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Perverse sheaves and knot contact homology

Faisceaux pervers et homologie de contact des nœuds

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ABSTRACT

In this paper, we give a new algebraic construction of knot contact homology in the sense of Ng [35]. For a link L in $\mathbb{R}^3$, we define a differential graded (DG) k-category $\tilde{\mathscr{A}}_L$ with finitely many objects, whose quasi-equivalence class is a topological invariant of L. In the case when L is a knot, the endomorphism algebra of a distinguished object of $\tilde{\mathscr{A}}_L$ coincides with the fully noncommutative knot DGA as defined by Ekholm, Etnyre, Ng, and Sullivan in [13]. The input of our construction is a natural action of the braid group B_n on the category of perverse sheaves on a two-dimensional disk with singularities at n marked points, studied by Gelfand, MacPherson, and Vilonen in [19]. As an application, we show that the category of finite-dimensional representations of the link k-category $\tilde{A}_L = H_0(\tilde{\mathscr{A}}_L)$ defined as the 0-th homology of $\tilde{\mathscr{A}}_L$ is equivalent to the category of perverse sheaves on $\mathbb{R}^3$ that are singular along the link L. We also obtain several generalizations of the category $\tilde{\mathscr{A}}_L$ by extending the Gelfand–MacPherson–Vilonen braid group action. Detailed proofs of results announced in this paper will appear in [4].

RÉSUMÉ

Dans cette Note, nous donnons une nouvelle construction algébrique de l'homologie de contact des nœuds, au sens de Ng [37]. Pour un entrelacs L dans $\mathbf{R}^3$, nous définissons une k-catégorie différentielle graduée $\tilde{\mathscr{A}}_L$ ayant un nombre fini d'objets, dont la classe de quasi-équivalence est un invariant topologique de L. Lorsque L est un nœud, l'algèbre des endomorphismes d'un objet distingué de $\tilde{\mathscr{A}}_L$ coïncide avec l'algèbre différentielle graduée, pleinement non commutative du nœud, définie par Ekholm, Etnyre, Ng et Sullivan dans [12]. Notre construction se base sur une action naturelle du groupe de tresses B_n sur la catégorie des faisceaux pervers sur un disque de dimension deux avec singularités en n points marqués, étudiée par Gelfand, McPherson et Vilonen dans [19]. Comme application, nous montrons que la catégorie des représentations de dimension finie de la k-catégorie d'entrelacs $\tilde{A}_L = H_0(\tilde{\mathscr{A}}_L)$, définie comme l'homologie de degré 0 de $\tilde{\mathscr{A}}_L$, est équivalente à la catégorie des faisceaux pervers sur $\mathbb{R}^3$ qui sont singuliers le long de l'entrelacs L. Nous

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obtenons également plusieurs généralisations de la catégorie $\tilde{\mathscr{A}_L}$ en étendant l'action du groupe de tresses de Gelfand–MacPherson–Vilonen.

1. Introduction

In a series of papers [35–39], L. Ng introduced and studied a new algebraic invariant of a link L in $\mathbb{R}^3$ represented by a semi-free differential graded (DG) algebra $\mathcal{A}_L$. The structure of this DG algebra (termed a combinatorial knot DGA) is determined by an element of a braid group B_n representing the link L. The homology of $\mathcal{A}_L$ is called the *knot contact homology* $HC_*(L)$, as it coincides with the Legendrian contact homology[1] of the unit conormal bundle $\Lambda_L \subseteq ST^*\mathbb{R}^3$ of L. This coincidence was conjectured in [35,36] and proved later in [13,14], where it was shown, in fact, that the entire combinatorial knot DGA is isomorphic to a geometrically defined DG algebra computing the Legendrian contact homology of Λ_L.

Our original motivation was to understand Ng's combinatorial proof of the invariance of $\mathcal{A}_L$ (up to quasi-isomorphism) under the Markov moves. We should remark that, although the differential of $\mathcal{A}_L$ is defined in [35] by an explicit formula, its combinatorial structure is fairly complicated and its algebraic origin seems mysterious. Even the fact that the 0-th homology of $\mathcal{A}_L$ is a link invariant is far from being obvious from the definition of [35] (*cf.* [35, Section 4.3]). As a result, we have come up with a different, more conceptual construction that makes the Markov invariance of $\mathcal{A}_L$ quite transparent[2] and, more importantly, places knot contact homology in one row with other classical invariants, such as knot groups and Alexander modules.

To clarify the ideas, we begin by recalling a classical theorem of E. Artin and J. Birman that gives a natural presentation of the link group $\pi_1(\mathbb{R}^3 \setminus L)$ in terms of a braid representing L. Let D be the unit disk in $\mathbb{R}^2$, and let $\{p_1, \dots, p_n\} \subset D$ be a set of distinct points in the interior of D. It is well known that the braid group on n-strands, B_n, can be identified with the mapping class group of $(D \setminus \{p_1, \dots, p_n\}, \partial D)$, and as such it acts naturally on the fundamental group $\pi_1(D \setminus \{p_1, \dots, p_n\}, p_0)$, where $p_0 \in D \setminus \{p_1, \dots, p_n\}$ is a basepoint (which we choose near the boundary of D). The fundamental group $\pi_1(D \setminus \{p_1, \dots, p_n\}, p_0)$ is a free group $\mathbb{F}_n$ of rank n based on generators $x_1, \dots, x_n$ that correspond to small loops in $D \setminus \{p_1, \dots, p_n\}$ around the points p_i. Explicitly, in terms of these generators, the action of B_n on $\pi_1(D \setminus \{p_1, \dots, p_n\}, p_0) \cong \mathbb{F}_n$ is given by

$$\sigma_i : \begin{cases} x_i & \mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} & \mapsto x_i \\ x_j & \mapsto x_j \quad (j \neq i, i+1) \end{cases} \tag{1.1}$$

where σ_i $(i = 1, 2, \dots, n-1)$ are the standard generators of B_n. This action is usually called the *Artin representation*, as it provides a faithful realization of B_n as a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$. Now, the Artin–Birman Theorem (see [6, Theorem 2.2]) asserts that the fundamental group of the complement of the link $L = \hat{\beta} \subset \mathbb{R}^3$ corresponding to a braid $\beta \in B_n$ has the presentation

$$\pi_1(\mathbb{R}^3 \setminus L) \cong \langle x_1, x_2, \dots, x_n \mid \beta(x_1) = x_1, \beta(x_2) = x_2, \dots, \beta(x_n) = x_n \rangle , \tag{1.2}$$

where $\beta(x_i)$ denotes the action of β on x_i via the Artin representation.

