INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

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Volume 360 (2022), p. 893-898

https://doi.org/10.5802/crmath.352

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

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https://doi.org/10.5802/crmath.352

Complex analysis and geometry / *Analyse et géométrie complexes*

Universal radial limits of meromorphic functions in the unit disk

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Abstract. We consider the space of meromorphic functions in the unit disk $\mathbb{D}$ and show that there exists a dense G_δ-subset of functions having universal radial limits. Our results complement known statements about holomorphic functions and further imply the existence of meromorphic functions having maximal cluster sets along certain subsets of $\mathbb{D}$.

2020 Mathematics Subject Classification. 30K15, 30D40, 30D35, 30D30.

Manuscript received 18 January 2022, accepted 6 March 2022.

1. Introduction

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ and $\mathbb{T}$ the unit circle $\{z : |z| = 1\}$. We further denote by $C(\mathbb{T})$ the set of continuous functions from $\mathbb{T}$ to $\mathbb{C}_\infty$, where $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is equipped with the chordal metric χ. On $C(\mathbb{T})$ we consider the metric $d_\mathbb{T}$ defined by $d_\mathbb{T}(f, g) := \max_\mathbb{T} \chi(f(z), g(z))$ for $f, g \in C(\mathbb{T})$. Finally, let $M(\mathbb{D})$ be the set of meromorphic functions in $\mathbb{D}$, including the function $f_\infty|_\mathbb{D}$, where $f_\infty \equiv \infty$. Endowed with the topology of spherically uniform convergence (i.e. uniform convergence with respect to the metric χ) on compact subsets of $\mathbb{D}$, the space $M(\mathbb{D})$ becomes a completely metrizable space. By d we denote a metric on $M(\mathbb{D})$ that induces its topology, such a metric can be defined via a compact exhaustion of $\mathbb{D}$ (see [7, Chap. VII]).

A result of Bayart [1] from 2005 implies the existence of functions f holomorphic in $\mathbb{D}$ having universal radial limits in the following sense: For every measurable function φ on $\mathbb{T}$, there exists an increasing sequence (ρ_l) in $[0, 1)$ with $\rho_l \to 1$ for $l \to \infty$, such that for every $z \in \mathbb{D}$ and almost every $\zeta \in \mathbb{T}$ we have

$$\lim_{l\to\infty} \left| f(\rho_l(\zeta - z) + z) - \varphi(\zeta) \right| = 0.$$

In a similar vein, the following result was recently proved by Charpentier [4], here, $f^{(k)}$ denotes the kth antiderivative of f for $k \in \mathbb{Z}$ with $k < 0$, and $f^{(0)} = f$ (see [5, 8] for related results).

Theorem ([4]). *Let $\mathcal{U}$ denote the set of functions f holomorphic in $\mathbb{D}$ having the following property: For every compact set $K \subset \mathbb{T}$ with $K \neq \mathbb{T}$, every continuous function $\varphi : K \to \mathbb{C}$, every compact set $L \subset \mathbb{D}$ and every $k \in \mathbb{Z}$, there exists an increasing sequence (ρ_l) in $[0,1)$ with $\rho_l \to 1$ for $l \to \infty$, such that*

$$\max_{\zeta \in K} \max_{z \in L} \left| f^{(k)}(\rho_l(\zeta - z) + z) - \varphi(\zeta) \right| \to 0 \quad \textit{for} \quad l \to \infty.$$

Then $\mathcal{U}$ is a dense G_δ-subset of the space of holomorphic functions in $\mathbb{D}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$.

The aim of this note is to show that similar universal radial limits are possible for functions in $M(\mathbb{D})$, where, in this case, the approximation property holds on the entire boundary of $\mathbb{D}$, i.e. for $K = \mathbb{T}$.

