = w(t+t_1) = w_2(t)$ for $0 < t \leqq t_2$. Hence the composition property 1.1 (*d*) is equivalent to (1.4) provided that $w_1 \star_{t_1} w_2$ is again in Ω for $w_1,\ w_2 \in \Omega$. But this is true because, using [1.2 (*a*)], the concatenation-closure of Ω in [1.1 (*a*)] is equivalent to the fact that for all $w_1,\ w_2 \in \Omega$ and $t_1,\ t_2 \in \mathbf{R}^+$ there is an $w \in \Omega$ satisfying (1.5).

In the cases $t_1 = 0$ or $t_2 = 0$ w_1 is restricted to $(0,\ 0]$ or w_2 is restricted to $(t_1,\ t_1]$. In both cases the interval is empty, so that $w_1\,|_{(0,\, 0]} = w_0 = w_2\,|_{(t_1,\, t_1]}$ where $w_0 : \emptyset \to U$ is the unique function from the empty set $\emptyset$ to U.

Hence, consistency in 1.1 (*c*) is equivalent to

1.6
$$\varphi(x,\ 0,\ w_0) = x \quad \text{for all } x \in X.$$

Summarizing the simplifications we can say (*cf.* section 6.1 in [7]) :

1.7 DEFINITION

A *constant, continuous-time system* is a construct

$$\Sigma = (X,\ \mathbf{R}^+ \underline{\times} \Omega,\ Y,\ \varphi,\ \eta)$$

where X and Y are sets, Ω a nonempty subset of functions from $(0,\ \infty)$ to a set U, $\mathbf{R}^+ \underline{\times} \Omega$ as defined in 1.3, and φ and η are functions

$$\varphi : \quad X \times (\mathbf{R}^+ \underline{\times} \Omega) \to X,$$
$$\eta : \quad X \to Y,$$

satisfying the following axioms:

(*a*) $\mathbf{R}^+ \underline{\times} \Omega$ is closed under concatenation, i. e. for all $(t_1,\ w_1)$, $(t_2,\ w_2) \in \mathbf{R}^+ \underline{\times} \Omega$ we also have $(t_1 + t_2,\ w_1 \star_{t_1} w_2) \in \mathbf{R}^+ \underline{\times} \Omega$ with $w_1 \star_{t_1} w_2$ defined in (1.5).

(*b*) (Consistency) $\varphi(x,\ 0,\ w_0) = x$, for all $x \in X$ and the unique function $w_0 : \emptyset \to U$.

(*c*) (Composition property) For all $(t_1,\ w_1)$, $(t_2,\ w_2) \in \mathbf{R}^+ \underline{\times} \Omega$:

$$\varphi(\varphi(x,\ t_1,\ w_1),\ t_2,\ w_2) = \varphi(x,\ t_1 + t_2,\ w_1 \star_{t_1} w_2).$$

Note, that direction of time [1.1 (*b*)] and causality [1.1 (*e*)] are trivially satisfied because the initial time τ is zero and $\varphi : X \times (\mathbf{R}^+ \underline{\times} \Omega) \to X$ is a total function. In view of the simplifications of definition 1.1 given in 1.3-1.6, we can state the following corollary:

1.8 Corollary

Given a set Ω of nonempty functions $w : \mathbf{R} \to U$, which is closed under the shift operator [cf. 1.1 (*a*)], *there is a 1.1 -correspondence between constant, continuous-time, dynamical systems* (*in the sense of* 1.1 *and* 1.2) *and constant continuous-time systems* (*in the sense of* 1.7).

Now, we will show that $\mathbf{R}^+ \underline{\times} \Omega$ has the structure of a monoid such that the state-transition-function φ becomes a right action of $\mathbf{R}^+ \underline{\times} \Omega$ on the state space X. (Note that this property essentially has been used in [14].) But first let us review the notions of a right action (*cf.* [8]) and a monoid automaton (*cf.* [3]). They are given for the category of sets in 1.9 in such a way that the same notation can be used in the general case (given in 2.4).

1.9 Definition

Given a monoid $(M, \star, e)$ with multiplication $\star : M \times M \to M$ and unit $e : M^0 \to M$ where M^0 is a set with one element, then we have in the category of sets:

(1) An object X together with a morphism $\varphi : X \times M \to X$ is called *right action*, in short $(X, \varphi) \in \mathbf{Ract}_M$, if the following diagrams are commutative:

(*a*)
$$\begin{tikzcd}
X \underline{\times} M^0 \arrow[r, "id_X \underline{\times} e"] \arrow[dr, "\sim"'] & X \underline{\times} M \arrow[d, "\varphi"] \\
& X
\end{tikzcd}$$

where $\xrightarrow{\sim}$ is used for natural isomorphisms,

(*b*)
$$\begin{tikzcd}
X \underline{\times} M \underline{\times} M \arrow[r, "id_X \underline{\times} \star"] \arrow[d, "\varphi \underline{\times} id_M"'] & X \underline{\times} M \arrow[d, "\varphi"] \\
X \underline{\times} M \arrow[r, "\varphi"] & X
\end{tikzcd}$$

(2) A *monoid automaton* $A = (M, X, Y, \varphi, \eta)$ consists of a right action $(X, \varphi) \in \mathbf{Ract}_M$ and an *output morphism* $\eta : X \to Y$.

1.10 THEOREM

(1) *The object* $\mathbf{R}^+ \underline{\times} \Omega$ *together with unit* $e=(0, \omega_0) \in \mathbf{R}^+ \underline{\times} \Omega$ *and multiplication* $\star : (\mathbf{R}^+ \underline{\times} \Omega) \times (\mathbf{R}^+ \underline{\times} \Omega) \to \mathbf{R}^+ \underline{\times} \Omega$ *defined by* (1.5) *and*

$$(t_1, \omega_1) \star (t_2, \omega_2) = (t_1 + t_2, \omega_1 \star_{t_1} \omega_2)$$

is a monoid.

(2) *Each constant, continuous-time system is a monoid automaton with monoid* $(\mathbf{R}^+ \underline{\times} \Omega, \star, e)$ *and vice versa.*

Proof: (1) It is straight forward to verify that the multiplication $\star$ is associative because $+$ in $\mathbf{R}^+$ and concatenation in Ω are associative. Moreover we have for all $(t, \omega) \in \mathbf{R}^+ \underline{\times} \Omega$:

$$(0, \omega_0) \star (t, \omega) = (t, \omega_0 \star_0 \omega) = (t, \omega)$$

and

$$(t, \omega) \star (0, \omega_0) = (t, \omega \star_t \omega_0) = (t, \omega)$$

showing that $\star$ is unit preserving.

(2) Obviously the closure under concatenation of $\mathbf{R}^+ \underline{\times} \Omega$ in 1.7 (*a*) corresponds to the fact that $\star$ is a function $\star: (\mathbf{R}^+ \underline{\times} \Omega) \times (\mathbf{R}^+ \underline{\times} \Omega) \to \mathbf{R}^+ \underline{\times} \Omega$. Consistency in 1.7 (*b*) and the composition property in 1.7 (*c*) are exactly the conditions 1.9.1 (*a*) and 1.9.1 (*b*) for φ to be a right action. □

Problems of reduction, reachability, observability and minimal realization for general systems and monoid automata will be studied in section 2 and for different types of constant, continuous-time systems in section 3.

2. MONOID AUTOMATA AND CONCEPTS OF MINIMAL REALIZATION

In section 1 we have shown that each constant continuous-time system can be regarded as a monoid automaton. As already mentioned in remark 5.7 of [3] all the constructions concerning reduction, minimization and realization known for Mealy and Moore automata in closed categories remain valid for monoid automata. Only the free monoid I^* has to be replaced by an arbitrary monoid M. Now, let us consider additional linear and topological structure on the state space X, on $\mathbf{R}^+$, and on the space of input functions Ω. Unfortunately, only in some cases there is a closed category $\mathbf{K}$ such that M becomes a monoid, $\varphi : X \times (\mathbf{R}^+ \underline{\times} \Omega) \to X$ a morphism, and the system a monoid automaton in $\mathbf{K}$. But in general there are different structures on X, $\mathbf{R}^+$ and Ω, and φ has different properties in these variables. In most of these cases, however, the general concept of minimal realization given in [2] and [4] can be applied. Hence, in this section we will give a short review of the concepts in [2] and [4] which will be applied to specific examples of constant, continuous-time systems in section 3. Moreover, we will show in this section explicitly that monoid automata in closed categories can be treated in the framework of [2] and [4]: It sufficies to construct a left and a right adjoint of the forgetful functor $U : \mathbf{Ract}_M \to \mathbf{K}$.

