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Partial differential equations

# On optimal Hölder regularity of solutions to the equation $\Delta u + b \cdot \nabla u = 0$ in two dimensions

## *Sur la régularité Hölder optimale pour les solutions de l'equation $\Delta u + b \cdot \nabla u = 0$ en dimension deux*

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ABSTRACT

We show that for an $L^2$ drift $b$ in two dimensions, if the Hardy norm of $\operatorname{div} b$ is small, then the weak solutions to $\Delta u + b \cdot \nabla u = 0$ have the same optimal Hölder regularity as in the case of divergence-free drift, that is, $u \in C^\alpha_{loc}$ for all $\alpha \in (0, 1)$.



RÉSUMÉ

Nous démontrons que, pour une dérive $b \in L^2_{loc}(\mathbb{R}^2; \mathbb{R}^2)$, si la norme de Hardy de div $b$ est petite, alors les solutions faibles de $\Delta u + b \cdot \nabla u = 0$ (en dimension deux) ont la même régularité Hölder que dans le cas de la dérive incompressible, c'est-à-dire que $u \in C^\alpha_{loc}$ pour tout $\alpha \in (0, 1)$.



### 1. Statement of the main result

In this note, we revisit the local (optimal) Hölder continuity of $W^{1,2}$ scalar solutions to

$$\Delta u + b \cdot \nabla u = 0 \tag{1.1}$$

in two dimensions where the drift $b = (b_1, b_2)$ is an $L^2_{loc}$ vectorfield in $\mathbb{R}^2$. When $\operatorname{div} b = 0$, Filonov [3, Theorem 1.2] shows that $u \in W^{2,q}_{loc}$ for all $q \in (1, 2)$ and hence the optimal Hölder regularity for $u$: $u \in C^\alpha_{loc}$ for all $\alpha \in (0, 1)$. In this note, we show that the same conclusions hold if the $\mathcal{H}^1(\mathbb{R}^2)$ Hardy norm of $\operatorname{div} b$ is small. The Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ will be recalled in Section 2.

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**Theorem 1.1.** *Let $\Omega$ be an open, bounded and connected domain in $\mathbb{R}^2$. Let $b \in L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ and let $u \in W^{1,2}(\Omega)$ satisfy (1.1) in $\Omega$ in the sense of distributions. There is a small, positive constant $\varepsilon_0$ such that if $\|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)} \le \varepsilon_0$, then $u \in W^{2,q}_{loc}(\Omega)$ for all $q \in (1,2)$ and hence, $u \in C^{\alpha}_{loc}(\Omega)$ for all $\alpha \in (0,1)$.*

If the condition $\operatorname{div} b = 0$ is dropped, then, as pointed out in [3], the solutions to (1.1) are not continuous nor bounded in general. Note that Eq. (1.1) with the divergence-free drift $b$ appears in various models in fluid mechanics; see, for example [4, 9] and the references therein. These papers also establish several regularity results, including Hölder continuity, for solutions to (1.1) when the divergence-free drift $b$ has low integrability. In 2D, in order to obtain the Hölder continuity of the solutions $u \in W^{1,2}$ to (1.1), that is, $u \in C^{\alpha}_{loc}(\Omega)$ for *some* $\alpha \in (0,1)$, it suffices to assume that $\operatorname{div} b \in \mathcal{H}^1(\mathbb{R}^2)$. This follows from revisiting the arguments of Bethuel [1].

Denote the rotation of the gradient vector in 2D by $\nabla^{\perp} v = (-\partial_2 v, \partial_1 v)$. Theorem 1.1 easily gives:

**Corollary 1.2.** *Let $\Omega$ be an open, bounded and connected domain in $\mathbb{R}^2$. Assume that $b = h\nabla^{\perp} v$ where $h \in L^{\infty}(\Omega) \cap W^{1,2}(\Omega)$ and $v \in W^{1,2}(\Omega)$. Let $u \in W^{1,2}(\Omega)$ satisfy (1.1) in $\Omega$. Then $u \in W^{2,q}_{loc}(\Omega)$ for all $q \in (1,2)$ and hence, $u \in C^{\alpha}_{loc}(\Omega)$ for all $\alpha \in (0,1)$.*

**Remark 1.3.** *The drift $b$ in Corollary 1.2 satisfies $\operatorname{div} b \in \mathcal{H}^1$ and thus the result of Bethuel [1] (see also [5] for system) already implies the Hölder continuity of $u$, that is, $u \in C^{\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$. The novelty of Corollary 1.2 is that it gives further and optimal regularity results for $u$.*

Our proof of Theorem 1.1 uses the Uhlenbeck–Rivière decomposition (also known as the nonlinear Hodge decomposition) and the integration by compensation to convert (1.1) into a *conservation law*. This circle of ideas was inspired by Rivière's proof of Heinz–Hildebrandt's conjecture [7]. We also give another proof of Theorem 1.1 using the uniqueness result for (1.1) following Filonov [3]. It is interesting to note that, *although (1.1) is linear, our arguments are non-linear.* A key ingredient in the proof of Theorem 1.1 is the local structure of the $L^2$ vectorfields $b$ whose $\operatorname{div} b$ have small Hardy norm.