We abstract this situation in the following way. Let $\mathcal{C}$ be a category with finite colimits. We assume that we are given a family of braid group actions $\varrho_n : B_n \to \mathrm{Aut}\, A^{(n)}$, $n \geq 1$, on objects of $\mathcal{C}$ having the properties:

(1) for each $n \geq 1$, $A^{(n)}$ is the n-fold coproduct of one and the same object A of $\mathcal{C}$;
(2) the actions ϱ_n are *local* and *homogeneous* in the sense that each $\sigma_i \in B_n$ acts only on the $(i, i+1)$-copy of $A^{(2)}$ in $A^{(n)}$ while keeping the rest fixed, and any two standard generators of B_n act in the same way on the corresponding copies of $A^{(2)}$ for all $n \geq 1$.

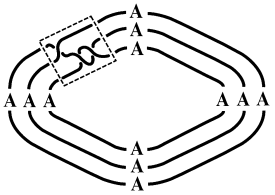
Such braid group actions are determined (generated) by a single morphism $\sigma : A \amalg A \to A \amalg A$ in the category $\mathcal{C}$ that we call a *co-Cartesian Yang–Baxter operator* (*cf.* Definition 2.1). For example, the Artin representations are generated by a co-Cartesian Yang–Baxter operator in the category of groups given by $\sigma : \mathbb{F}_1 \amalg \mathbb{F}_1 \to \mathbb{F}_1 \amalg \mathbb{F}_1$, $x_1 \mapsto x_1 x_2 x_1^{-1}$, $x_2 \mapsto x_1$.

[1] In the sense of [16] (see also [11,12]).
[2] In fact, the invariance of our construction under type I Markov moves follows directly from its definition.

Now, for an arbitrary co-Cartesian Yang–Baxter operator (A,σ), we define a universal construction $\mathcal{L}(A,\sigma)$ that associates with each braid $\beta \in B_n$ the coequalizer of the endomorphisms id and β of the object $A^{(n)}$, or equivalently, the following pushout in $\mathcal{C}$:

$$\mathcal{L}(A,\sigma)[\beta] := \operatorname{coeq}\Big[A^{(n)} \underset{\mathrm{id}}{\overset{\beta}{\rightrightarrows}} A^{(n)}\Big] = \operatorname{colim}\big[\, A^{(n)} \xleftarrow{(\beta,\,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\,\mathrm{id})} A^{(n)} \,\big]. \tag{1.3}$$

We call (1.3) the *categorical closure of the braid β on the object A* with respect to the Yang–Baxter operator σ. This terminology can be justified by the following "picture"[3] of the pushout (1.3) that manifestly exhibits it as "a braid closure on A":

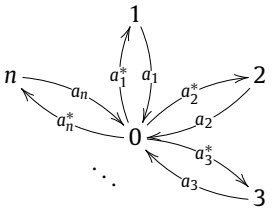


$$\operatorname{colim}\big[\, A^{(3)} \xleftarrow{(\beta,\,\mathrm{id})} A^{(3)} \amalg A^{(3)} \xrightarrow{(\mathrm{id},\,\mathrm{id})} A^{(3)} \,\big].$$

In the case of Artin representations, the Artin–Birman Theorem (1.2) implies that $\mathcal{L}(\mathbb{F}_1,\sigma)[\beta] \cong \pi_1(\mathbb{R}^3 \setminus L)$. This means, in particular, that $\mathcal{L}(\mathbb{F}_1,\sigma)[\beta]$ is a link invariant.

In general, we show that, if a co-Cartesian Yang–Baxter operator (A,σ) satisfies some natural conditions, which we call the *Reidemeister conditions* (see Definition 2.15), then the isomorphism class of the categorical closure of any braid with respect to (A,σ) is stable under the Markov moves, and hence defines a link invariant (*cf.* Theorem 2.10 and Theorem 2.17). Apart from the group $\pi_1(\mathbb{R}^3 \setminus L)$, many classical link invariants arise in this way (see, for example, Theorem 2.8 that represents as a categorical braid closure the Alexander module).

Next, we consider the category $\mathsf{Perv}(D,\{p_1,\dots,p_n\})$ of perverse sheaves on the disk D with only possible singularities at the points $\{p_1,\dots,p_n\}$. In [19], Gelfand, MacPherson and Vilonen showed that $\mathsf{Perv}(D,\{p_1,\dots,p_n\})$ is equivalent to the category $\tilde{\mathscr{Q}}^{(n)}$ of finite-dimensional k-linear representations of the following quiver

$Q^{(n)} =$ 

such that the operators $T_i := e_0 + a_i a_i^*$ act as isomorphisms for all $i = 1, 2, \dots, n$. More formally, $\tilde{\mathscr{Q}}^{(n)}$ can be described as the category $\operatorname{Mod}\tilde{A}^{(n)}$ of finite-dimensional modules over the k-category

$$\tilde{A}^{(n)} := k\langle Q^{(n)}\rangle[T_1^{-1}, \dots, T_n^{-1}] \tag{1.4}$$

which is obtained by localizing the path category of $Q^{(n)}$ at the set of morphisms $\{T_1,\dots,T_n\}$. Now, the braid group B_n acts on the disk D with n marked points $\{p_1,\dots,p_n\}$ as a mapping class group, and this naturally induces an action on the category $\mathsf{Perv}(D,\{p_1,\dots,p_n\})$. It was shown in [19] that, under the equivalence $\mathsf{Perv}(D,\{p_1,\dots,p_n\}) \simeq \tilde{\mathscr{Q}}^{(n)}$, the action of B_n on the category of perverse sheaves corresponds to a *strict* action on the category $\tilde{\mathscr{Q}}_n$ (*cf.* [19, Proposition 1.3]). This, in turn, induces an action of B_n on the k-category $\tilde{A}^{(n)}$, which is given explicitly (on generating morphisms of $\tilde{A}^{(n)}$) by the following formulas

$$\sigma_i : \begin{cases} a_i \mapsto T_i\, a_{i+1} & \\ a_{i+1} \mapsto a_i & \\ a_j \mapsto a_j & (j \neq i, i+1) \\ a_i^* \mapsto a_{i+1}^*\, T_i^{-1} & \\ a_{i+1}^* \mapsto a_i^* & \\ a_j^* \mapsto a_j^* & (j \neq i, i+1). \end{cases} \tag{1.5}$$

We call (1.5) the *Gelfand–MacPherson–Vilonen (GMV) braid action*.