2. Main result

Using similar ideas as in [1, 4], we prove the following result:

Theorem 1. *Let $\mathcal{U}$ denote the set of functions $f \in M(\mathbb{D})$ having the following property: For every function $\varphi \in C(\mathbb{T})$ and every compact set $L \subset \mathbb{D}$, there exists an increasing sequence (ρ_l) in $[0,1)$ with $\rho_l \to 1$ for $l \to \infty$, such that*

$$\max_{\zeta \in \mathbb{T}} \max_{z \in L} \chi(f(\rho_l(\zeta - z) + z), \varphi(\zeta)) \to 0 \quad \textit{for} \quad l \to \infty.$$

Then $\mathcal{U}$ is a dense G_δ-subset of $M(\mathbb{D})$.

Proof. We first note that the space $C(\mathbb{T})$ is separable (this follows e.g. from [15, Thm. 13.3] and [9, Thm. 3]) and denote by (φ_j) a sequence that is dense in $C(\mathbb{T})$. Furthermore, we set $L_s := \{z : |z| \leq \frac{s}{s+1}\}$ for $s \in \mathbb{N}$, and consider an increasing sequence (ρ_l) in $[0,1)$ with $\rho_l \to 1$ for $l \to \infty$. For $j, s, l \in \mathbb{N}$, we define the set

$$U(j,s,l) := \left\{ f \in M(\mathbb{D}) : \max_{\zeta \in \mathbb{T}} \max_{z \in L_s} \chi(f(\rho_l(\zeta - z) + z), \varphi_j(\zeta)) < \frac{1}{s} \right\}.$$

We then have that

$$\mathcal{U} = \bigcap_{j,s} \bigcup_{l} U(j,s,l).$$

Indeed, suppose that f belongs to the set on the right hand side and let be given a function $\varphi \in C(\mathbb{T})$, a compact set $L \subset \mathbb{D}$ and some $\varepsilon > 0$. There exists $s_0 \in \mathbb{N}$ such that $L \subset L_{s_0}$ and $\frac{1}{s_0} < \frac{\varepsilon}{2}$. Furthermore, since (φ_j) is dense in $C(\mathbb{T})$, there exists $j_0 \in \mathbb{N}$ such that $\max_{\mathbb{T}} \chi(\varphi_{j_0}(\zeta), \varphi(\zeta)) < \frac{\varepsilon}{2}$. By assumption, there exists $l_0 \in \mathbb{N}$ such that $f \in U(j_0, s_0, l_0)$, and hence

$$\max_{\zeta \in \mathbb{T}} \max_{z \in L_{s_0}} \chi(f(\rho_{l_0}(\zeta - z) + z), \varphi_{j_0}(\zeta)) < \frac{1}{s_0}.$$

It follows

$$\begin{aligned} &\max_{\zeta \in \mathbb{T}} \max_{z \in L} \chi(f(\rho_{l_0}(\zeta - z) + z), \varphi(\zeta)) \\ &\qquad \leq \max_{\zeta \in \mathbb{T}} \max_{z \in L_{s_0}} \chi(f(\rho_{l_0}(\zeta - z) + z), \varphi_{j_0}(\zeta)) + \max_{\zeta \in \mathbb{T}} \chi(\varphi_{j_0}(\zeta), \varphi(\zeta)) < \varepsilon, \end{aligned}$$

so that $f \in \mathcal{U}$, and hence $\mathcal{U} \supset \bigcap_{j,s} \bigcup_l U(j,s,l)$. The other inclusion is obvious.