2.1 General assumption

For this section let $\mathbf{D}$ be a category, called category of *dynamics* (*cf.* [3]) and $V : \mathbf{D} \to \mathbf{K}$ a functor, called *forgetful functor*, from $\mathbf{D}$ to a base category $\mathbf{K}$. For 2.2-2.6 we assume that $\mathbf{D}$ has an $\mathfrak{C}$-$\mathfrak{M}$ factorization (*cf.* [3]) while for 2.7-2.9 it sufficies to consider classes $\mathfrak{C}$ of epi- and $\mathfrak{M}$ of monomorphisms in $\mathbf{D}$ which are closed under composition.

A *system* or *machine* (in a very general sense) is a construct $\Sigma = (Q, I, Y, \tau, \eta)$ where:

- Q is an object in $\mathbf{D}$ (*state object with dynamics structure*);
- I (*initial states*) and Y (*output*) are objects in $\mathbf{K}$;
- τ (*initial state morphism*) and η (*output morphism*) are $\mathbf{K}$-morphisms of the form

$$I \xrightarrow{\tau} VQ \xrightarrow{\eta} Y.$$

A morphism $f : \Sigma \to \Sigma'$ of systems (with fixed I and Y) is a $\mathbf{D}$ morphism $f : Q \to Q'$ satisfying $Uf \circ \tau = \tau'$ and $\eta' \circ Uf = \eta$.

In the terminology of definition 1.7 the state object with dynamics structure Q is given by the pair (X, φ) such that $VQ = V(X, \varphi) = X$ is the state set X. I can be considered as a one-element set such that $\tau : I \to X$ defines an initial state $\tau(I) \in X$.

2.2 Reachable and observable systems

Now, let us assume that $V : \mathbf{D} \to \mathbf{K}$ has a left and a right adjoint, written $\square^{\S} : \mathbf{K} \to \mathbf{D}$ and $\square_{\S} : \mathbf{K} \to \mathbf{D}$ respectively ($\square$ is used for a blank or an empty word):

$$\square^{\S} \dashv V \dashv \square_{\S}.$$

Given a system $\Sigma = (Q, I, Y, \tau, \eta)$ then there are unique extensions $\rho : I^{\S} \to Q$, called *reachability map*, of $\tau : I \to VQ$ and $\sigma : Q \to Y_{\S}$, called *observability map*, of $\eta : VQ \to Y$ such that the following diagram commutes

$$\begin{tikzcd}
VI^{\S} \arrow[dr, "V\rho"] & & VY_{\S} \arrow[d, "v_Y"] \\
I \arrow[u, "u_I"] \arrow[r] & VQ \arrow[ur, "V\sigma"] \arrow[r] & Y
\end{tikzcd}$$

By u_I and v_Y we denote the universal and couniversal morphisms of the adjunction $\square^{\S} \dashv V$ and $V \dashv \square_{\S}$ respectively.

A system is called *reachable* if ρ belongs to the class $\mathfrak{C}$ and *observable* if σ belongs to $\mathfrak{M}$.

In order to get the idea of the constructions above let us consider the well-known case of Moore automata.

2.3 Example (moore automata)

Let **K** be the category **Sets**, **D** the category **Medv** of Medvedew automata $X \times U \xrightarrow{\delta} X$ with states X and fixed input-alphabet U, and V : **Medv** $\to$ **Sets** be given by $V(X \times U \to X) = X$.

Then a system in the sense of 2.1 is a Moore automaton

$$I \xrightarrow{\tau} X, \qquad X \times U \xrightarrow{\delta} X, \qquad X \xrightarrow{\eta} Y.$$

Then left adjoint $I^{\S}$ is the free Medvedew automaton $(I \times U^*) \times U \xrightarrow{\delta_I} (I \times U^*)$ with $\delta_I(i, w, u) = (i, wu)$ and $u_I(i) = (i, \square)$ for $i \in I$, $w \in U^*$, the free monoid on U, and $u \in U$. In fact, $\rho : I \times U^* \to X$ becomes the usual reachability map defined by $\rho(i, w) = \delta^*(\tau(i), w)$ for $i \in I$, $w \in U^*$ where $\delta^* : X \times U^* \to X$ is the well-known extension of δ.

The right adjoint $Y_{\S}$ is the cofree Medvedew automaton $\langle U^*, Y \rangle \times U \xrightarrow{L} \langle U^*, Y \rangle$ with $L(f, u) = f \circ L_u \in \langle U^*, Y \rangle$ and $L_u(w) = uw$ and $v_Y(f) = f(\square)$ for all $f \in \langle U^*, Y \rangle$, $u \in U$, $w \in U^*$ ($\langle U^*, Y \rangle$ is the set of all function from U^* to Y). Hence, $\sigma : X \to \langle U^*, Y \rangle$ is the usual observability map defined by $\sigma(x)(w) = \eta(\delta^*(x, w))$ for $x \in X$, $w \in U^*$.

Note, that the extension $\delta^* : X \times U^* \to X$ is a right action of the monoid U^* on X in the sense of 1.9 such that Moore automata are special monoid automata. Now, we will give the general definition of monoid automata in closed categories:

2.4 DEFINITION (*right actions and monoid automata*)

Let $(\mathbf{K}, \otimes)$ be a closed category where the internal hom-functor is denoted by $\langle K, - \rangle : \mathbf{K} \to \mathbf{K}$, $ev : \langle K, Y \rangle \otimes K \to Y$ is the evaluation, and E the unit object in $(\mathbf{K}, \otimes)$.

Given a monoid M in $(\mathbf{K}, \otimes)$ with multiplication $\star : M \otimes M \to M$ and unit $e : E \to M$ (*cf.* [3], 5.6) right actions and monoid automata are defined as in 1.9 where all objects and morphisms are in $\mathbf{K}$, $M^0 = E$ and the cartesian product " $\times$ " has to be replaced by $\otimes$.

A morphism $f : (X, \varphi) \to (X', \varphi')$ of right actions (M fixed) is a $\mathbf{K}$ morphism $f : X \to X'$ satisfying

$$\begin{tikzcd} X \otimes M \arrow[r, "\varphi"] \arrow[d, "f \otimes M"'] \arrow[dr, phantom, "="] & X \arrow[d, "f"] \\ X' \otimes M \arrow[r, "\varphi'"] & X' \end{tikzcd}$$

In this way we obtain the category $\mathbf{Ract}_M$ of right actions and a forgetful functor $V : \mathbf{Ract}_M \to \mathbf{K}$.

For initial Moore automata we have in addition a (fixed) initial states object I and a K-morphism $\tau : I \to X$ such that initial Moore automata become systems in the sense of 2.1. Now we are going to give the explicit constructions and proofs for the left and the right adjoints of $V : \mathbf{Ract}_M \to \mathbf{K}$. These constructions, implicitly known in the literature, are generalizing those in 2.3 and will be used in section 3.

2.5 THEOREM

The forgetful functor $V : \mathbf{Ract}_M \to \mathbf{K}$ *has a left adjoint* $\square^{\S} : \mathbf{K} \to \mathbf{Ract}_M$, *given by*

$$I^{\S} = (I \otimes M, I \otimes \star : I \otimes M \otimes M \to I \otimes M)$$

and a right adjoint $\square_{\S} : \mathbf{K} \to \mathbf{Ract}_M$, *given by*

$$Y_{\S} = (\langle M, Y \rangle, L : \langle M, Y \rangle \otimes M \to \langle M, Y \rangle)$$

where the left shift L *is the adjoint morphism of*

$$\langle M, Y \rangle \otimes M \otimes M \xrightarrow{\langle M, Y \rangle \otimes \star} \langle M, Y \rangle \otimes M \xrightarrow{ev} Y.$$

Proof: $I \otimes \star$ and L are satisfying (*a*) and (*b*) of 1.9 using the monoid axioms for $\star$ and e.