[3] In fact, this "picture" of a categorical braid closure can be formalized by using the diagrammatic tensor calculus developed by A. Joyal, R. Street and others (see [26,27,41]). We briefly discuss it at the end of Section 2.

The GMV braid actions are generated by a single co-Cartesian Yang–Baxter operator in the category of (small) *pointed* k-categories Cat_k^*. Specifically, for each $n \geq 1$, the k-category $\tilde{A}^{(n)}$ is the coproduct (fusion product) in Cat_k^* of n copies of the k-category $\tilde{A} = k\langle Q\rangle[T^{-1}]$, where $k\langle Q\rangle$ is the path category of the quiver $Q = \big[1 \underset{a^*}{\overset{a}{\rightleftarrows}} 0\big]$ with the distinguished object 0. The corresponding Yang–Baxter operator $\sigma : \tilde{A} \amalg \tilde{A} \to \tilde{A} \amalg \tilde{A}$ is given by

$$(a_1,\, a_1^*) \mapsto (T_1\, a_2,\, a_2^*\, T_1^{-1})\,, \quad (a_2,\, a_2^*) \mapsto (a_1,\, a_1^*)\,. \tag{1.6}$$

Just as in the case of Artin actions, it is easy to check that (1.6) satisfies the Reidemeister conditions, and hence the categorical braid closure with respect to $(\tilde{A}, \sigma)$ is a link invariant. For a given $\beta \in B_n$, this invariant is represented by the equivalence class of the k-category $\tilde{A}_L := \mathcal{L}(\tilde{A}, \sigma)[\beta]$, which we call the *(fully noncommutative) link k-category*[4] of $L = \hat{\beta}$. In Section 7, we will show that the k-category $\tilde{A}_L$ is a natural extension of the fully noncommutative cord algebra of [13,39] in the sense that the latter can be identified with the endomorphism algebra of an object in $\tilde{A}_L$. Thus, in our algebraic formalism, the link category $\tilde{A}_L$ arises exactly the same way as the link group $\pi_1(\mathbb{R}^3 \setminus L)$, provided we take as an input the Gelfand–MacPherson–Vilonen braid action instead of the Artin representation.

At this point, we pause to remark that the notion of a categorical braid closure has already appeared in the literature: explicitly – in the case of groups (see [52,8]), and in a somewhat disguised form, in the theory of quandles (see, for example, [18,7]). From this last perspective, our results give a precise interpretation of such geometric knot invariants as a cord algebra in combinatorial terms of racks and quandles (see Remark 2.18 below).

However, our main observation is that the simple categorical formalism we outlined above admits an interesting generalization to homotopical contexts. Specifically, if the category $\mathcal{C}$ that we work with has a natural class $\mathcal{W}$ of weak equivalences (e.g., $\mathcal{C}$ is a Quillen model category or a homotopical category in the sense of [10]), then the operation of a categorical braid closure is usually not invariant under weak equivalences, *i.e.* it is not well defined[5] in the homotopy category $\mathrm{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$. In abstract homotopy theory, there is a standard way to remedy this problem: namely, replace a homotopy non-invariant functor F by its derived functor, which gives a universal approximation to F on the level of homotopy categories (see, e.g., [9, Section 9]). In our situation, we can define a "derived" version of the categorical braid closure by simply replacing the 'colim' in the definition (1.3) by its derived functor: the homotopy colimit 'hocolim'. To be precise, given a co-Cartesian Yang–Baxter operator (A, σ) in (say) a model category $\mathcal{C}$, we define the *homotopy braid closure* of $\beta \in B_n$ with respect to (A, σ) by

$$h\mathcal{L}(A, \sigma)[\beta] := \mathrm{hocolim}\,[\, A^{(n)} \xleftarrow{(\beta,\,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\,\mathrm{id})} A^{(n)}\,]\,. \tag{1.7}$$

One of our main results (Theorem 3.4) states that if (A, σ) satisfies the Reidemeister conditions (and A is flat in an appropriate sense), then the weak equivalence class of the homotopy braid closure on A, *i.e.* the isomorphism class of (1.7) in the homotopy category $\mathrm{Ho}(\mathcal{C})$, is invariant under the Markov moves, and hence defines a link invariant. This last invariant is more refined than the one given by the usual categorical braid closure in the same way as the homotopy type of a topological space is a more refined invariant of the space than just its fundamental group.

Now, let us return to our basic example of the co-Cartesian Yang–Baxter operator $(\tilde{A}, \sigma)$ associated with the GMV action, see (1.6). To define the homotopy braid closure with respect to this operator, we will regard the k-category $\tilde{A}$ as an object of the category dgCat_k^* comprising all (small) pointed DG categories. The category dgCat_k^* has a natural model structure, in which the weak equivalences are the quasi-equivalences[6] of DG categories (see [48]). Its homotopy theory has been extensively studied in recent years with a view towards applications in algebraic geometry and representation theory (see, e.g., [31,50] and references therein).