Let now $j_1 \in \mathbb{N}$ and $s_1 \in \mathbb{N}$ be given. We show that the set $\bigcup_l U(j_1, s_1, l)$ is open and dense in $M(\mathbb{D})$. Consider therefore $l \in \mathbb{N}$ and suppose that $f \in U(j_1, s_1, l)$. Since the set

$$K_{l,s_1} := \{\rho_l(\zeta - z) + z : \zeta \in \mathbb{T}, z \in L_{s_1}\}$$

is a compact subset of $\mathbb{D}$, there then exists $\varepsilon > 0$ such that

$$\max_{\zeta\in\mathbb{T}}\max_{z\in L_{s_1}}\chi(f(\rho_l(\zeta-z)+z),\varphi_{j_1}(\zeta)) < \frac{1}{s_1}-\varepsilon.$$

Furthermore, according to Lemma 1.7 in [7, Chap. VII], there exists $\delta > 0$, such that for every $g \in M(\mathbb{D})$ with $d(g,f) < \delta$ we have that

$$\max_{z\in K_{l,s_1}}\chi(g(z),f(z)) < \varepsilon.$$

Hence, for $g \in M(\mathbb{D})$ with $d(g,f) < \delta$, we obtain

$$\begin{aligned}
&\max_{\zeta\in\mathbb{T}}\max_{z\in L_{s_1}}\chi(g(\rho_l(\zeta-z)+z),\varphi_{j_1}(\zeta))\\
&\qquad\le \max_{\zeta\in\mathbb{T}}\max_{z\in L_{s_1}}\chi(g(\rho_l(\zeta-z)+z),f(\rho_l(\zeta-z)+z)) + \max_{\zeta\in\mathbb{T}}\max_{z\in L_{s_1}}\chi(f(\rho_l(\zeta-z)+z),\varphi_{j_1}(\zeta))\\
&\qquad= \max_{z\in K_{l,s_1}}\chi(g(z),f(z)) + \max_{\zeta\in\mathbb{T}}\max_{z\in L_{s_1}}\chi(f(\rho_l(\zeta-z)+z),\varphi_{j_1}(\zeta)) < \frac{1}{s_1},
\end{aligned}$$

so that $g \in U(j_1,s_1,l)$. This implies that for $l \in \mathbb{N}$, the set $U(j_1,s_1,l)$ is open in $M(\mathbb{D})$, and the same then obviously holds for the set $\bigcup_l U(j_1,s_1,l)$.

In order to show that the set $\bigcup_l U(j_1,s_1,l)$ is dense in $M(\mathbb{D})$, we consider $f \in M(\mathbb{D})$ and $\varepsilon > 0$. By Lemma 1.7 in [7, Chap. VII], there exists $s_2 \in \mathbb{N}$ with $s_2 \ge s_1$ and $\delta > 0$, such that for every $g \in M(\mathbb{D})$ with $\max_{L_{s_2}}\chi(g(z),f(z)) < \delta$, we have that $d(g,f) < \varepsilon$. Consider now the set $K := L_{s_2} \cup \mathbb{T}$ and the function

$$h(z) := \begin{cases} f(z), & \text{for } z \in L_{s_2},\\ \varphi_{j_1}(z), & \text{for } z \in \mathbb{T}.\end{cases}$$

Then K is a compact subset of $\mathbb{C}$ and $\mathbb{C}\setminus K$ has two components, so that a combination of Mergelyan's Theorem on rational approximation (e.g. [15, Thm. 13.3]) and [9, Thm. 3] yields the existence of a rational function $r(z)$, such that

$$\max_{z\in K}\chi(r(z),h(z)) < \min\left\{\delta,\frac{1}{2s_1}\right\},$$

so that we obtain

$$\max_{z\in L_{s_2}}\chi(r(z),f(z)) < \delta \quad\text{and}\quad \max_{\zeta\in\mathbb{T}}\chi(r(\zeta),\varphi_{j_1}(\zeta)) < \frac{1}{2s_1}.$$

Note that by the first inequality, we have that $d(r,f) < \varepsilon$. It remains to show that $r \in \bigcup_l U(j_1,s_1,l)$. Clearly, r is uniformly continuous on compact subsets of $\mathbb{C}$, hence there exists $\delta_{s_1} > 0$ such that for $z_1,z_2 \in \overline{\mathbb{D}}$, we have that

$$|z_1-z_2| < \delta_{s_1} \quad\text{implies}\quad \chi(r(z_1),r(z_2)) < \frac{1}{2s_1}.$$