Hence, it sufficies to prove the universal properties of $I^{\S}$ and the couniversal ones of $Y_{\S}$:

1. Let $u_I :=(I \xrightarrow{\sim} I\otimes E \xrightarrow{I\otimes e} I\otimes M)$ and for given right action (X, φ) and $\tau: I \to X$ define $\rho := I\otimes M \xrightarrow{\tau\otimes M} X\otimes M \xrightarrow{\varphi} X$.

We have to show that ρ is the unique $\mathbf{Ract}_M$-morphism $\rho: I^{\S} \to (X, \varphi)$ satisfying $V\rho \circ u_I = \tau$.

The necessity of the construction of ρ follows from

$$\begin{tikzcd}
I\otimes M \arrow[rrr, "id"] \arrow[dr, "\sim"] \arrow[dd, "id"'] & & & I\otimes M \arrow[dd, "\rho"] \\
& I\otimes E\otimes M \arrow[r, "I\otimes e\otimes M"] & I\otimes M\otimes M \arrow[ur, "I\otimes *"'] \arrow[d, "\rho\otimes M"] & \\
I\otimes M \arrow[urr, "U_I\otimes M"] \arrow[rr, "\tau\otimes M"'] & & X\otimes M \arrow[r, "\varphi"'] & X
\end{tikzcd}$$

and sufficiency from

$$\begin{tikzcd}
I\otimes M\otimes M \arrow[r, "I\otimes *"] \arrow[d, "\tau\otimes M\otimes M"'] & I\otimes M \arrow[d, "\tau\otimes M"] & I\otimes E \arrow[l, "I\otimes e"'] \arrow[d, "\tau\otimes E"] & I \arrow[l, "\sim"'] \arrow[d, "\tau"] \arrow[ll, bend right, "u_I"'] \\
X\otimes M\otimes M \arrow[r, "X\otimes *"] \arrow[d, "\varphi\otimes M"'] & X\otimes M \arrow[d, "\varphi"] & X\otimes E \arrow[l, "X\otimes e"'] & X \arrow[l, "\sim"'] \arrow[dll, "id"] \\
X\otimes M \arrow[r, "\varphi"] & X & &
\end{tikzcd}$$

where all our assumptions are frequently used.

2. To prove the properties of the right adjoint let

$$v_Y :=(\langle M, Y\rangle \xrightarrow{\sim} \langle M, Y\rangle \otimes E \xrightarrow{\langle M, Y\rangle\otimes e} \langle M, Y\rangle \otimes M \xrightarrow{ev} Y)$$

and for given right action (X, φ) and $\eta: X \to Y$ let $\sigma: X \to \langle M, Y\rangle$ be the adjoint morphism of $\eta\circ\varphi: X\otimes M \to Y$. Now, we have to show that σ is the unique $\mathbf{Ract}_M$-morphism $\sigma: (X, \varphi) \to Y_{\S}$ satisfying $v_Y \circ V\sigma = \eta$.

In order to prove the necessity it sufficies to show $\eta\circ\varphi = ev\circ(\sigma\otimes M)$. But this follows from

$$\begin{tikzcd}
X\otimes M \arrow[r, "\varphi"] \arrow[d, "\sigma\otimes M"'] & X \arrow[d, "\sigma"] \arrow[dr, "\eta"] & \\
<M,Y>\otimes M \arrow[r, "L"] & <M,Y> \arrow[r, "v_Y"'] & Y
\end{tikzcd}$$

using $v_Y \circ L = ev$ which is straightforward by definition of L. Vice versa, we obtain $v_Y \circ \sigma = \eta$ and $\sigma \circ \varphi = L \circ (\sigma \otimes M)$ from the following diagrams respectively. In the second case we use the couniversal properties of ev.

□

$$\begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[d, "s"] \arrow[ddd, bend right=60, "id"'] & & {<M,Y>} \arrow[dl, "s"'] \arrow[ddd, "v_Y"] \\
X \otimes E \arrow[r, "\sigma \otimes E"] \arrow[d, "X \otimes e"'] & {<M,Y> \otimes E} \arrow[d, "{<M,Y> \otimes e}"] & \\
X \otimes M \arrow[r, "\sigma \otimes M"] \arrow[d, "\varphi"'] & {<M,Y> \otimes M} \arrow[dr, "ev"] & \\
X \arrow[rr, "\eta"'] & & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes M \otimes M \arrow[rr, "\varphi \otimes M"] \arrow[dr, "X \otimes *"] \arrow[dd, "\sigma \otimes id"'] & & X \otimes M \arrow[r, "\sigma \otimes M"] \arrow[d, "\varphi"] & {<M,Y> \otimes M} \arrow[ddd, "ev"] \\
& X \otimes M \arrow[r, "\varphi"] \arrow[d, "\sigma \otimes M"] & X \arrow[ddr, "\eta"] & \\
{<M,Y> \otimes M \otimes M} \arrow[r, "id \otimes *"] \arrow[d, "L \otimes M"'] & {<M,Y> \otimes M} \arrow[drr, "ev"] & & \\
{<M,Y> \otimes M} \arrow[rrr, "ev"'] & & & Y
\end{tikzcd}$$

2.6 THEOREM (minimal-realization)

Given $\square^{\S} \dashv V \dashv \square_{\S}$ *and a system* $\Sigma = (Q, I, Y, \tau, \eta)$ *with reachability map* $\rho: I^{\S} \to Q$ *and observability map* $\sigma: Q \to Y_{\S}$ *let us call the composition* $\sigma \circ \rho: I^{\S} \to Y_{\S}$ *the* behavior *of* Σ. *Then we have*

1. *The* $\mathfrak{C}$-$\mathfrak{M}$ *factorization* $I^{\S} \xrightarrow{\rho'} Q' \xrightarrow{f'} Q$ *of* ρ *leads to a unique (up to isom.) equivalent reachable system* $\Sigma' = (Q', I, Y, V\rho' \circ u_I, \eta \circ Vf')$ *such that* f' *becomes an* $\mathfrak{M}$*-morphism from* Σ' *to* Σ.

2. *The* $\mathfrak{C}$-$\mathfrak{M}$ *factorization* $Q \xrightarrow{\bar{f}} \bar{Q} \xrightarrow{\bar{\sigma}} Y_{\S}$ *of* σ *leads to a unique (up to isom.) equivalent observable system* $\bar{\Sigma} = (\bar{Q}, I, Y, V\bar{f} \circ \tau, v_Y \circ V\bar{\sigma})$ *such that* $\bar{f}$ *becomes an* $\mathfrak{C}$ *morphism from* Σ *to* $\bar{\Sigma}$.

3. *The* $\mathfrak{C}$-$\mathfrak{M}$ *factorization* $I^{\S} \xrightarrow{\hat{\rho}} \hat{Q} \xrightarrow{\hat{\sigma}} Y_{\S}$ *of* $\sigma \circ \rho$ *leads to a "minimal" realization* $\hat{\Sigma} = (\hat{Q}, I, Y, V\hat{\rho} \circ u_I, v_Y \circ V\hat{\sigma})$ *which is (up to isom.) the unique reachable and observable system equivalent to S.*

$$\begin{tikzcd}
VI^{\S} \arrow[r, "V\hat{\rho}"] \arrow[dr, "V\rho"] & V\hat{Q} \arrow[r, "V\hat{\sigma}"] & VY^{\S} \arrow[d, "v_Y"] \\
I \arrow[u, "u_I"] \arrow[r, "\tau"'] & VQ \arrow[ur, "V\sigma"] \arrow[r, "\eta"'] & Y
\end{tikzcd}$$

Proof: 1. By construction Σ' is reachable because $\rho' \in \mathfrak{C}$. The behavior of Σ' is just the composition $\sigma' \circ \rho'$ where σ' is defined by $v_Y \circ V\sigma' = \eta \circ Vf'$. But on the other hand we have $(v_Y \circ V\sigma) \circ Vf' = \eta \circ Vf'$, hence by adjointness $\sigma' = \sigma \circ f'$. So the behavior of Σ' is just the same as the behavior of Σ:

$$\sigma' \circ \rho' = \sigma \circ f' \circ \rho' = \sigma \circ \rho.$$

The uniqueness of Σ' as a reachable system equivalent to Σ with an $\mathfrak{M}$-morphism from Σ' to Σ follows from the uniqueness of the $\mathfrak{C}$-$\mathfrak{M}$ factorization (up to isomorphism).