The homotopy braid closure of the GMV action in the model category dgCat_k^* gives a new link invariant, which is a quasi-equivalence class of DG categories. For a given $\beta \in B_n$, formula (1.7) allows us, in fact, to construct an explicit representative for the corresponding quasi-equivalence class that we call the *fully noncommutative link DG category* $\mathscr{A}_L$ (see Definition 7.1). If we assume, for simplicity, that L is a knot (*i.e.*, a link with a single component), then $\mathscr{A}_L$ contains a distinguished object, and the endomorphism DG algebra of that object is isomorphic to the fully noncommutative knot DGA constructed in [13]. This observation is part of Theorem 7.4 that we state in full generality (for links with an arbitrary number of components) but do not prove in this paper. Instead, we sketch a proof of an analogous result – Theorem 5.6 – that identifies the *framed* knot DGA (originally introduced in [37]) with the DG endomorphism algebra of a distinguished object in the homotopy braid closure of a modified GMV action. The modification amounts to collapsing all objects of the GMV k-category $\tilde{A}^{(n)}$, except for the base object '0', to a single object '1', while preserving all the generating morphisms

[4] Strictly speaking, the categorical braid closure gives a specialization of the fully noncommutative link k-category, with all longitude parameters set to be 1 (see Remark 7.9). For a general definition of $\tilde{A}_L$, we refer to Section 7, Definition 7.1.

[5] The problem is that pointwise weak equivalences of diagrams do not necessarily induce weak equivalences of colimits, so the objects defined by colimits of diagrams defined up to homotopy are not well defined, even up to homotopy type.

[6] Recall that a *quasi-equivalence* of DG categories is a DG functor $F : \mathscr{A} \to \mathscr{B}$ such that $F : \mathscr{A}(X, Y) \to \mathscr{B}(FX, FY)$ is a quasi-isomorphism of k-complexes for all objects $X, Y \in \mathrm{Ob}(\mathscr{A})$ and the induced functor on the 0th homology $H_0(F) : H_0(\mathscr{A}) \xrightarrow{\sim} H_0(\mathscr{B})$ is an equivalence of categories.

a_i and a_i^*. We also impose n extra relations $a_i^* a_i = (\mu - 1)\, e_1$, one for each $i = 1, 2, \ldots, n$, which depend on an invertible central parameter μ in the ground ring k. The resulting k-category $A^{(n)}$ with two objects $\{0, 1\}$ inherits the GMV braid action (1.5), and one can still define its homotopy braid closure by formula (1.7).[7]

Now, as in the case of topological spaces, to compute the homotopy colimit of a diagram like (1.7), one should first 'resolve' the objects by their cofibrant models $R \overset{\sim}{\twoheadrightarrow} A^{(n)}$, then replace one of the arrows by a (weakly equivalent) cylinder cofibration, and then take the usual colimit in the underlying category $\mathcal{C}$:

$$\operatorname{hocolim}[\, A^{(n)} \leftarrow A^{(n)} \amalg A^{(n)} \to A^{(n)}] \cong \operatorname{colim}[\, R \leftarrow R \amalg R \hookrightarrow \operatorname{Cyl}(R)\,]\,. \tag{1.8}$$

In the category of DG categories with finitely many objects, there is a canonical cylinder object $\operatorname{Cyl}_{\mathrm{BL}}(R)$ defined for any semi-free DG category R. We call this object the *Baues–Lemaire cylinder* as it was originally constructed (in the case of chain DG algebras) in [3]. The differentials of $\operatorname{Cyl}_{\mathrm{BL}}(R)$ are defined by explicit formulas in terms of differentials of R, while for a semi-free resolution $R \overset{\sim}{\twoheadrightarrow} A^{(n)}$, the differentials of R are determined explicitly by the relations of $A^{(n)}$. Thus, taking the colimit (1.8) with the help of the Baues–Lemaire cylinder $\operatorname{Cyl}_{\mathrm{BL}}(R)$, we find an explicit presentation for the knot DG category $\mathcal{A}$, given in Definition 5.3. An elementary calculation then shows that the DG algebra $\mathcal{A}(1,1)$ consisting of all endomorphisms of the object '1' in the DG k-category $\mathcal{A}$ is precisely the knot DGA defined in [37]. This explains the 'mysterious' algebraic formula for the differentials in Ng's combinatorial knot DGA: it arises from the Baues–Lemaire cylinder on the natural DG resolution of the k-category $A^{(n)}$.

In [36,37], Ng has also given an explicit description of the 0th homology of his knot DGA in terms of the knot group $\pi_1(\mathbb{R}^3 \setminus K)$ and the peripheral pair (m, l) of a meridian and longitude in $\pi_1(\mathbb{R}^3 \setminus K)$. We extend this description to the 0th homology of the knot DG category, both in the framed and fully noncommutative cases (see Theorem 6.2 and Theorem 7.7). Our proof is purely algebraic, in contrast to a topological proof given in [36,37].

Finally, we mention one interesting application of our results that brings us back to topology. Given a link $L \subset \mathbb{R}^3$, we consider the category $\mathsf{Perv}(\mathbb{R}^3, L)$ of perverse sheaves on $\mathbb{R}^3$ constructible with respect to the stratification $L \hookrightarrow \mathbb{R}^3 \hookleftarrow \mathbb{R}^3 \setminus L$ with perversity given by $p(1) = 0$ and $p(3) = -1$. Our Theorem 7.10 states that $\mathsf{Perv}(\mathbb{R}^3, L)$ is equivalent to the category of finite-dimensional left modules over the fully noncommutative link k-category $\tilde{A}_L$. This leads to an algebraic description of the category $\mathsf{Perv}(\mathbb{R}^3, L)$ in terms of groups and quivers, similar, in spirit, to the Gelfand–MacPherson–Vilonen description of the category $\mathsf{Perv}(D, \{p_1, \ldots, p_n\})$.