**Theorem 1.4.** *Assume that $\Omega = B_1(0) \subset \mathbb{R}^2$ and $b \in L^2_{loc}(\mathbb{R}^2;\mathbb{R}^2)$. There exists a positive constant $\varepsilon_1$ with the following property. If $\|b\|_{L^2(\Omega)} + \|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)} \le \varepsilon_1$, then there exist $A \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ and $B \in W^{1,2}(\Omega)$ such that $A^{-1} \in L^{\infty}(\Omega)$, $A = 1$ on $\partial\Omega$, $\int_{\Omega} B = 0$, and*

$$b = A^{-1}\nabla A + A^{-1}\nabla^{\perp} B.$$

The rest of the note is organized as follows. We recall Hardy spaces and related Wente's estimates in Sect. 2. Assuming Theorem 1.4, we prove Theorem 1.1 in Sect. 3. We prove Theorem 1.4 in Sect. 4.

## 2. Hardy spaces and related Wente's estimates

First, we recall the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$, following Hélein [6, Section 3.2]. For any function $f \in L^1(\mathbb{R}^2)$, we denote the Riesz transforms $R_j f$ $(j = 1,2)$ of $f$ by $\widehat{R_j f}(\xi) = \frac{\xi_j}{|\xi|}\hat{f}(\xi)$ $(\xi = (\xi_1,\xi_2) \in \mathbb{R}^2)$, where $\hat{f}$ is the Fourier transform of $f$: $\hat{f}(\xi) = (2\pi)^{-1}\int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx$.

**Definition 2.1.** The Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ with norm $\|\cdot\|_{\mathcal{H}^1(\mathbb{R}^2)}$ is defined by

$$\mathcal{H}^1(\mathbb{R}^2) = \{f \in L^1(\mathbb{R}^2) : \|f\|_{\mathcal{H}^1(\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2)} + \|R_1 f\|_{L^1(\mathbb{R}^2)} + \|R_2 f\|_{L^1(\mathbb{R}^2)} < \infty\}.$$

A basic observation is the following theorem, due to Coifman–Lions–Meyer–Semmes [2]:

**Theorem 2.2.** *([6, Theorem 3.2.2]) If $u, v \in W^{1,2}(\mathbb{R}^2)$ then $\nabla u \cdot \nabla^{\perp} v \in \mathcal{H}^1(\mathbb{R}^2)$. Moreover, there is a uniform constant $C$ such that the following estimate holds:*

$$\|\nabla u \cdot \nabla^{\perp} v\|_{\mathcal{H}^1(\mathbb{R}^2)} \le C\|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla v\|_{L^2(\mathbb{R}^2)}.$$

We recall the following regularity result concerning solutions to the Laplace equation with Hardy right-hand side. The following theorem follows through combining Theorems 3.3.4, 3.3.8 and together Eq. (3.38) in [6].

**Theorem 2.3.** *Let $\Omega$ be an open subset of $\mathbb{R}^2$, with $C^1$ boundary. Let $f \in \mathcal{H}^1(\mathbb{R}^2)$, and $\phi \in L^1_{loc}(\Omega)$ be a solution to*

$$\Delta\phi = f \text{ in } \Omega, \text{ and } \phi = 0 \text{ on } \partial\Omega.$$

*Then $\phi \in W_0^{1,2}(\Omega) \cap C(\overline{\Omega})$ and there is a constant depending only on $\Omega$, $C(\Omega)$, such that*

$$\|\phi\|_{L^\infty(\Omega)} + \|\nabla\phi\|_{L^2(\Omega)} \le C(\Omega)\|f\|_{\mathcal{H}^1(\mathbb{R}^2)}.$$

Finally, we recall the following estimates for boundary value problems with Jacobian structure right-hand side in the theory of integration by compensation, due to Wente's [10]; see also [2].