2. Is proved dually to 1. The proof of 3 is similar. (Confer also [4] theorem 1.7, 1.7* and 1.9 or the Axiomatic Minimal Realization theorem in [2].) □

Note, that part 1 of the theorem remains true if we have only the left adjoint and part 2 if we have only the right adjoint of V. Now, let us consider the case that we have neither a left nor a right adjoint. More results concerning the mixed cases are given in [4].

2.7 Definition

Remember that we have assumed for 2.7-2.9 classes $\mathfrak{C}$ (of epimorphisms) and $\mathfrak{M}$ (of monomorphisms) closed under composition in $\mathbf{D}$. Let $\mathfrak{C}_{\mathrm{IN}}$ be the class of all "IN-V morphisms" $\tau : I \to VQ$ such that for each factorization $\tau = Vf \circ \tau'$ with f in $\mathfrak{M}$ we have f in $\mathfrak{C}$. Dually, let $\mathfrak{M}_{\mathrm{OUT}}$ be the class of all "OUT-V morphisms" η: $VQ \to Y$ such that for each factorization $\eta = \eta' \circ Vf$ with f in $\mathfrak{C}$ we have f in $\mathfrak{M}$.

(Note, that these conditions for $\mathfrak{C}_{\mathrm{IN}}$ and $\mathfrak{M}_{\mathrm{OUT}}$ are in general slightly weaker than that in [4], but they are equivalent if there is an $\mathfrak{C}$-$\mathfrak{M}$ factorization in $\mathbf{D}$.)

Now a system $\Sigma = (Q, I, Y, \tau, \eta)$ is called *simple*, if τ belongs to $\mathfrak{C}_{\mathrm{IN}}$, and is called *reduced*, if η belongs to $\mathfrak{M}_{\mathrm{OUT}}$.

2.8 Theorem (*construction of $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$ and $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorizations*).

We obtain unique $\mathfrak{C}_{IN}$-$\mathfrak{M}$ (*resp.* $\mathfrak{C}$-$\mathfrak{M}_{OUT}$) *factorizations of* IN-V (*resp.* OUT-V) *morphisms in each of the following three cases:*

1. *Given an* IN-V *morphism* $\tau: I \to VQ$ *we construct the intersection* Q' *of all* $\mathfrak{M}$ *subobjects* $f_i: Q_i \to Q$ *of* Q *for which there is a* $\tau_i: I \to VQ_i$ *with* $Vf_i \circ \tau_i = \tau$. *If* **D** *has (large) intersections and these are preserved by* V *then* $f': Q' \to Q$ *is in* $\mathfrak{M}$ *and there is a* $\tau': I \to VQ'$ *such that* $Vf' \circ \tau' = \tau$ *is (up to isomorphism) a unique* $\mathfrak{C}_{IN}$-$\mathfrak{M}$ *factorization of* τ.

In most examples Q' *is the subobject of* Q *generated by* $\tau(I) \leqq Q$.

2. *Dually, given an* OUT-V *morphism* $\eta: VQ \to Y$ *we construct the cointersection* $\overline{Q}$ *of all* $\mathfrak{C}$-*quotient objects* $f_i: Q \to Q_i$ *of* Q *for which there is a* $\eta_i: VQ_i \to Y$ *with* $\eta_i \circ Vf_i = \eta$. *If* **D** *has (large) cointersections and these are preserved by* V *then* $\overline{f}: Q \to \overline{Q}$ *is in* $\mathfrak{C}$ *and there is a* $\overline{\eta}: V\overline{Q} \to Y$ *such that* $\overline{\eta} \circ V\overline{f} = \eta$ *is (up to isomorphism) a unique* $\mathfrak{C}$-$\mathfrak{M}_{OUT}$ *factorization of* η.

3. *Let* $K \rightrightarrows Q$ *be the (relative) kernel pair of* $\eta: VQ \to Y$ *and* $Q \xrightarrow{\overline{f}} \overline{Q}$ *the coequalizer of* $K \rightrightarrows Q$ *in* **D**. *If this coequalizer is preserved by* V *and* $\mathfrak{C}$ *the class of all coequalizers then there is a unique* $\overline{\eta}: V\overline{Q} \to Y$ *such that* $\overline{\eta} \circ V\overline{f} = \eta$ *is a (up to isomorphism) unique* $\mathfrak{C}$-$\mathfrak{M}_{OUT}$ *factorization of* η.

Proof: We only prove 2 and 3, because 1 and 2 are dual to each other. We start with the proof of case 2:

Given an OUT-V morphism $y: VQ \to Y$ let $f_i: Q \to Q_i$ $(i \in I)$ be the set of all **D** morphisms $f': Q \to Q'$ in $\mathfrak{C}$ such that there is an OUT-V morphism $y': VQ' \to Y$ satisfying $y = y' \circ Vf'$. Now, let $f: Q \to \overline{Q}$ be the cointersection of the family f_i $(i \in I)$. Using the universal properties of the cointersection and the $\mathfrak{C}$-$\mathfrak{M}$ factorization in **D** it is easy to see that also f belongs to $\mathfrak{C}$ (*cf.* proof of prop. 7.3 in [3]). Since V preserves cointersections there is also a unique $\overline{y}: V\overline{Q} \to Y$ satisfying $y = \overline{y} \circ U f$. In order to show $\overline{y} \in \mathfrak{M}_{OUT}$ let us consider an arbitrary OUT-V-factorization $\overline{y} = y' \circ V f'$ of $\overline{y}$. We have to show: $f' \in \mathfrak{M}$. Now, let $f' = m \circ e$ be an $\mathfrak{C}$-$\mathfrak{M}$ factorization of f'. Hence we have $y = y' \circ V(m) \circ V(e \circ f)$ with $e \circ f \in \mathfrak{C}$ such that $e \circ f = f_{i_0}$ for some $i_0 \in I$ without loss of generality. But then using the construction of f we have that e is an isomorphism and hence $f' = m \circ e \in \mathfrak{M}$. Given another $\mathfrak{C}$-$\mathfrak{M}_{OUT}$ factorization $y = y_0 \circ V f_0$ there is a unique $\varphi: Q_0 \to \overline{Q} \in \mathfrak{C}$ with $\varphi \circ f_0 = f$ and $\overline{y} \circ V\varphi = y_0$ by construction of the cointersection. But $y_0 \in \mathfrak{M}_{OUT}$ implies also $\varphi \in \mathfrak{M}$ such that φ becomes a **D** isomorphism. In order to show that $\mathfrak{M}_{OUT}$ is closed under composition with $\mathfrak{M}$ from left let $y_2 \circ V f_2 = y \circ V f_1$, with $y \in \mathfrak{M}_{OUT}$ and $f_1 \in \mathfrak{M}$. We have to show that f_2 is a monomorphism. Hence, let f_3, $f_4: Q_3 \to Q_1$ arbitrary **D** morphisms with $f_2 \circ f_3 = f_2 \circ f_4$ and $d: Q \to Q_4$ the coequalizer of $f_1 \circ f_3$ and $f_1 \circ f_4$. Since V

preserves coequalizers there is a unique y_4: $VQ_4 \to Y$ satisfying $y = y_4 \circ Vd$. Now, $y \in \mathfrak{M}_{\mathrm{OUT}}$ implies $d \in \mathfrak{M}$. Using also $f_1 \in \mathfrak{M}$ and $d \circ f_1 \circ f_3 = d \circ f_1 \circ f_4$ we obtain $f_3 = f_4$.

$$\begin{tikzcd}
VQ_3 \arrow[r, shift left, "Vf_3"] \arrow[r, shift right, "Vf_4"'] & VQ_1 \arrow[r, "Vf_1"] \arrow[dd, "Vf_2"] & VQ \arrow[r, "y"] \arrow[d, "Vd"'] & Y \\
& & VQ_4 \arrow[ur, "y_4"'] & \\
& VQ_2 \arrow[uurr, "y_2"'] & &
\end{tikzcd}$$