The paper is organized as follows. In Section 2, we define co-Cartesian Yang–Baxter operators and the associated categorical braid closure, and give two criteria – the Wada condition (Definition 2.9) and the Reidemeister condition (Definition 2.15) – for the categorical braid closure to be a link invariant (Theorem 2.10 and Theorem 2.17). In Section 3, we extend the construction of a categorical braid closure to the homotopical setting. The main result in this section is Theorem 3.4. In Section 4, we introduce our main example of the co-Cartesian Yang–Baxter operator associated with the GMV braid action. In Section 5, we calculate the homotopy braid closure with respect to the GMV operator, and show that the resulting DG category is an extension of Ng's knot DGA (see Theorem 5.5 and Theorem 5.6). The main tool in this calculation is the Baues–Lemaire cylinder on a semi-free DG category; for the reader's convenience, we review its construction is some detail. In Section 6, we compute the 0th homology of the knot DG category, called the *knot k-category*, and give a description of this category in terms of the knot group together with a peripheral pair (see Theorem 6.2). In Section 7, we define the fully noncommutative link DG category and extend the main results of Sections 5 and 6 to this case (see Theorem 7.4 and Theorem 7.7). While the input for the knot DG category introduced in Sections 5 and 6 is the modified GMV action, the input for the fully noncommutative case is the original GMV action. This allows us to relate the corresponding module category to perverse sheaves (see Theorem 7.10). Finally, in Section 8, we give two natural generalizations of the GMV operator, inspired by the work of Wada [52] and Crisp–Paris [8] in the group case. These generalizations satisfy the Reidemeister conditions, and hence the corresponding homotopy braid closures give link invariants generalizing the link DG category associated with the original GMV action. We will discuss these new link invariants elsewhere.

2. Yang–Baxter operators and categorical braid closure

Let $\mathcal{C}$ be a category closed under finite colimits. Let $A \in \mathcal{C}$ be an object of $\mathcal{C}$. For an integer $n \geq 2$, we denote the n-fold coproduct of copies of A in $\mathcal{C}$ by $A^{(n)} := A \amalg \overset{n}{\ldots} \amalg A$. If $f : A \to B$ is a morphism in $\mathcal{C}$, we denote its n-fold coproduct by $f^{(n)} : A^{(n)} \to B^{(n)}$. Now, suppose that we are given an object A and a morphism $\sigma : A \amalg A \to A \amalg A$ in $\mathcal{C}$. Then, for each $n \geq 2$ and $i = 1, 2 \ldots, n-1$, σ induces a morphism $\sigma_{i,i+1} : A^{(n)} \to A^{(n)}$ defined by

$$\sigma_{i,i+1} := \mathrm{id}^{(i-1)} \amalg \sigma \amalg \mathrm{id}^{(n-i-1)} : A^{(n)} \to A^{(n)}.$$

Definition 2.1. A *co-Cartesian Yang–Baxter operator* on A is an invertible morphism

$$\sigma : A \amalg A \to A \amalg A$$

[7] To introduce the second central parameter $\lambda \in k^\times$ we also modify the arrow (β, id) in the homotopy colimit (1.7) by appropriately twisting the action map $\beta : A^{(n)} \to A^{(n)}$ (see Section 4).

satisfying the equation

$$\sigma_{23}\,\sigma_{12}\,\sigma_{23} = \sigma_{12}\,\sigma_{23}\,\sigma_{12} \qquad \text{in} \quad \mathrm{Hom}_{\mathcal{C}}(A^{(3)},\, A^{(3)})\,. \tag{2.2}$$

We will often use the term "Yang–Baxter" as an adjective for an invertible morphism σ satisfying (2.2).

Any co-Cartesian Yang–Baxter operator σ on A extends in a natural way to a *left* action of the Artin braid group B_n on $A^{(n)}$ for each $n \geq 2$. We refer to this action as the action generated by σ.

We give two basic examples of co-Cartesian Yang–Baxter operators corresponding to two classical representations of the braid group B_n.

Example 2.3. Let $\mathcal{C} = \mathsf{Gr}$ be the category of groups, and let $A = \mathbb{F}_1 \in \mathcal{C}$ be the free group on one generator. Consider the map $\sigma : A \amalg A \to A \amalg A$ given by

$$\sigma : \mathbb{F}_2 \to \mathbb{F}_2 \qquad x_1 \mapsto x_1 x_2 x_1^{-1}, \qquad x_2 \mapsto x_1.$$

As mentioned in the Introduction, this is a Yang–Baxter map generating the Artin representations.[8]

Example 2.4. Let $\mathcal{C} = \mathrm{Mod}(R)$ be the category of modules over the commutative ring $R = \mathbb{Z}[t, t^{-1}]$. Take $A = R$ to be the free R-module of rank one, and define the map $\sigma : R^{\oplus 2} \to R^{\oplus 2}$ by left multiplication by the matrix $\begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}$. This map is a co-Cartesian Yang–Baxter operator in the category $\mathrm{Mod}(R)$ generating the classical (unreduced) Burau representations.

Now, given a co-Cartesian Yang–Baxter operator $\sigma : A \amalg A \to A \amalg A$, we denote the resulting braid group action on the n-fold coproduct by

$$\phi_n^{(A,\sigma)} : B_n \to \mathrm{Aut}\, A^{(n)}.$$

Abusing notation, for a braid $\beta \in B_n$, we will often write the automorphism $\phi_n^{(A,\sigma)}(\beta)$ simply as β if the underlying Yang–Baxter operator is understood to be (A, σ).

Definition 2.5. The *categorical braid closure* of a braid $\beta \in B_n$ with respect to a co-Cartesian Yang–Baxter operator $\sigma : A \amalg A \to A \amalg A$ is defined to be the coequalizer

$$\mathcal{L}(A,\sigma)[\beta] := \mathrm{coeq}\Big[\, A^{(n)} \underset{\mathrm{id}}{\overset{\beta}{\rightrightarrows}} A^{(n)} \,\Big]\,,$$

or equivalently, the following pushout in $\mathcal{C}$:

$$\mathcal{L}(A,\sigma)[\beta] = \mathrm{colim}\,[\, A^{(n)} \xleftarrow{(\beta,\,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\,\mathrm{id})} A^{(n)} \,]\,.$$

Recall that the coequalizer of two morphisms $f : X \to Y$ and $g : X \to Y$ in $\mathcal{C}$ is an object $E \in \mathcal{C}$ given together with a morphism $p : Y \to E$ such that the pair (E, p) is universal among all pairs satisfying $pf = pg$. In practice, computing the coequalizer amounts to taking a quotient of the object Y by the relations $f(x) = g(x)$ for all $x \in X$. Thus, in Example 2.3, the categorical closure of $\beta \in B_n$ is the group presented by

$$\mathcal{L}(A,\sigma)[\beta] = \langle\, x_1,\, \ldots,\, x_n \mid \beta(x_1) = x_1,\, \ldots,\, \beta(x_n) = x_n \,\rangle\,.$$

The next theorem is a classical result first stated by E. Artin in [2] and proved by J. Birman in [6].