**Lemma 2.4.** *Let $\Omega$ be an open subset of $\mathbb{R}^2$, with $C^1$ boundary. Suppose that $u, v \in W^{1,2}(\Omega)$. Let $w$ be the unique solution in $W^{1,p}(\Omega)$ for $1 \le p < 2$ to the equation $\Delta w = \nabla u \cdot \nabla^\perp v$ in $\Omega$, either with the Dirichlet condition $w = 0$ on $\partial\Omega$, or with the Neumann boundary condition $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ and $\int_\Omega w = 0$. Then $w$ belongs to $C(\overline{\Omega}) \cap W^{1,2}(\Omega)$, and there is a constant $C$ depending on $\Omega$ such that*

$$\|w\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^2(\Omega)} + \|D^2 w\|_{L^1(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}.$$

## 3. Proofs of Theorem 1.1 and Corollary 1.2

Assuming Theorem 1.4, we prove Theorem 1.1 and Corollary 1.2 in this section. Let $\varepsilon_1$ be the positive number given by Theorem 1.4. Let $\varepsilon_0 = \varepsilon_1/2$. Let $u \in W^{1,2}(\Omega)$ satisfy (1.1) in $\Omega$.

### 3.1. Rescaling

Theorem 1.1 and Corollary 1.2 are of local nature, so it suffices to prove the optimal Hölder continuity of $u$ in a small ball $B_r(x_0) \subset \Omega$ round each $x_0 \in \Omega$. We rescale Eq. (1.1) in $B_r(x_0)$ to $B_1(0)$ by setting

$$\tilde{u}(x) = u(x_0 + rx), \; \tilde{b}(x) = rb(x_0 + rx).$$

Then $\tilde{u} \in W^{1,2}(B_1(0))$ solves

$$-\Delta\tilde{u} = \tilde{b} \cdot \nabla\tilde{u} \text{ in } B_1(0). \tag{3.2}$$

Furthermore,

$$\|\tilde{b}\|_{L^2(B_1(0))} = \|b\|_{L^2(B_r(x_0))} \text{ and } \|\text{div } \tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} = \|\text{div}\, b\|_{\mathcal{H}^1(\mathbb{R}^2)}. \tag{3.3}$$

By the Dominated Convergence theorem, $\|b\|_{L^2(B_r(x_0))} \to 0$ as $r \to 0$. Thus, we can fix a radius $r > 0$ so that $\|b\|_{L^2(B_r(x_0))} < \varepsilon_0$.

From now on, we can assume $\Omega = B_1(0)$ with the following smallness condition on $\|\tilde{b}\|_{L^2(B_1(0))}$:

$$\|\tilde{b}\|_{L^2(B_1(0))} < \varepsilon_0. \tag{3.4}$$

### 3.2. Proofs of Theorem 1.1

**Proof of Theorem 1.1 via conservation laws.** Suppose that $\|\text{div}\, b\|_{\mathcal{H}^1(\mathbb{R}^2)} \le \varepsilon_0$. Then from (3.3) and (3.4), we have

$$\|\tilde{b}\|_{L^2(B_1(0))} + \|\text{div } \tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} \le 2\varepsilon_0 = \varepsilon_1.$$

Applying Theorem 1.4, we find $A \in W^{1,2}(B_1(0)) \cap L^\infty(B_1(0))$ and $B \in W^{1,2}(B_1(0))$ such that $A^{-1} \in L^\infty(B_1(0))$, $A = 1$ on $\partial B_1(0)$, $\int_{B_1(0)} B = 0$, and

$$\tilde{b} = A^{-1}\nabla A + A^{-1}\nabla^\perp B.$$

This together with (3.2) gives

$$\text{div}\,(A\nabla\tilde{u} - B\nabla^\perp\tilde{u}) = 0.$$

Thus, we have just converted (1.1) into a conservation law. By [8, Theorem 4.3] or the proof of [7, Theorem 1.1], $\tilde{u} \in W^{1,p}_{\text{loc}}(B_1(0))$ for all $p \in (1, \infty)$. Now, the right-hand side of (3.2) belongs to $L^q_{\text{loc}}(B_1(0))$ for all $q \in (1, 2)$, and hence $\tilde{u} \in W^{2,q}_{\text{loc}}(B_1(0))$ for all $q \in (1, 2)$. Therefore, by the Sobolev embedding theorem, $\tilde{u} \in C^\alpha_{\text{loc}}(B_1(0))$ for all $\alpha \in (0, 1)$. Rescaling back, we obtain the desired regularity for $u$. □

**Proof of Theorem 1.1 via uniqueness.** Instead of using [7,8], we can give a direct and short proof of Theorem 1.1 using the uniqueness approach of Filonov [3]. In [3, Theorem 1.2], Filonov proved[1] that $W^{1,2}$ solutions to (3.2) belong to $W^{2,q}_{\text{loc}}(B_1(0))$ for all $1 < q < 2$ provided that the equation

[1] Our sign is different from [3]. Filonov considered $-\Delta v + \tilde{b} \cdot \nabla v = 0$.