Proof of case 3: Let (f_1, f_2): $Q_1 \rightrightarrows Q$ be the kernel pair relative V of y: $VQ \to Y$ and $\bar{f}$: $Q \to \bar{Q}$ the coequalizer of (f_1, f_2). Since V preserves this coequalizer there is a unique y: $V\bar{Q} \to Y$ satisfying $y = \bar{y} \circ V\bar{f}$. In order to show $\bar{y} \in \mathfrak{M}_{\mathrm{OUT}}$ let $\bar{y} = y' \circ Vf'$ and $f' = m \circ e$ an $\mathfrak{C}$-$\mathfrak{M}$ factorization of f'. It remains to show that e is an isomorphism. By assumption $e \circ \bar{f} \in \mathfrak{C}$ is again a coequalizer, say of (f_3, f_4): $Q_3 \rightrightarrows Q_4$. Hence, by construction of (f_1, f_2) there is a unique f_5: $Q_3 \to Q_1$ satisfying $f_1 \circ f_5 = f_3$ and $f_2 \circ f_5 = f_4$. Thus $\bar{f} \circ f_3 = \bar{f} \circ f_4$ such that there is a unique f'': $Q'' \to \bar{Q}$ with $f'' \circ e \circ \bar{f} = \bar{f}$. Now $\bar{f} \in \mathfrak{C}$ implies that e is an isomorphism showing that y is in $\mathfrak{M}_{\mathrm{OUT}}$. Using a similar proof we can show that the $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorization $\bar{y} = y \circ V\bar{f}$ is unique up to isomorphism.

$$\begin{tikzcd}
VQ_1 \arrow[r, shift left, "Vf_1"] \arrow[r, shift right, "Vf_2"'] & VQ \arrow[rr, "y"] \arrow[d, "V\bar{f}"] & & Y \\
VQ_2 \arrow[u, dashed, "Vf_5"] \arrow[ur, shift left, "Vf_3"] \arrow[ur, shift right, "Vf_4"'] & V\bar{Q} \arrow[urr, "\bar{y}"] \arrow[rr, "Vf'"] \arrow[dr, "Ve"] & & VQ' \arrow[u, "y'"'] \\
& & VQ'' \arrow[ul, dashed, "Vf''"] \arrow[ur, "Vm"'] &
\end{tikzcd}$$

As immediate consequences of theorem 2.8 we obtain the following constructions:

2.9 Corollary (simple and reduced systems)

Assuming that there are unique $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$ and $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorizations we have for each system $\Sigma = (Q, I, Y, \tau, \eta)$:

1. *The $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$ factorization $I \xrightarrow{\tau'} VQ' \xrightarrow{Vf'} VQ$ of τ leads to a unique (up to isom.) simple system $\Sigma' = (Q', I, Y, \tau', \eta \circ Vf')$ such that f' becomes an $\mathfrak{M}$ morphism from Σ' to Σ.*

2. *The $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorization $VQ \xrightarrow{V\bar{f}} V\bar{Q} \xrightarrow{\bar{\eta}} Y$ of η leads to a unique (up to isom.) reduced system $\bar{\Sigma} = (\bar{Q}, I, Y, V\bar{f} \circ \tau, \bar{\eta})$ such that $\bar{f}$ becomes an $\mathfrak{C}$ morphism from Q to $\bar{Q}$.*

3. *The constructions of step 1 and 2 can be combined leading to a simple and reduced system which is a "subquotient" of Σ, provided that $\mathfrak{C}_{\mathrm{IN}}$ is closed under composition with $\mathfrak{C}$ from right.*

Remark: The existence of unique $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$-factorizations implies that the "reduction problem" is solvable, i. e. each "reduction" f: $\Sigma \to \Sigma'$ with f in $\mathfrak{C}$ can be extended uniquely by a reduction f': $\Sigma' \to \bar{\Sigma}$ to obtain the "universal" reduction $\bar{f}$: $\Sigma \to \bar{\Sigma}$. Dually the simplification" problem is solvable if we have unique $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$-factorizations (*cf.* [4]).

3. APPLICATIONS TO CONSTANT, CONTINUOUS-TIME SYSTEMS

In this section the theory of monoid automata and the concepts of minimal realization will be applied to several types of constant, continuous-time systems in the sense of section 1. We will start with ordinary systems in the sense of definition 1.7. Then additional topological and linear structure will be considered on the state space X, the output space Y and on the monoid $\mathbf{R}^+ \underline{\times} \Omega$, and we will make different assumptions for φ: $X \times \mathbf{R}^+ \underline{\times} \Omega \to X$ and η: $X \to Y$ in these variables. Moreover, we will consider constant, continuous-time systems with initial states I given by a function τ: $I \to X$ such that an initialized system is a construct $\Sigma = (X, \mathbf{R}^+ \underline{\times} \Omega, I, Y, \varphi, \tau, \eta)$. Especially, our constructions can be applied to bilinear systems in the sense of ([10] and [12] and smooth nonlinear systems in [11]).

3.1 Constant, continuous-time systems

Let $\mathbf{K}$ be the category **Sets**, $\mathbf{D}$ the category $\mathbf{Ract}_{\mathbf{R}^+ \underline{\times} \Omega}$ of right actions (X, φ) of the monoid $\mathbf{R}^+ \underline{\times} \Omega$ and V: $\mathbf{Ract}_{\mathbf{R}^+ \underline{\times} \Omega} \to \mathbf{Sets}$ the forgetful functor defined by $V(X, \varphi) = X$. Then a system in the sense of 2.1 is an initialized, constant, continuous-time system. For $\mathfrak{C}$ and $\mathfrak{M}$ we take the classes of surjective and injective right action morphisms respectively (*cf.* 2.4). According to 2.5 the left adjoint $I^{\S}$ is given by

$$I^{\S} = (I \times (\mathbf{R}^+ \underline{\times} \Omega), I \times \star: I \times (\mathbf{R}^+ \underline{\times} \Omega) \times (\mathbf{R}^+ \underline{\times} \Omega) \to I \times (\mathbf{R}^+ \underline{\times} \Omega)$$

where $\star$ is defined in 1.4 and 1.7.

Given a system $\Sigma=(X, \mathbf{R}^+ \underline{\times} \Omega, I, Y, \varphi, \tau, \eta)$ the reachability map (*cf.* 2.2) becomes a function

$$\rho: \quad I \times \mathbf{R}^+ \underline{\times} \Omega \rightarrow X$$

defined by $\rho(i, (t, w))=\varphi(\tau(i), (t, w))$ for all $i \in I$, $(t, w) \in \mathbf{R}^+ \underline{\times} \Omega$. Now, Σ is reachable if ρ is surjective. If not, the image factorization $f' \circ \rho' = \rho$ in 2.6.1 leads to a unique (up to isom.) equivalent reachable system Σ' such that f': $\Sigma' \rightarrow \Sigma$ becomes an injective morphism of systems. Again by 2.5 the right adjoint $\mathrm{Y}_\S$ is given by

$$Y_\S=(\langle \mathbf{R}^+ \underline{\times} \Omega, Y\rangle, L: \langle \mathbf{R}^+ \underline{\times} \Omega, Y\rangle \times \mathbf{R}^+ \underline{\times} \Omega \rightarrow \langle \mathbf{R}^+ \underline{\times} \Omega, Y\rangle)$$

where the left shift L is defined by $L\ (f, (t, w))=f \circ L_{(t, w)}$ and $L_{(t, w)}(t', w')=(t+t', w \star_t w')$ for all $f: \mathbf{R}^+ \underline{\times} \Omega \rightarrow Y$ and $(t, w), (t', w') \in \mathbf{R}^+ \underline{\times} \Omega$. The observability map (*cf.* 2.2) becomes a function

$$\sigma: \quad X \rightarrow \langle \mathbf{R}^+ \underline{\times} \Omega, Y\rangle$$

defined by $\sigma(x)(t, w)=\eta(\varphi(x, (t, w)))$ for $x \in X$, $(t, w) \in \mathbf{R}^+ \underline{\times} \Omega$. Now, the image factorization $\bar{\sigma} \circ \bar{f}=\sigma$ in 2.6.2 leads to a unique (up to isom.) equivalent observable system $\bar{\Sigma}$ such that $\bar{f}$: $\Sigma \rightarrow \bar{\Sigma}$ becomes a surjective morphism.