Since $\rho_l \to 1$ for $l \to \infty$, there further exists $l_1 \in \mathbb{N}$ such that

$$\max_{\zeta\in\mathbb{T}}\max_{z\in L_{s_1}}\left|\rho_{l_1}(\zeta-z)+z-\zeta\right| = \max_{\zeta\in\mathbb{T}}\max_{z\in L_{s_1}}\left|\zeta(\rho_{l_1}-1)-z(\rho_{l_1}-1)\right| < \delta_{s_1}.$$

Finally, we obtain

$$\begin{aligned}
&\max_{\zeta\in\mathbb{T}}\max_{z\in L_{s_1}}\chi(r(\rho_{l_1}(\zeta-z)+z),\varphi_{j_1}(\zeta))\\
&\qquad\le \max_{\zeta\in\mathbb{T}}\max_{z\in L_{s_1}}\chi(r(\rho_{l_1}(\zeta-z)+z),r(\zeta)) + \max_{\zeta\in\mathbb{T}}\chi(r(\zeta),\varphi_{j_1}(\zeta)) < \frac{1}{s_1},
\end{aligned}$$

so that $r \in U(j_1,s_1,l_1)$ and the set $\bigcup_l U(j_1,s_1,l)$ is dense in $M(\mathbb{D})$.

Hence, for any $j,s \in \mathbb{N}$, the set $\bigcup_l U(j,s,l)$ is open and dense in $M(\mathbb{D})$. By Baire's Theorem, the set $\bigcap_{j,s}\bigcup_l U(j,s,l) = \mathscr{U}$ is thus a dense G_δ-subset of $M(\mathbb{D})$, which proves the theorem. □

We explicitly state the case $L = \{0\}$ in Theorem 1:

Corollary 2. *There exists a dense G_δ-subset $\mathcal{U}$ of $M(\mathbb{D})$, such that every $f \in \mathcal{U}$ has the following property: For every function $\varphi \in C(\mathbb{T})$, there exists an increasing sequence (ρ_l) in $[0,1)$ with $\rho_l \to 1$ for $l \to \infty$, such that*

$$\max_{\zeta\in\mathbb{T}} \chi(f(\rho_l\zeta), \varphi(\zeta)) \to 0 \quad \textit{for} \quad l \to \infty.$$

3. Properties of functions $f \in \mathcal{U}$

In the following, we will study some properties of functions $f \in \mathcal{U}$, where $\mathcal{U}$ is the set defined in Theorem 1. We start by showing that in contrast to the holomorphic case, the universal approximation property in Theorem 1 does not hold for the derivatives of f.

Proposition 3. *Consider a function $f \in M(\mathbb{D})$. Then we have $f' \notin \mathcal{U}$.*

Proof. Suppose that there exists $f \in M(\mathbb{D})$ such that $f' \in \mathcal{U}$. Then, there exists an increasing sequence (ρ_l) in $[0,1)$ with $\rho_l \to 1$ for $l \to \infty$, such that

$$\max_{\zeta\in\mathbb{T}} \chi\left(f'(\rho_l\zeta), \frac{1}{\zeta}\right) \to 0 \quad \text{for} \quad l \to \infty.$$

Note that this implies that for l sufficiently large, the function $f'(\rho_l\zeta)$ has no poles on $\mathbb{T}$. Now, since $\frac{1}{\zeta}$ is bounded on $\mathbb{T}$, we obtain that the sequence $(f'(\rho_l\zeta))$ converges uniformly to $\frac{1}{\zeta}$ on $\mathbb{T}$ for $l \to \infty$, which in turn implies

$$\frac{1}{2\pi i}\int_{\mathbb{T}} f'(\rho_l\zeta)\,\mathrm{d}\zeta \to \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{1}{\zeta}\,\mathrm{d}\zeta = 1 \quad \text{for} \quad l \to \infty.$$