$$\Delta v + \tilde{b} \cdot \nabla v = 0 \text{ in } B_1(0), \text{ with } v = 0 \text{ on } \partial B_1(0). \tag{3.5}$$

has a unique solution $v = 0$ in $W_0^{1,2}(B_1(0))$. We prove that this is indeed the case in the context of Theorem 1.1. As above, we have $\tilde{b} = A^{-1}(\nabla A + \nabla^\perp B)$ and hence (3.5) becomes

$$\begin{cases} \operatorname{div}(A\nabla v) + \nabla^\perp B \cdot \nabla v = 0 & \text{in } B_1(0), \\ v = 0 & \text{on } \partial B_1(0). \end{cases} \tag{3.6}$$

Since $A^{-1} \in L^\infty(B_1(0))$, we prove that $v = 0$ by showing $\int_{B_1(0)} A|\nabla v|^2 = 0$. Following the idea of the proof of [3, Lemma 2.6], we choose a sequence $\psi_n \in C_0^\infty(B_1(0))$ such that $\psi_n \to v$ in $W^{1,2}(B_1(0))$. From

$$\begin{aligned} \int_{B_1(0)} A|\nabla v|^2 &= \int_{B_1(0)} A\nabla v \cdot \nabla \psi_n + \int_{B_1(0)} A\nabla v \cdot (\nabla v - \nabla \psi_n) \\ &\le \int_{B_1(0)} A\nabla v \cdot \nabla \psi_n + \|A\|_{L^\infty(B_1(0))} \|\nabla v\|_{L^2(B_1(0))} \|\nabla v - \nabla \psi_n\|_{L^2(B_1(0))}, \end{aligned}$$

and $\|\nabla v - \nabla \psi_n\|_{L^2(B_1(0))} \to 0$ when $n \to \infty$, it remains to show that

$$\int_{B_1(0)} A\nabla v \cdot \nabla \psi_n \to 0 \text{ when } n \to \infty. \tag{3.7}$$

To see this, multiplying both sides of (3.6) by $\psi_n$ and using that

$$\int_{B_1(0)} \nabla^\perp B \cdot \nabla \psi_n \psi_n = \int_{B_1(0)} \nabla^\perp B \cdot \nabla(\psi_n^2/2) = 0$$

which follows from integrating by parts and $\operatorname{div} \nabla^\perp B = 0$, we obtain

$$\int_{B_1(0)} A\nabla v \cdot \nabla \psi_n = \int_{B_1(0)} \nabla^\perp B \nabla v \psi_n = \int_{B_1(0)} \nabla^\perp B \cdot \nabla(v - \psi_n)\psi_n.$$

Since $\operatorname{div}(\nabla^\perp B) = 0$, by [3, Lemma 2.4], and the fact that $\psi_n \to v$ in $W^{1,2}(B_1(0))$, we have

$$\int_{B_1(0)} \nabla^\perp B \cdot \nabla(v - \psi_n)\psi_n \le C\|\nabla B\|_{L^2(B_1(0))} \|\nabla v - \nabla \psi_n\|_{L^2(B_1(0))} \|\nabla \psi_n\|_{L^2(B_1(0))} \to 0 \text{ when } n \to \infty.$$

Therefore, we obtain (3.7) and the proof of Theorem 1.1 is complete. □

**Proof of Corollary 1.2.** Suppose that $b = h\nabla^\perp v$ where $h \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ and $v \in W^{1,2}(\Omega)$. After having rescaled $u$ and $b$ in Section 3.1, we also rescale $h$ and $v$ as follows:

$$\tilde{h}(x) = h(x_0 + rx) - c_1, \; \tilde{v}(x) = v(x_0 + rx) - c_2$$

where $c_1$ and $c_2$ are constants such that $\int_{B_1(0)} \tilde{h} = \int_{B_1(0)} \tilde{v} = 0$. Note that $\|\nabla \tilde{v}\|_{L^2(B_1(0))} = \|\nabla v\|_{L^2(B_r(x_0))}$ and that, by Poincaré's inequality, we have

$$\|\tilde{v}\|_{W^{1,2}(B_1(0))} \le C\|\nabla \tilde{v}\|_{L^2(B_1(0))} \le C\|\nabla v\|_{L^2(B_r(x_0))}. \tag{3.8}$$

Similarly, we have

$$\|\tilde{h}\|_{W^{1,2}(B_1(0))} \le C\|\nabla h\|_{L^2(B_r(x_0))} \tag{3.9}$$

We can extend $\tilde{h}$ and $\tilde{v}$ to be compactly supported functions in $\mathbb{R}^2$ such that $\tilde{h} \in L^\infty(\mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^2)$, $\tilde{v} \in W^{1,2}(\mathbb{R}^2)$, $\|\tilde{h}\|_{L^\infty(\mathbb{R}^2)} \le C\|\tilde{h}\|_{L^\infty(B_1(0))}$, and

$$\|\tilde{h}\|_{W^{1,2}(\mathbb{R}^2)} \le C\|\tilde{h}\|_{W^{1,2}(B_1(0))}, \;\; \|\tilde{v}\|_{W^{1,2}(\mathbb{R}^2)} \le C\|\tilde{v}\|_{W^{1,2}(B_1(0))}. \tag{3.10}$$