Finally, by 2.6.3 the image-factorization

$$I \times \mathbf{R}^+ \underline{\times} \Omega \xrightarrow{\hat{\rho}} \hat{X} \xrightarrow{\hat{\sigma}} \langle \mathbf{R}^+ \underline{\times} \Omega, Y\rangle$$

of the composition $\sigma \circ \rho$ leads to a minimal realization $\hat{\Sigma}=(\hat{X}, \mathbf{R}^+ \underline{\times} \Omega, I, Y, \hat{\varphi}, \hat{\tau}, \hat{\eta})$ which is (up to isom.) the unique reachable and observable system equivalent to Σ. In more detail we have:

$$\hat{X}=\{f_{i,(t, w)}: \mathbf{R}^+ \underline{\times} \Omega \rightarrow Y \mid f_{i,(t, w)}=\sigma\rho(i, (t, w))\ i \in I, (t, w) \in \mathbf{R}^+ \underline{\times} \Omega\}$$

$\hat{\varphi}$: $\hat{X} \times \mathbf{R}^+ \underline{\times} \Omega \rightarrow \hat{X}$ is the restriction of the left shift L, $\hat{\tau}: I \rightarrow \hat{X}$ is given by $\hat{\tau}(i)=\hat{\rho}(i, (0, w_0))$ (*cf.* 1.7), $\hat{\eta}$: $\hat{X} \rightarrow Y$ is given by $\hat{\eta}(f_{i,(t, w)})=\eta(\varphi(\tau(i), (t, w)))$.

3.2 Bicontinuous systems

A system $\Sigma=(X, \mathbf{R}^+ \underline{\times} \Omega, I, Y, \varphi, \tau, \eta)$ is called *bicontinuous* if X, I and Y are topological spaces, $(\mathbf{R}^+ \underline{\times} \Omega, \star)$ is a bicontinuous topological monoid (*see* below) and $\tau: I \rightarrow X$ and $\eta: X \rightarrow Y$ are continuous, and $\varphi: X \times(\mathbf{R}^+ \underline{\times} \Omega) \rightarrow X$ is bicontinuous, i.e. continuous in each of both components separately. This means that for fixed time and control function the state transition is a continuous map from X to itself, and also for fixed state the state transition depends continuously on t and w simultaneously, where $\mathbf{R}^+ \times \Omega$ carries quotient topology. Similarly a *bicontinuous topological monoid* is a monoid such that the monoid multiplication

$$\star: \quad (\mathbf{R}^+ \underline{\times} \Omega) \times (\mathbf{R}^+ \underline{\times} \Omega) \to \mathbf{R}^+ \underline{\times} \Omega$$

is bicontinuous.

Using the bitopology $M \otimes N$ on the cartesian product $M \times N$ (i.e. the final topology of the family of injections $\{m\} \times N, \to M \times N, M \times \{n\} \to M \times N$) a map f: $M \otimes N \to P$ is continuous if and only if f: $M \times N \to P$ is bicontinuous. It is easy to see that such an f induces uniquely a continuous map $\bar{f}$: $M \to C(N, P)$ where $C(N, P)$ is the space of all continuous functions from N to P furnished with the topology of pointwise convergence. So the category of all topological spaces (**Top,** $\otimes$) becomes a closed category (*cf.* [5]).

Now let us give an example of a suitable control space Ω such that $(\mathbf{R}^+ \underline{\times} \Omega, \star)$ becomes a bicontinuous topological monoid. Let U be a 1st countable topological space and

$$\Omega = \{ w: (0, \infty) \to U / w \text{ piecewise continuous} \}.$$

and $\mathbf{R}^+ \underline{\times} \Omega$ be the quotient space (as in 3.1) by the equivalence relation

$$(t, w) \sim (t', w') \quad \Leftrightarrow \quad t = t' \qquad \text{and} \qquad w\,|_{(0, t]} = w'\,|_{(0, t]}.$$

$w_n(x) \xrightarrow[n \to \infty]{} w(x)$ for all but a countable number of $x \in (0, \infty)$. This leads to the topology of pointwise convergence up to a countable set. By easy calculation the concatenation $\star$ regarded as a map

$$\star: \quad (\mathbf{R}^+ \times \Omega) \times (\mathbf{R}^+ \times \Omega) \to \mathbf{R}^+ \times \Omega$$

is continuous. Getting to the quotient the concatenation

$$\star: \quad (\mathbf{R}^+ \underline{\times} \Omega) \times (\mathbf{R}^+ \underline{\times} \Omega) \to \mathbf{R}^+ \underline{\times} \Omega$$

is bicontinuous.

Of course, there are other weak topologies on suitable function spaces (*cf.* [12]).

By definition the action-part of a bicontinuous system can be regarded as a monoid-automaton in the closed category (**Top,** $\otimes$). Hence we can apply theorems 2.5 and 2.6 to bicontinuous systems leading to the following result:

for each bicontinuous system Σ the equivalent reachable, the equivalent observable system, and the minimal realization are bicontinuous, too.

3.3 Compactly generated systems

A system Σ is called *compactly generated* if X, Y, I, and $\mathbf{R}^+ \underline{\times} \Omega$ are compactly generated Hausdorff-spaces and φ: $X \pi (\mathbf{R}^+ \underline{\times} \Omega) \to X$, τ and η are continuous, and moreover $(\mathbf{R}^+ \underline{\times} \Omega, \star)$ is a topological monoid in the sense that $\star : (\mathbf{R}^+ \underline{\times} \Omega) \pi (\mathbf{R}^+ \underline{\times} \Omega) \to \mathbf{R}^+ \underline{\times} \Omega$ is continuous. Here π means the Kelleyfication of the topological product leading to a compactly generated space.

The Kelleyfication kX of a space X is the same underlying set furnished with the following topology: a set is closed in kX iff its intersections with each compact subset C of X is closed in C. Let us remark that each metric space and each locally compact space is compactly generated, and that the Kelleyfication of the topological product is just the topological product provided that it is already compactly generated (*cf.* [5]).

Now let us construct a suitable compactly generated topological monoid $(\mathbf{R}^+ \underline{\times} \Omega, \star)$. Let U be a 1st countable Hausdorff space and

$$\Omega' = \{ w \colon (0, \infty) \to U / w \text{ piecewise continuous, and } w \text{ left-continuous} \}$$

with subspace topology relative to the Ω in 3.2.

Ω' is a Hausdorff space, so let Ω be the Kelleyfication of Ω', and from the continuity of $\star$: $(\mathbf{R}^+ \times \Omega) \times (\mathbf{R}^+ \times \Omega) \to \mathbf{R}^+ \times \Omega$ (*cf.* 3.2) it follows that $\star$: $(\mathbf{R}^+ \underline{\times} \Omega) \pi (\mathbf{R}^+ \underline{\times} \Omega) \to \mathbf{R}^+ \underline{\times} \Omega$ is continuous, too, because the functor $\square \pi \square$ preserves quotients (This is a general fact in each cartesian closed category, *cf.* [9], p. 7.) Of course, the quotient topology is to be taken in the category of compactly generated Hausdorff spaces.

Hence the action-part of a compactly generated system can be regarded as a monoid automaton in the closed category $(\mathbf{CG}, \pi)$, so we can apply theorems 2.5 and 2.6 to such systems.

3.4 Semilinear systems

A system Σ is called *semilinear* if X, Y and I are R-modules for some ring R and τ, η, and, for each $(t, w) \in \mathbf{R}^+ \underline{\times} \Omega$, $\varphi(-, (t, w)) : X \to X$ are R-linear.

Since φ is not assumed to be linear in the other components it is not possible to interprete the action parts of semilinear systems as monoid automata in the category $\mathbf{Mod}_R$.

But the forgetful functor $V : \mathbf{D} \to \mathbf{Mod}_R$ from the category $\mathbf{D}$ of action-parts of semilinear systems into $\mathbf{Mod}_R$ has a right adjoint $\square_{\S}$ and a left adjoint $\square^{\S}$ which will be shown in the following:

We define for $Y \in \mathbf{Mod}_R$:

$$Y_{\S} : = \langle \mathbf{R}^+ \underline{\times} \Omega,\ Y \rangle$$

where $\langle A, B \rangle$ denotes the set of all functions $f : A \to B$. $Y_{\S}$ carries in a natural way the structure of an R-module. Together with the left shift

$$\begin{aligned} L : \quad & Y_{\S} \times (\mathbf{R}^+ \underline{\times} \Omega) \to Y_{\S}, \\ & (f, p) \mapsto f_p, \\ & f_p(q) : = f(p \star q), \end{aligned}$$

it is an action-part of a semilinear system.