According to the Residue Theorem, the integral on the left hand side equals 0 for every $l \in \mathbb{N}$, hence we get a contradiction. □

We further mention that Theorem 1 implies the existence of functions in $M(\mathbb{D})$ having maximal cluster sets along certain subsets of $\mathbb{D}$, as well as functions having no asymptotic values. We recall the corresponding definitions. A value $a \in \mathbb{C}_\infty$ is said to be an asymptotic value of $f \in M(\mathbb{D})$, if there exists a continuous curve $\gamma : [0,1) \to \mathbb{D}$ with $\lim_{r\to 1^-} |\gamma(r)| = 1$, such that $\lim_{r\to 1^-} f(\gamma(r)) = a$. The curve γ is then called an asymptotic curve for f. Furthermore, for a function $f \in M(\mathbb{D})$ and a set $A \subset \mathbb{D}$ with $\overline{A} \cap \mathbb{T} \neq \emptyset$, the cluster set $C(f,A)$ of f along A is defined by

$$C(f,A) := \left\{a \in \mathbb{C}_\infty : \exists\ (z_n) \text{ in } A \text{ and } \zeta \in \mathbb{T} : z_n \to \zeta \text{ and } \chi(f(z_n), a) \to 0 \text{ for } n \to \infty\right\},$$

and the cluster set is said to be maximal, if $C(f,A) = \mathbb{C}_\infty$. Given $f \in \mathcal{U}$ and setting $A_{z,\zeta} := \{\rho(\zeta - z) + z : \rho \in [0,1)\}$ for $z \in \mathbb{D}$ and $\zeta \in \mathbb{T}$, it is easily seen that for every $z \in \mathbb{D}$ and every $\zeta \in \mathbb{T}$, the radial cluster set $C(f, A_{z,\zeta})$ is maximal and the curve $\gamma_{z,\zeta} : [0,1) \to \mathbb{D}$ with $\gamma_{z,\zeta}(r) := r(\zeta - z) + z$ is not an asymptotic curve for f. Hence, as a consequence of Theorem 1, we obtain that the set of functions in $M(\mathbb{D})$ having this property contains a dense G_δ-subset of $M(\mathbb{D})$. In fact, we have the following stronger statement (see [2,3,11–13] for related results):

Corollary 4. *Let be given a function $f \in \mathcal{U}$. Then the following hold:*

(i) *the cluster set $C(f, \gamma([0,1)))$ is maximal for every continuous curve $\gamma : [0,1) \to \mathbb{D}$ with $\limsup_{r\to 1^-} |\gamma(r)| = 1$.*
(ii) *the function f does not have an asymptotic value. Furthermore, f takes every value $a \in \mathbb{C}_\infty$ infinitely many times in $\mathbb{D}$ and for every $\zeta \in \mathbb{T}$ and every $\varepsilon > 0$, we have that the set $\mathbb{C}_\infty \setminus f(U_\varepsilon(\zeta) \cap \mathbb{D})$ contains at most two points, where $U_\varepsilon(\zeta) := \{z : |z - \zeta| < \varepsilon\}$.*

In particular, the set of functions in $M(\mathbb{D})$ having the above properties contains a dense G_δ-subset of $M(\mathbb{D})$.

Proof. (i). Let $f \in \mathscr{U}$ and $a \in \mathbb{C}_\infty$ be given and consider a continuous curve $\gamma : [0,1) \to \mathbb{D}$ with $\limsup_{r\to 1^-} |\gamma(r)| = 1$. By assumption, there exists an increasing sequence (ρ_l) in $[0,1)$ with $\rho_l \to 1$ for $l \to \infty$, such that

$$\max_{\zeta\in\mathbb{T}} \chi\left(f(\rho_l\zeta), a\right) \to 0 \quad \text{for} \quad l \to \infty.$$