With these extensions, we have, by Theorem 2.2, $\operatorname{div} \tilde{b} = \nabla \tilde{h} \cdot \nabla^\perp \tilde{v} \in \mathcal{H}^1(\mathbb{R}^2)$ and

$$\|\operatorname{div} \tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} \le C\|\nabla \tilde{h}\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{v}\|_{L^2(\mathbb{R}^2)}.$$

It follows from (3.8), (3.9) and (3.10) that

$$\|\operatorname{div}\tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} \le C\|\nabla h\|_{L^2(B_r(x_0))}\|\nabla v\|_{L^2(B_r(x_0))}.$$

Using the Dominated Convergence theorem, we can now further reduce the small radius $r$ in Section 3.1 so that $\|\operatorname{div}\tilde{b}\|_{\mathcal{H}^1(\mathbb{R}^2)} \le \varepsilon_0$. Applying Theorem 1.1 to (3.2), we obtain the conclusion of the corollary. □

## 4. Proof of Theorem 1.4

### 4.1. Uhlenbeck–Rivière decomposition

Recall that $\Omega = B_1(0)$ and $\operatorname{div} b \in \mathcal{H}^1(\mathbb{R}^2)$. Let $\tau = (-y, x)$, and $\nu = (x, y)$ denote the unit tangential and normal vector-fields on $\partial\Omega$. We use the Hodge decomposition

$$b = \nabla^{\perp}\xi - \nabla p, \text{ where } p = 0 \text{ on } \partial\Omega. \tag{4.11}$$

To do this, let $p \in L^1_{\text{loc}}(\Omega)$ solve

$$-\Delta p = \operatorname{div} b \text{ in } \Omega, \; p = 0 \text{ on } \partial\Omega.$$

Because $\operatorname{div} b \in \mathcal{H}^1(\mathbb{R}^2)$, by Theorem 2.3, we have $p \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$. Furthermore,

$$\|p\|_{L^\infty(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \le C\|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)}. \tag{4.12}$$

With the above $p$, we have $\operatorname{div}(b + \nabla p) = 0$, so we can find $\xi \in W^{1,2}(\Omega)$ such that (4.11) holds. Now, suppose that we have a smallness condition on $b$, precisely, for some small $\varepsilon_1 > 0$ to be determined,

$$\|b\|_{L^2(\Omega)} + \|\operatorname{div} b\|_{\mathcal{H}^1(\mathbb{R}^2)} \le \varepsilon_1. \tag{4.13}$$

From (4.12) and (4.11), we have

$$\|p\|^2_{L^\infty(\Omega)} + \int_\Omega |\nabla\xi|^2 + \int_\Omega |\nabla p|^2 \le C\left(\int_\Omega |b|^2 + \|\operatorname{div} b\|^2_{\mathcal{H}^1(\mathbb{R}^2)}\right). \tag{4.14}$$

Inspired by Rivière [7,8], we now rewrite (4.11) into the Uhlenbeck–Rivière decomposition (also known as the nonlinear Hodge decomposition). Let $P = e^p$. Then (4.11) becomes a nonlinear decomposition

$$b = \nabla^{\perp}\xi - P^{-1}\nabla P. \tag{4.15}$$

We will use (4.15) to convert (1.1) into a *conservation law*. Note that, $\nabla P = e^p\nabla p$, $\nabla P^{-1} = -e^{-p}\nabla p$.

Let $\varepsilon \in (0, \frac{1}{100})$ be a small constant to be chosen in Lemma 4.1 below. With this $\varepsilon$, we choose $\varepsilon_1$ small so that from (4.13) and (4.14), we have

$$\int_\Omega |\nabla\xi|^2 + |\nabla P|^2 + |\nabla P^{-1}|^2 < \varepsilon \tag{4.16}$$

and

$$\|P\|_{L^\infty(\Omega)} \le 1 + \varepsilon. \tag{4.17}$$

It follows that

$$1/10 \le \|P\|_{L^\infty(\Omega)}, \|P^{-1}\|_{L^\infty(\Omega)} \le 10. \tag{4.18}$$

Recall that $P = 1$ on $\partial\Omega$. To prove Theorem 1.4, it remains to prove the following lemma.