The left adjoint is defined for $I \in \mathbf{Mod}_R$ by

$$I^{\S} : = I \otimes_R F(\mathbf{R}^+ \underline{\times} \Omega)$$

where $F(.)$ denotes the free R-module. Here the action φ is defined by

$$\varphi\,(a \otimes \sum_i \alpha_i (t_i, w_i), (t, w)) : = (a \otimes (\sum_i \alpha_i (t_i, w_i) \star (t, w)))$$

on the generators of the tensorproduct.

To show the universal properties of $Y_{\S}$ and $I^{\S}$ define

$$v_Y : \quad VY_{\S} \to Y$$

by evaluation at the zero element e of the monoid $\mathbf{R}^+ \underline{\times} \Omega$.

Let $\eta : VQ \to Y$ be arbitrary. Then a necessary condition for $\sigma : Q \to Y_{\S}$ to make the following diagram commutative

$$\begin{tikzcd} & VQ \arrow[rr, "\eta"] \arrow[drr, "V\sigma"'] & & Y \\ Q \arrow[dr, dashed, "\sigma"'] & & & VY_{\S} = \langle \mathbb{R}^+ \underline{\times}\, \Omega, Y \rangle \arrow[u, "v_Y"'] \\ & Y_{\S} & & \end{tikzcd}$$

is that the following equations hold (m denotes the action of Q):

$$\sigma(q)(e)=(v_Y\circ\sigma)(q)=\eta(q),$$

$$\begin{aligned}\sigma(q)(t,\,w)=L(\sigma(q),\,(t,\,w))(e)&=\sigma(m(q,\,(t,\,w)))(e)\\&=\eta(m(q,\,(t,\,w)))\qquad\text{for}\quad q\in Q.\end{aligned}$$

These equations are also sufficient, because obviously σ can be defined uniquely as a mapping, and it turns out that the so defined σ is linear. This completes the construction of the right adjoint $\square_{\S}$ of V.

Now let us consider the left adjoint $\square^{\S}$: For given $\tau : I\to VQ$ necessary conditions for $\rho : I^{\S}\to Q$ making the diagram commutative

$$\begin{tikzcd}
I \arrow[r, "\tau"] \arrow[d, "u_I"'] & VQ & \\
VI^{\S} \arrow[ur, "V\rho"'] & & Q \\
& I^{\S} \arrow[ur, dashed, "\rho"'] &
\end{tikzcd}$$

are the following:

$$\rho(x\otimes e)=\rho\circ u_I(x)=\tau(x),$$

$$\begin{aligned}\rho(x\otimes(t,\,w))=\rho(\varphi(x\otimes e,\,(t,\,w)))&=m(\rho(x\otimes e),\,(t,\,w))\\&=m(\tau(x),\,(t,\,w))\qquad\text{for}\quad x\in I.\end{aligned}$$

For an arbitrary generator of the tensor product one gets a similar condition using the linearity of ρ. On the other hand this shows also the existence of ρ. Having shown the left and the right adjoint of $V:\mathbf{D}\to\mathbf{Mod}_R$ we can apply theorem 2.6 leading to a minimal realization of semilinear systems.

3.5 Bilinear systems

A bilinear system in the internal sense (*cf.* [10], [11]) is a semi-linear system with the special property that X and Y are finite-dimensional R-vector spaces, the state transition φ is differentiable in t, and the vector $(\partial\varphi/\partial t)(x_0,\,t_0,\,w_0)$ is a linear function in x_0 and w_0 separately. Applying our theorem 2.6 to bilinear systems regarding them as semilinear systems we get an equivalent reachable or observable semilinear system. It remains to verify that bilinearity carries over from the given system to the so constructed one.

This is clearly true for the equivalent reachable system (2.6.1) and in case of the equivalent observable system (2.6.2) the reduction $e:\Sigma\to\overline{\Sigma}$ carries over bilinearity in the above sense. This follows directly from the equation $\overline{\varphi} \circ (2 \times id) = e \circ \varphi$. So for example proposition 2 in [10] is a corollary of our theorem 2.6.2.

Of course, each semilinear system induces a unique representation $\mathbf{R}^+ \underline{\times} \Omega \to \mathrm{Lin}(X, X)$ of the monoid $\mathbf{R}^+ \underline{\times} \Omega$. Now assume that X is a topological vector space and φ is bicontinuous. This implies that the representation above is continuous if $\mathrm{Lin}(X, X)$ is furnished with the topology of pointwise convergence which does agree with the natural topology of $\mathrm{Lin}(X, X)$ regarded as a topological vector space provided that X is finite dimensional. Vice versa each continuous representation of that kind induces exactly one action-part of a semilinear system with φ being bicontinuous.

Under the additional assumption of U being a compact convex subset of $\mathbf{R}^n$ with nonempty interior theorem 1 in [12] shows that all these continuous representations are in natural bijection to the class of all bilinear systems. In view of that bijection let us consider in 3.6 semilinear systems with additional topological structure:

3.6 Semilinear bicontinuous systems

A system Σ is called *semilinear bicontinuous* if X, Y and I are topological vector spaces and Σ is semilinear in the sense of 3.4 and simultaneously bicontinuous in the sense of 3.2.

Let us take the same Ω and $\mathbf{R}^+ \underline{\times} \Omega$ as in 3.2.

Then the right adjoint $\square_{\S}$ to the forgetful functor V from the corresponding category of action parts into the category of topological vector space is given by

$$Y_{\S} : = C(\mathbf{R}^+ \underline{\times} \Omega, Y),$$

which is the topological vector space of all continuous functions from $\mathbf{R}^+ \underline{\times} \Omega$ to Y furnished with the topology of pointwise convergence. It is easy to see that the left shift L is bicontinuous, and the universal property of $Y_{\S}$ follows from the fact that there is the same situation for bicontinuous systems and semilinear systems. Note, that in the bicontinuous case the function space $\langle \mathbf{R}^+ \underline{\times} \Omega, Y \rangle$ carries topology of pointwise convergence, too.

Instead of a left adjoint let us construct an $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$ factorization of a given morphism $I \xrightarrow{f} VQ$ (*cf.* 2.7).

We define $\overline{I} : = \mathrm{Lin}(\varphi(f(I) \times (\mathbf{R}^+ \underline{\times} \Omega)))$ where φ is the action in Q, and $\mathrm{Lin}(.)$ means the linear hull. Clearly I is a linear topological subspace of VY which by linearity of φ is closed under the action φ.

$$\begin{tikzcd} I \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & & VQ \\ & \bar{I} = VQ' \arrow[ur, "m"'] & \end{tikzcd}$$

This construction of $\bar{I}$ is the same as the $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$ factorization given in 2.8.1 using intersections. Hence we can omit the proof that this is indeed an $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$ factorization of f.

Hence we can apply theorem 2.6.2 and 2.9.1 to semilinear bicontinuous systems. Note, that the same remains true if we, in analogy to 3.3, consider compactly generated semilinear systems.

Remark: In the special case of semilinear systems over suitable locally convex vector spaces it is possible to express the differential equation generating the state-transition in a categorical framework. This has been done in [6] where on the level of differential equations left and right adjoints for the functors $F_K : DSG(K) \to K$ are constructed where K denotes a subcategory of the category of locally convex spaces. This concept leads to results similar to that of theorem 2.6, and moreover it gives information about the behavior of the differential equations.

3.7 Linear systems

A system Σ is called *linear* if X, Y, I and Ω are R-modules, τ and η are R-linear, and for each $t \in \mathbf{R}^+$ $\varphi(-, t, -) : X \times \Omega_t \to X$ is R-linear, with

$$\Omega_t := \{ w \,|\, (t, w) \in \mathbf{R}^+ \underline{\times} \Omega \}$$

This definition is equivalent with that of [7], definition 1.5.

The forgetful functor V from the corresponding action-category into $\mathbf{Mod}_R$ has no left adjoint and no right adjoint because it does not preserve products or coproducts. But we are able to construct and $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$ factorization ($\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorization) of a given linear map:

$$\tau : \; I \to VQ \qquad (\eta : VQ \to Y)$$

using intersections and cointersections (*cf.* 2.8).