Since $\limsup_{r\to 1^-} |\gamma(r)| = 1$, there exists $l_0 \in \mathbb{N}$ such that for every $l > l_0$, we have that $\gamma([0,1)) \cap \{z : |z| = \rho_l\} \neq \emptyset$. Thus, there exists a sequence (z_s) on $\gamma([0,1))$ with $|z_s| = \rho_{l_0+s}$ for $s \in \mathbb{N}$, and hence $|z_s| \to 1$ for $s \to \infty$. We finally obtain

$$\chi(f(z_s), a) \le \max_{\{z:|z|=\rho_{l_0+s}\}} \chi(f(z), a) = \max_{\zeta\in\mathbb{T}} \chi(f(\rho_{l_0+s}\zeta), a) \to 0 \quad \text{for} \quad s \to \infty,$$

which implies $a \in C(f, \gamma([0,1)))$, hence $C(f, \gamma([0,1)))$ is maximal.

(ii). Since by the first statement, the cluster set $C(f, \gamma([0,1)))$ is maximal for every continuous curve $\gamma : [0,1) \to \mathbb{D}$ with $\lim_{r\to 1^-} |\gamma(r)| = 1$, it is clear that f can not have an asymptotic value. The other statements then immediately follow from classical results in [6, 14]. □

The above corollary shows that $f \in \mathscr{U}$ does not have a Picard value. We finish with a slightly stronger statement, namely that $f \in \mathscr{U}$ can not have a deficient value in the sense of Nevanlinna. We therefore introduce some standard terminology from Nevanlinna theory, see [10] for the corresponding definitions. For a function $f \in M(\mathbb{D})$, a value $a \in \mathbb{C}_\infty$ and $r \in [0,1)$, we denote by $m(r,a,f)$ and $N(r,a,f)$ the proximity function and the (integrated) counting function, respectively. We write $T(r,f)$ for the characteristic function of f and we recall that in case that $T(r,f) \to \infty$ for $r \to 1^-$, the deficiency $\delta(a,f)$ of a is defined by

$$\delta(a,f) = 1 - \limsup_{r\to 1^-} \frac{N(r,a,f)}{T(r,f)} = \liminf_{r\to 1^-} \frac{m(r,a,f)}{T(r,f)},$$

and a is called a deficient value of f, if $\delta(a,f) > 0$.

Proposition 5. *Consider a function $f \in \mathscr{U}$. Then f has no deficient value.*

Proof. Let be given a function $f \in \mathscr{U}$. Then, the radial limit $\lim_{\rho\to 1^-} f(\rho\zeta)$ does not exist for any $\zeta \in \mathbb{T}$, and it is an immediate consequence of Nevanlinna's extension of Fatou's Theorem (e.g. [10, Thm. 6.12]) that $T(r,f) \to \infty$ for $r \to 1^-$. By assumption, there exists an increasing sequence (ρ_l) in $[0,1)$ with $\rho_l \to 1$ for $l \to \infty$, such that

$$\max_{\zeta\in\mathbb{T}} \chi\left(f(\rho_l\zeta), 0\right) \to 0 \quad \text{for} \quad l \to \infty.$$

This implies the uniform convergence of $(f(\rho_l\zeta))$ to 0 on $\mathbb{T}$ for $l \to \infty$, from which we obtain $m(\rho_l, a, f) = \mathscr{O}(1)$ for $l \to \infty$ and every $a \in \mathbb{C}_\infty \setminus \{0\}$, and hence

$$\delta(a,f) = \liminf_{r\to 1^-} \frac{m(r,a,f)}{T(r,f)} \le \lim_{l\to\infty} \frac{m(\rho_l,a,f)}{T(\rho_l,f)} = 0.$$

By a very similar argumentation, we obtain an increasing sequence (ρ_l) in $[0,1)$ with $\rho_l \to 1$ for $l \to \infty$, and $m(\rho_l, 0, f) = \mathscr{O}(1)$ for $l \to \infty$, which implies $\delta(0,f) = 0$. Hence, no value $a \in \mathbb{C}_\infty$ is deficient for f. □

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