**Lemma 4.1.** *Assume that (4.16) and (4.17) hold. If $\varepsilon$ is sufficiently small, then there exist $A \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $B \in W^{1,2}(\Omega)$ such that $A^{-1} \in L^\infty(\Omega)$, $A = 1$ on $\partial\Omega$, $\int_\Omega B = 0$, and*

$$Ab = \nabla A + \nabla^{\perp} B, \tag{4.19}$$

*with*

$$\|AP - 1\|^2_{L^\infty(\Omega)} + \|\nabla(AP)\|^2_{L^2(\Omega)} + \|\nabla B\|^2_{L^2(\Omega)} \le C\varepsilon.$$

*4.2. Proof of Lemma 4.1*

**Proof of Lemma 4.1.** Notice that, by an approximation argument using the standard mollifications, it suffices to prove the lemma for smooth vector fields $b$. In this case, we have on $\partial\Omega$

$$b\cdot\tau = (\nabla^\perp\xi - \nabla p)\cdot\tau = \frac{\partial\xi}{\partial\nu}. \tag{4.20}$$

The function $\xi$ in (4.11) can be chosen to be the smooth solution to

$$-\Delta\xi = \operatorname{curl} b = \partial_2 b_1 - \partial_1 b_2 \text{ in } \Omega,\ \frac{\partial\xi}{\partial\nu} = b\cdot\tau \text{ on } \partial\Omega \text{ and } \int_\Omega \xi = 0.$$

In what follows, we will use Eq. (4.20). We use a fixed point argument as in Rivière [7,8]. Let $P$ be as in Sect. 4.1. With each $A\in W^{1,2}(\Omega)$, we associate $\tilde A = AP$. Suppose that $A$ and $B$ are solutions to (4.19). Then, recalling (4.15), and noting that $\nabla P P^{-1} = -P\nabla P^{-1}$, we have

$$\nabla\tilde A - \tilde A\nabla^\perp\xi + \nabla^\perp B.P = 0 \text{ and } \nabla^\perp B = Ab - \nabla A = A\nabla^\perp\xi + AP\nabla P^{-1} - \nabla A.$$

Taking the divergence of the first equation and taking the curl ($= -\nabla^\perp$) of the second equation yield:

$$\Delta\tilde A = \nabla\tilde A\cdot\nabla^\perp\xi - \nabla^\perp B\cdot\nabla P \text{ and } \Delta B = \nabla^\perp(A\nabla^\perp\xi + AP\nabla P^{-1}) = \operatorname{div}(A\nabla\xi) + \nabla^\perp\tilde A\cdot\nabla P^{-1}.$$

We now proceed as follows.

**Step 1.** We prove, provided $\varepsilon$ is sufficiently small, the existence of a solution $(\tilde A, B)$ to the system

$$\begin{cases} \Delta\tilde A = \nabla\tilde A\cdot\nabla^\perp\xi - \nabla^\perp B\cdot\nabla P & \text{in } \Omega,\\ \Delta B = \operatorname{div}(\tilde A\nabla\xi P^{-1}) + \nabla^\perp\tilde A\cdot\nabla P^{-1} & \text{in } \Omega,\\ \tilde A = 1 \text{ and } \frac{\partial B}{\partial\nu} = b\cdot\tau & \text{on } \partial\Omega,\\ \int_\Omega B = 0, \end{cases} \tag{4.21}$$

with

$$\|\tilde A - 1\|^2_{L^\infty(\Omega)} + \|\nabla\tilde A\|^2_{L^2(\Omega)} + \|\nabla B\|^2_{L^2(\Omega)} \le C\varepsilon.$$

**Step 2.** We show that (4.21) implies

$$\nabla\tilde A - \tilde A\nabla^\perp\xi + \nabla^\perp B{\cdot}P = 0. \tag{4.22}$$

*Let us indicate how Steps 1 and 2 complete the proof of Lemma 4.1.* Assuming (4.22), we find from $\tilde A = AP$ that $P\nabla A + A\nabla P - AP\nabla^\perp\xi + \nabla^\perp B{\cdot}P = 0$. Since $P$ is invertible, we obtain

$$\nabla A + A\nabla P P^{-1} - A\nabla^\perp\xi + \nabla^\perp B = 0.$$

Therefore, recalling (4.15), we obtain (4.19). The last estimate in Lemma 4.1 follows from the last estimate in Step 1 and from the fact that $\tilde A = AP$. The proof of Lemma 4.1 is complete.