Here these intersections and cointersections are constructed first in $\mathbf{Mod}_R$. Then they are furnished with the unique structure of an $(\mathbf{R}^+ \underline{\times} \Omega)$-action.

Hence corollary 2.9 can be applied to linear systems, or more precisely to continuous-time constant linear systems. We assume, however, that stronger results can be obtained from section 2 using another representation for linear systems similar to the discrete-time case in [1].

3.8 Smooth systems

Let us call a system Σ *smooth* if X is a smooth manifold (i. e. finite-dimensional, separable, connected C^∞-manifold with or without boundary) and for each $w \in \Omega$ $\varphi(-, -, w) : X \times \mathbf{R}^+ \to X$ is smooth. Moreover let I, Y and $\mathbf{R}^+ \underline{\times} \Omega$ be topological spaces and for each $x \in X$ $\varphi(x, -, -) : \mathbf{R}^+ \underline{\times} \Omega \to X$ be continuous, and τ and η be continuous, too.

Hence a smooth system is bicontinuous in the sense of 3.2 and it is similar to smooth systems in the sense of [7].

On the other hand, forward complete nonlinear systems considered by Sussmann in [11] are smooth in our sense. This follows from a general theorem in the theory of differential equations showing that the solution depends differentiably on the initial values.

However, most of the constructions for observability, orbit-minimality (reachability) and minimal realization, given in [11], are done on the lower level of ordinary (or alternatively bicontinuous) systems (*cf.* 3.1). In a second step, the quotient of the state space for example, is furnished with a structure of a C^∞-manifold.

Moreover, the notion of homomorphisms and weak isomorphisms in [11] are exactly the same as in our ordinary case. But, by lemma 5 in [11] weak isomorphisms restricted to orbits become C^∞-diffeomorphisms, which are isomorphisms for smooth systems in our sense. These constructions will be used in the theorem below.

Unfortunately, our general approach in section 2 is too weak, up to now, to allow such a two level strategy for smooth systems directly.

Since we cannot expect that the forgetful functor from the action part of smooth systems $\mathbf{D}$ to sets or topological spaces $\mathbf{K}$ has a left or a right adjoint we have to construct the $\mathfrak{C}_{IN}$-$\mathfrak{M}$ and $\mathfrak{C}$-$\mathfrak{M}_{OUT}$ factorization directly. For $\mathfrak{C}$ we take surjective submersions and for $\mathfrak{M}$ injective immersions. We restrict ourselves to I being a one-element space. Then we construct the $\mathfrak{C}_{IN}$-$\mathfrak{M}$ factorization of a function $f : I \to VQ$ defining S to be the orbit of the single element $f(I)$.

Clearly, S is a submanifold of VQ (with boundary) which is closed under the action φ, hence the inclusion $m : (S, \varphi) \to Q$ belongs to the class $\mathfrak{M}$ of injective immersions. Furthermore the restriction $f' : I \to S = V(S, \varphi)$ is an $\mathfrak{C}_{IN}$ morphism because for every factorisation $f'=m'\circ e'$, with m' injective immersion, m' is already surjective and hence a diffeomorphism. The uniqueness of the factorization $f=m\circ f'$ up to strong isomorphisms in the action-category follows from the rank theorem for smooth mappings.

Of course, only for special mappings $VQ \xrightarrow{\eta} Y$ it is possible to construct an $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorization. Let VQ be a manifold without boundary (we don't know whether the following remains true for manifolds with boundary). Consider the machine morphism $M : VQ \to \langle \mathbf{R}^+ \underline{\times} \Omega, Y\rangle$ in **Sets**.

The quotient VQ/R where R denotes the equivalence relation induced by M, carries the structure of a C^∞-manifold such that the natural map becomes a submersion, if (1) and (2) hold:

(1) R is closed in $VQ \times VQ$;

(2) the set of everywhere defined symmetry vector fields of R is weakly transitive, meaning that it spans at each point x the whole tangent space $T_x M$.

This has been proved as theorem 9 in [13].

In our case (1) is satisfied because we can regard M as the continuous machine-morphism in **Top**. Let us assume that (2) is satisfied, too, then

$$\begin{tikzcd} VQ \arrow[rr, "M"] \arrow[dr, "\mathrm{nat}"'] & & \langle \mathbb{R}^+ \underline{\times} \Omega , Y \rangle \\ & VQ/R \arrow[ur, "m"'] & \end{tikzcd}$$

is an $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorization: first of all m is an $\mathfrak{M}_{\mathrm{OUT}}$ morphism because for each factorization $m=m'\circ e'$ with e' surjective submersion, e' is injective because m is injective, and hence e' is a diffeomorphism. The uniqueness of the $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorization up to strong isomorphisms in the action category follows similar to that of the $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$ factorization from the rank theorem.

Now let $v_Y : \langle \mathbf{R}^+ \underline{\times} \Omega, Y\rangle \to Y$ be the evaluation on the unit element (*cf.* 2.5). Then it is easy to see that m is an $\mathfrak{M}_{\mathrm{OUT}}$ morphism if and only if $v_Y\circ m : VQ/R \to Y$ is an $\mathfrak{M}_{\mathrm{OUT}}$ morphism, and hence $(v_Y\circ m)\circ e=M$ is in fact an $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorization for $\eta=v_Y\circ M$.

Since smooth systems are also systems in the sense of 3.1, reachability and observability can be defined as in 3.1.

The set of vector fields F defined by

$$F(x)=\frac{\partial}{\partial t}\varphi(x,\,t,\,w)$$

is called the family of *associated vector fields* (denoted by $\mathfrak{F}_\Sigma$ in [13]), where w runs over the controls $w \in \Omega$.

Furthermore a system Σ is called *regular* if the set of associated vector fields has maximal rank at each point $x \in M$, i. e. it spans the whole tangent space $T_x M$ (this condition is slightly stronger than the "accessibility property" in the sense of [11]).

Now we are able to formulate the following theorem corresponding to some of the main results in [11]. On the other hand part (i) and (ii) are more or less theorem 2.9.1 and 2.9.2 respectively:

THEOREM: (i) *For each smooth system Σ with $I = \{ p \}$ (one point) there exists an equivalent reachable smooth system Σ' and a morphism $f : \Sigma' \to \Sigma$ which is an injective immersion.*

Moreover Σ' is unique with this property up to isomorphism.

(ii) *For each smooth regular system Σ with M being a connected manifold without boundary there exists an equivalent observable system $\overline{\Sigma}$ and a morphism $e : \Sigma \to \overline{\Sigma}$ which is a surjective submersion. Moreover $\overline{\Sigma}$ is unique with this property up to isomorphism.*

(iii) *Equivalent reachable and observable smooth systems are strongly isomorphic (i.e. the isomorphism is a diffeomorphism).*

Proof: The proof of (i) follows from 2.9.1 using the $\mathfrak{C}_{\mathrm{IN}}$-$\mathfrak{M}$ factorization of the function $\tau : I \to VQ$ above. This is the same construction as in paragraph 8 in [11].

Similarly (ii) follows from the existence of an $\mathfrak{C}$-$\mathfrak{M}_{\mathrm{OUT}}$ factorization of the output function $\eta : VQ \to Y$ (*cf.* 2.9.2). We only have to verify the condition (2) above, namely the weak transitivity of the set of everywhere defined symmetry vector fields of R. Now the family of associated vector fields is contained in the set of all everywhere defined symmetry vector fields (*cf.* Lemma 3 (*c*) in [11]), and by assumption that Σ is regular, this family of associated vector fields is weakly transitive.

(iii) holds, of course, in the category of ordinary systems (*cf.* 3.1 and 2.6). Hence two equivalent reachable and observable systems are weak isomorphic, in the sense that the homomorphism is bijective. Applying lemma 5 in [11] we obtain that it is smooth in both directions and hence a strong isomorphism. □

Remark: Of course, one can compose (i) and (ii) to construct an equivalent reachable and observable system. But this works only if the state space of the reachable system constructed in (i) is a manifold without boundary.

We have stated this theorem here in the present form to apply the general concept of $\mathfrak{C}_{IN}$-$\mathfrak{M}$ factorization and $\mathfrak{C}$-$\mathfrak{M}_{OUT}$ factorization. Of course, in special cases there may hold stronger versions.

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