Theorem 2.6 *(Artin–Birman). The categorical closure of a braid $\beta \in B_n$ with respect to the Artin representation is the fundamental group of the link complement $\mathbb{R}^3 \setminus L$, where $L = \hat{\beta}$ is the closure of the braid β.*

Similarly, in Example 2.4, the categorical closure of $\beta \in B_n$ is the module over $R = \mathbb{Z}[t, t^{-1}]$ given by

$$\mathcal{L}(R,\sigma)[\beta] = \mathrm{coker}[R^{\oplus n} \xrightarrow{\mathrm{id}-\beta} R^{\oplus n}]\,. \tag{2.7}$$

In this case, we have the following theorem due to D. Goldschmidt [21].

[8] In the literature (see, e.g., [6]), it is more common to extend σ to a *right* braid action. Thus, if $\Phi : B_n \to B_n$ is the anti-isomorphism of B_n where $\Phi(\sigma_i) = \sigma_i$, then the automorphism in the convention of [6] corresponding to the element $\beta \in B_n$ is equal to the automorphism in our present convention corresponding to the element $\Phi(\beta) \in B_n$.

Theorem 2.8. *The categorical closure of a braid with respect to the Burau action is the Alexander module of the unlinked disjoint union $L \cup O$ of the braid closure $L = \hat{\beta}$ with the unknot O.*

Thus, the categorical braid closure of both the Artin and Burau examples are link invariants. This raises the natural question: when does the categorical braid closure of a co-Cartesian Yang–Baxter operator produce a link invariant? To address this question, we begin with the following definition.

Given a map $\sigma : A \amalg A \to A \amalg A$, we consider the coequalizer

$$E := \mathrm{coeq}\Big[\, A \underset{i_2}{\overset{\sigma\circ i_2}{\rightrightarrows}} A \amalg A \,\Big]$$

where $i_2 : A \to A \amalg A$ is the canonical map identifying A with its second copy in $A \amalg A$. We let $p : A \amalg A \to E$ denote the universal map such that $p \circ \sigma \circ i_2 = p \circ i_2$.

Definition 2.9. We say that the map σ is *Wada* if the following composition is an isomorphism in $\mathcal{C}$:

$$j' : A \xrightarrow{i_1} A \amalg A \xrightarrow{p} E.$$

For a Wada map σ, we consider the map

$$j : A \xrightarrow{i_1} A \amalg A \xrightarrow{\sigma} A \amalg A \xrightarrow{p} E$$

and define the *torsion* of σ to be the map

$$\chi(\sigma) = (j')^{-1} \circ j : A \to A.$$

We say that a Wada map σ has *trivial torsion* if $\chi(\sigma) = \mathrm{id}_A$ is the identity map.

As an easy exercise for the reader, we recommended to check that both the Artin and Burau Yang–Baxter operators have trivial torsion. The next theorem explains why the categorical braid closures of these operators give link invariants.

Theorem 2.10 *(Wada). Suppose that a co-Cartesian Yang–Baxter operator $\sigma : A \amalg A \to A \amalg A$ is Wada with trivial torsion, then the isomorphism type of the categorical braid closure is invariant under Markov moves, and hence give a link invariant.*

This theorem was proved in [52] in the special case when $\mathcal{C}$ is the category Gr of groups, and the object $A \in \mathsf{Gr}$ is the free group $\mathbb{F}_1$ on one generator. However, the arguments of [52] can be easily formalized and extended to a proof in the general case.

The Wada condition involves coequalizers, which makes its verification somewhat clumsy in practice (especially, in the homotopical setting which we will discuss in the next section). We therefore introduce another condition on σ that, among other things, turns the Wada condition into a simpler form.

Definition 2.11. We say that a map $\sigma : A \amalg A \to A \amalg A$ is *dualizable* if

(1) The map $\sigma : A \amalg A \to A \amalg A$ is invertible.
(2) The map $\sigma_U^R = (\sigma \circ i_2, i_2) : A \amalg A \to A \amalg A$ is invertible.
(3) The map $\sigma_U^L = (\sigma \circ i_1, i_1) : A \amalg A \to A \amalg A$ is invertible.

With this definition, we have the following proposition.

Proposition 2.12. *Assume that $\sigma : A \amalg A \to A \amalg A$ is dualizable. Consider the composition of maps*

$$(j, j') : A \amalg A \xrightarrow{\sigma_U^L} A \amalg A \xrightarrow{(\sigma_U^R)^{-1}} A \amalg A \xrightarrow{\nabla} A\,, \tag{2.13}$$

where ∇ is the canonical folding map. Then, σ is Wada if and only if the map j' is an isomorphism. In this case, the torsion of σ is given by

$$\chi(\sigma) = (j')^{-1} \circ j. \tag{2.14}$$

Next, we introduce our main definition.

Definition 2.15. We say a map $\sigma : A \amalg A \to A \amalg A$ is *Reidemeister* if it is Yang–Baxter, dualizable, and Wada with invertible torsion $\chi(\sigma)$ (see Proposition 2.12).

In Theorem 2.10, we required the torsion of σ to be trivial. However, in our main example of the co-Cartesian Yang–Baxter operator associated with the GMV action (see Section 4), the torsion is not trivial, but invertible. It turns out, if σ is Reidemeister, with not necessarily trivial torsion, then an analogue of Theorem 2.10 holds, provided we modify the categorical braid closure in an appropriate way. From now on, for simplicity, we will work with knots (*i.e.*, links with one component), while all that follows holds for the general case of links (see [4] for more details).