**Proof of Step 1.** To prove the existence of a solution $(\tilde A, B)$ to (4.21), we will use a fixed-point argument as in Rivière [7,8]. Let us denote for $g\in H^{1/2}(\partial\Omega)$ the space $W^{1,2}_g(\Omega) = \{u\in W^{1,2}(\Omega), u = g \text{ on } \partial\Omega\}$. Consider the map $f(\hat A,\hat B) = (\tilde A, B)$ from

$$X = \left(W^{1,2}_1(\Omega)\cap L^\infty(\Omega)\right)\times W^{1,2}(\Omega)$$

into itself, where for given $(\hat A,\hat B)\in X$, the pair $(\tilde A, B)$ solves the system

$$\begin{cases} \Delta(\tilde A - 1) = \nabla(\hat A - 1)\cdot\nabla^\perp\xi - \nabla^\perp\hat B\cdot\nabla P & \text{in } \Omega,\\ \Delta(B - B_0) = \operatorname{div}((\hat A - 1)\nabla\xi P^{-1}) + \nabla^\perp(\hat A - 1)\cdot\nabla P^{-1} & \text{in } \Omega,\\ \tilde A = 1 \text{ and } \frac{\partial B}{\partial\nu} = b\cdot\tau & \text{on } \partial\Omega,\\ \int_\Omega B = 0. \end{cases} \tag{4.23}$$

Here, the function $B_0\in W^{1,2}(\Omega)$ is the solution to

$$\begin{cases} \Delta B_0 = \operatorname{div}(\nabla\xi P^{-1}) & \text{in } \Omega, \\ \frac{\partial B_0}{\partial\nu} = b\cdot\tau & \text{on } \partial\Omega, \\ \int_\Omega B_0 = 0. \end{cases} \tag{4.24}$$

Clearly, a fixed point of (4.23) is a solution to (4.21).

By (4.20), Eq. (4.24) has a unique solution. Multiplying both sides of the first equation of (4.24) by $B_0$, integrating by parts, we find from $P=1$ on $\partial\Omega$ and (4.20) that $\int_\Omega |\nabla B_0|^2 = \int_\Omega \nabla\xi P^{-1}\cdot\nabla B_0$, from which we can estimate the gradient of $B_0$ by

$$\int_\Omega |\nabla B_0|^2 \le \int_\Omega |P^{-1}\nabla\xi|^2 \le \|P^{-1}\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla\xi|^2. \tag{4.25}$$

Applying Lemma 2.4 to $\tilde{A}-1$ and recalling the first and third equations in (4.23), we find

$$\|\tilde{A}-1\|^2_{L^\infty(\Omega)} + \|\nabla\tilde{A}\|^2_{L^2(\Omega)} \le C\int_\Omega |\nabla\hat{A}|^2 \int_\Omega |\nabla\xi|^2 + C\int_\Omega |\nabla\hat{B}|^2 \int_\Omega |\nabla P|^2. \tag{4.26}$$

Note that, by the second and last equations in (4.23), $B-B_0\in W^{1,2}(\Omega)$ satisfies

$$\begin{cases} \Delta(B-B_0) = \operatorname{div}\left((\hat{A}-1)\nabla\xi P^{-1}\right) + \nabla^\perp\hat{A}\cdot\nabla P^{-1} & \text{in } \Omega, \\ \frac{\partial(B-B_0)}{\partial\nu} = 0 & \text{on } \partial\Omega, \\ \int_\Omega (B-B_0) = 0. \end{cases} \tag{4.27}$$

At this point, we can use Theorem 2.4 and argue as in (4.25) to obtain the estimate

$$\|\nabla(B-B_0)\|^2_{L^2(\Omega)} \le C\|P^{-1}\|^2_{L^\infty(\Omega)}\|\hat{A}-1\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla\xi|^2 + C\int_\Omega |\nabla\hat{A}|^2 \int_\Omega |\nabla P^{-1}|^2. \tag{4.28}$$

This, combined with (4.25), gives

$$\|\nabla B\|^2_{L^2(\Omega)} \le C\|P^{-1}\|^2_{L^\infty(\Omega)}\|\hat{A}-1\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla\xi|^2 + C\int_\Omega |\nabla\hat{A}|^2 \int_\Omega |\nabla P^{-1}|^2 + C\|P^{-1}\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla\xi|^2. \tag{4.29}$$

The above arguments show that for $(\hat{A},\hat{B}),(\hat{A}_1,\hat{B}_1)\in X$, the pairs $(\tilde{A},B)=f(\hat{A},\hat{B}),(\tilde{A}_1,B_1)=f(\hat{A}_1,\hat{B}_1)$ satisfy the estimates

$$\|\tilde{A}-\tilde{A}_1\|^2_{L^\infty(\Omega)} + \|\nabla(\tilde{A}-\tilde{A}_1)\|^2_{L^2(\Omega)} \le C\int_\Omega |\nabla(\hat{A}-\hat{A}_1)|^2 \int_\Omega |\nabla\xi|^2 + C\int_\Omega |\nabla(\hat{B}-\hat{B}_1)|^2 \int_\Omega |\nabla P|^2,$$

and

$$\|\nabla(B-B_1)\|^2_{L^2(\Omega)} \le C\|P^{-1}\|^2_{L^\infty(\Omega)}\|(\hat{A}-\hat{A}_1)\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla\xi|^2 + C\int_\Omega |\nabla(\hat{A}-\hat{A}_1)|^2 \int_\Omega |\nabla P^{-1}|^2.$$