Let $\sigma : A \amalg A \to A \amalg A$ be a Reidemeister operator with (invertible) torsion $\chi : A \to A$. Suppose that $\beta \in B_n$ is a braid that closes to a knot $\hat{\beta} = K$. Write $w = w(\beta)$ for the writhe of β.

Definition 2.16. The *normalized categorical closure* of β with respect to σ is defined by

$$\bar{\mathcal{L}}(A,\sigma)[\beta] = \operatorname{coeq}\Big[\, A^{(n)} \underset{\mathrm{id}}{\overset{\Psi_0}{\rightrightarrows}} A^{(n)} \,\Big],$$

where Ψ_0 is the composition of morphisms: $A^{(n)} \xrightarrow{\chi^{-w} \amalg \mathrm{id}^{(n-1)}} A^{(n)} \xrightarrow{\beta} A^{(n)}$.

Theorem 2.17. *For any Reidemeister operator $\sigma : A \amalg A \to A \amalg A$, the isomorphism type of the normalized categorical braid closure is invariant under Markov moves, and hence gives a knot invariant.*

There is a conceptual way to prove Theorems 2.10 and 2.17 by interpreting the categorical braid closure as an abstract trace in the sense of [27]. To this end, we extend the category $\mathcal{C}$ to a larger category $\widehat{\mathcal{C}}$ with the same object set $\mathrm{Ob}(\widehat{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$. The hom-set $\mathrm{Hom}_{\widehat{\mathcal{C}}}(A,B)$ in $\widehat{\mathcal{C}}$ is given by the set of cospans $[B \to X \leftarrow A]$ modulo isomorphisms that are identity on A and B. The composition in $\widehat{\mathcal{C}}$ is defined by pushouts in the obvious way. The category $\widehat{\mathcal{C}}$ has a monoidal product $\boxtimes$ induced by the coproduct in $\mathcal{C}$; hence the monoidal structure of $\widehat{\mathcal{C}}$ extends the co-Cartesian monoidal structure on $\mathcal{C}$ in the sense that there is a faithful, strongly monoidal functor $\iota : (\mathcal{C}, \amalg) \hookrightarrow (\widehat{\mathcal{C}}, \boxtimes)$. Moreover, the monoidal category $(\widehat{\mathcal{C}}, \boxtimes)$ has a canonical pivotal structure, and therefore an abstract trace axiomatized in [27] and [44]. This construction allows us to interpret the categorical braid closure as an abstract trace in $\widehat{\mathcal{C}}$, and the Wada condition (Definition 2.9) can then be interpreted as a condition on the partial trace of σ viewed as a morphism in $\widehat{\mathcal{C}}$ under the faithful embedding $\iota : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$. We remark that the Wada condition reinterpreted this way is analogous to a condition on partial trace for "enriched Yang–Baxter operators" introduced by Turaev in [51]. This interpretation allows us to prove Theorem 2.10 and Theorem 2.17 by diagrammatic tensor calculus. In fact, starting with a Reidemeister operator, one can construct a ribbon category (in the sense of [41]), whose associated link invariant, which lives in the set $\mathrm{Hom}_{\widehat{\mathcal{C}}}(\phi,\phi)$ of isomorphism classes of objects in $\mathcal{C}$, coincides with the categorical braid closure. One can see this as another justification for the term "categorical braid closure". For details, we refer the reader to [4].

Remark 2.18. The notion of a co-Cartesian Yang–Baxter operator is closely related to biracks and biquandles.[9] To make this relation precise, we recall the (dual) Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathsf{Set})$ for a category $\mathcal{C}$ that associates with an object $A \in \mathcal{C}$ the corepresentable functor $h^A := \mathrm{Hom}(A, -) : \mathcal{C} \to \mathsf{Set}$. A cobirack structure on A can then be defined by factoring h^A into a composition of functors $\mathcal{C} \to \mathsf{Birack} \xrightarrow{\text{forget}} \mathsf{Set}$. Similarly, a cobiquandle structure on A is the factorization of h^A into a composition of functors $\mathcal{C} \to \mathsf{Biquandle} \xrightarrow{\text{forget}} \mathsf{Set}$. Then, one can show that, giving a cobirack structure on A is equivalent to giving a dualizable co-Cartesian Yang–Baxter operator on A. Similarly, giving a cobiquandle structure on A is equivalent to giving a Reidemeister operator on A with trivial torsion. Thus, in particular, given a Reidemeister operator on A with trivial torsion, the set $X_B = \mathrm{Hom}_{\mathcal{C}}(A,B)$ has a natural biquandle structure for any object $B \in \mathcal{C}$. In fact, many examples of biquandles in the literature arise in this manner. (In particular, almost all examples of biquandles given in [18] are of this form.)

Given a biquandle X and a link L, one can define a combinatorial link invariant called $\mathrm{Col}_X(L)$, which is the set of colorings of a link diagram of L by the biquandle X (see [18]). If the link L is the closure of a braid $\beta \in B_n$, and if the biquandle $X = X_B$ arises from a cobiquandle structure on an object A in the sense above, then we have

$$\mathrm{Col}_{X_B}(L) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}(A,\sigma)[\beta], B)\ .$$

This last formula gives a combinatorial interpretation of the categorical braid closure in terms of arcs of a link diagram that we alluded to in the Introduction.

3. Homotopy braid closure

In this section, we will work with model categories and assume the reader to have some familiarity with the theory of model categories and derived functors. For an excellent introduction, we recommend the article by Dwyer and Spalinski

[9] For the definition and basic examples of biracks and biquandles we refer to [18,7].

[9], which covers enough material for the present paper. For a more comprehensive study of model categories, we refer to [22,23].

If $\mathcal{C}$ is a model category, then the notion of a categorical braid closure associated with a co-Cartesian Yang–Baxter operator in $\mathcal{C}$ admits a natural generalization, which is obtained by replacing colimits in Definition 2.5 and Definition 2.16 by homotopy colimits. The analogues of Theorem 2.10 and Theorem 2.17 hold in this homotopical setting, provided that the object A satisfies a pseudoflatness condition (which roughly says that the n-fold homotopy coproduct of A coincides with the n-fold coproduct of A). Since the translation to the homotopical context is fairly straightforward, we omit here formal details. Instead, we will give explicit definitions and statements only in the case where the notion of a normalized braid closure is further refined by allowing the knot in question to be colored by certain maps.