Since $\int_\Omega (B-B_1)=0$, we have by the Poincaré inequality $\int_\Omega |B-B_1|^2 \le C\int_\Omega |\nabla(B-B_1)|^2$. Hence, if $\int_\Omega |\nabla\xi|^2 + |\nabla P|^2 + |\nabla P^{-1}|^2 \le \varepsilon$ is sufficiently small, then a standard fixed-point argument in the space $X = \left(L^\infty(\Omega)\cap W^{1,2}_1(\Omega)\right)\times W^{1,2}(\Omega)$ yields the existence of a solution $(\tilde{A},B)$ to the system (4.21).

Furthermore, from (4.26), (4.29) together with (4.16) and (4.18), the solution $(\tilde{A},B)$ to the system (4.21) satisfies

$$\begin{aligned} \|\tilde{A}-1\|^2_{L^\infty(\Omega)} + \|\nabla\tilde{A}\|^2_{L^2(\Omega)} &\le C\varepsilon\|\nabla\tilde{A}\|^2_{L^2(\Omega)} + C\varepsilon|\nabla B|^2_{L^2(\Omega)} \\ &\le C\varepsilon\|\nabla\tilde{A}\|^2_{L^2(\Omega)} + C\varepsilon^2(1+\|\tilde{A}-1\|^2_{L^\infty(\Omega)} + \|\nabla\tilde{A}\|^2_{L^2(\Omega)}). \end{aligned}$$

Thus, if $C\varepsilon\le 1/4$, we have

$$\|\tilde{A}-1\|^2_{L^\infty(\Omega)} + \|\nabla\tilde{A}\|^2_{L^2(\Omega)} \le C\varepsilon.$$

By (4.29), we then have $\|\nabla B\|^2_{L^2(\Omega)} \le C\varepsilon$. The proof of Step 1 is complete.

**Proof of Step 2.** To show (4.22), we introduce the Hodge decomposition

$$\nabla\tilde{A} - \tilde{A}\nabla^\perp\xi + \nabla^\perp B.P = \nabla E + \nabla^\perp D \tag{4.30}$$

where $E = 0$ on $\partial\Omega$. Taking divergence of both sides of (4.30), and recalling the first equation of (4.21), we get $\Delta E = 0$ in $\Omega$. Hence, $E \equiv 0$ and (4.30) reduces to

$$\nabla\tilde{A} - \tilde{A}\nabla^{\perp}\xi + \nabla^{\perp}B.P = \nabla^{\perp}D. \tag{4.31}$$

It remains to show that $D$ is a constant. By (4.31), and recalling (4.21), we have

$$\nabla D = -\nabla^{\perp}\tilde{A} - \tilde{A}\nabla\xi + P\nabla B \text{ and } \Delta D = P(\tilde{A}\nabla\xi\cdot\nabla P^{-1} + \nabla^{\perp}\tilde{A}\cdot\nabla P^{-1} + \nabla B\cdot\nabla P P^{-1}).$$

It follows from a simple calculation that

$$\operatorname{div}(\nabla D P^{-1}) = (-\nabla^{\perp}\tilde{A} - \tilde{A}\nabla\xi + P\nabla B)\cdot\nabla P^{-1} + \Delta D P^{-1} = \nabla B\cdot\nabla(P P^{-1}) = 0. \tag{4.32}$$

With (4.32), we complete the proof of Step 2 as follows. Taking dot product with $\tau$ on both sides of (4.31), and recalling that $P = \tilde{A} = 1$ on $\partial\Omega$, we find that on $\partial\Omega = \partial B_1(0)$:

$$D_{\nu} = \nabla^{\perp}D\cdot\tau = \tilde{A}_{\tau} - \tilde{A}\xi_{\nu} + B_{\nu}P = B_{\nu} - \xi_{\nu} = b\cdot\tau - b\cdot\tau = 0.$$

Multiplying both sides of (4.32) by $D$ and integrating by parts, we find

$$0 = \int_{\Omega}\operatorname{div}(\nabla D\cdot P^{-1})D = -\int_{\Omega}P^{-1}|\nabla D|^2 + \int_{\partial\Omega}P^{-1}D\nabla D\cdot\nu = -\int_{\Omega}P^{-1}|\nabla D|^2.$$

Recalling (4.18), we obtain $\nabla D = 0$ in $\Omega$ and hence $D$ is a constant in $\Omega$. □

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