Definition 3.1. Given a co-Cartesian Yang–Baxter operator $\sigma : A \amalg A \to A \amalg A$, we say that a map $\theta : A \to A$ is *σ-natural* if the following two diagrams commute

$$\begin{array}{ccc} A \amalg A & \xrightarrow{\sigma} & A \amalg A \\ {\scriptstyle \theta \amalg \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \amalg \theta} \\ A \amalg A & \xrightarrow{\sigma} & A \amalg A \end{array} \qquad \begin{array}{ccc} A \amalg A & \xrightarrow{\sigma} & A \amalg A \\ {\scriptstyle \mathrm{id} \amalg \theta}\downarrow & & \downarrow{\scriptstyle \theta \amalg \mathrm{id}} \\ A \sqcup A & \xrightarrow{\sigma} & A \amalg A \end{array}$$

Now, let $\sigma : A \amalg A \to A \amalg A$ be a Reidemeister operator with (invertible) torsion $\chi : A \to A$, and let $\theta : A \to A$ be a σ-natural map. Suppose that $\beta \in B_n$ is a braid that closes to a knot $\hat{\beta} = K$.

Definition 3.2. The *θ-colored normalized homotopy closure* of β with respect to σ is defined to be the homotopy coequalizer in $\mathcal{C}$:

$$h\bar{\mathcal{L}}_\theta(A,\sigma)[\beta] = \mathrm{hocoeq}\Big[\, A^{(n)} \underset{\mathrm{id}}{\overset{\Psi}{\rightrightarrows}} A^{(n)} \,\Big]$$

where $\Psi : A^{(n)} \to A^{(n)}$ is the composition

$$A^{(n)} \xrightarrow{\theta\chi^{-w} \amalg \mathrm{id}^{(n-1)}} A^{(n)} \xrightarrow{\beta} A^{(n)}. \tag{3.3}$$

When $\theta = \mathrm{id}_A$ is the identity map, which is always σ-natural, this construction obviously reduces to the original one without coloring.

Theorem 3.4. *Let $\sigma : A \amalg A \to A \amalg A$ be a Reidemeister operator on a pseudoflat object A in a model category $\mathcal{C}$, and let $\theta : A \to A$ is a σ-natural map. Then, the isomorphism type in* $\mathrm{Ho}(\mathcal{C})$ *(i.e., the weak equivalence type in $\mathcal{C}$) of the θ-colored normalized homotopy braid closure is invariant under Markov moves, and hence gives a knot invariant.*

This theorem allows one to refine many classical link invariants defined by categorical braid closure. To illustrate this we will return to Examples 2.3 and 2.4 in Section 2.

Example 3.5. To compute the homotopy braid closure of the Burau representations (see Example 2.4), we embed the module category $\mathrm{Mod}(R)$ into the category $\mathrm{Ch}(R)$ of chain complexes in the usual way. The module $A = R$ is then identified with the chain complex $A = [0 \to R \to 0]$, with R concentrated in degree 0. The category $\mathrm{Ch}(R)$ has a natural (projective) model structure, with weak equivalences being the quasi-isomorphisms and the fibrations being the degreewise surjective morphisms of complexes (see [23, Section 2.3]). Every object in this model category is pseudoflat. The corresponding homotopy category $\mathrm{Ho}(\mathcal{C})$ is the (unbounded) derived category $\mathcal{D}(R)$ of R-modules. Now, the homotopy closure of a braid $\beta \in B_n$ with respect to the Burau operator $\sigma : R^{\oplus 2} \to R^{\oplus 2}$ is given by the mapping cone of the morphism $\mathrm{id} - \beta$ in the derived category $\mathcal{D}(R)$, *i.e.*

$$h\mathcal{L}(R,\sigma)[\beta] = \mathrm{Cône}(\mathrm{id} - \beta) := [\, 0 \to R^{\oplus n} \xrightarrow{\mathrm{id}-\beta} R^{\oplus n} \to 0 \,],$$

where the two copies of $R^{\oplus n}$ are concentrated in homological degrees 0 and 1. The isomorphism class of $h\mathcal{L}(R,\sigma)[\beta]$ in $\mathcal{D}(R)$ is a link invariant by Theorem 3.4. Note that the homology of this complex in degree 0 is the cokernel of $\mathrm{id} - \beta$, which is precisely the categorical braid closure $\mathcal{L}(R,\sigma)[\beta]$, see (2.7).

Example 3.6. To compute the homotopy braid closure of the Artin representations (see Example 2.3), we embed the category of groups into the model category $\mathtt{sGr}$ of *simplicial* groups. The model structure on $\mathtt{sGr}$ is inherited from the category $\mathtt{sSet}$ of simplicial sets, so that the weak equivalences and fibrations of simplicial groups are the weak equivalences and fibrations of the underlying simplicial sets (see, e.g., [20]). The category of simplicial groups has a rich homotopy theory, which is

classically known to be equivalent to that of topological spaces. Precisely, there is a Quillen equivalence between $\mathtt{sGr}$ and the category $\mathtt{Top}_{0,*}$ of connected pointed topological spaces given by the composition of functors

$$\mathtt{sGr} \xrightarrow{\overline{W}} \mathtt{sSet} \xrightarrow{|-|} \mathtt{Top}_{0,*} \tag{3.7}$$

where $\overline{W}$ is Kan's bar construction assigning to a simplicial group its classifying simplicial space and $|-|$ is Milnor's geometric realization functor. The functor (3.7) induces an equivalence of the homotopy categories $\mathrm{Ho}(\mathtt{sGr}) \cong \mathrm{Ho}(\mathtt{Top}_{0,*})$, which gives a bijective correspondence between the homotopy classes of simplicial groups and the homotopy classes of pointed connected CW complexes (see, e.g., [20, V.6.4]). In this way, every simplicial group can be thought of as representing a topological space (up to homotopy).

Now, if we regard $\mathbb{F}_1$ as a discrete simplicial group in $\mathtt{sGr}$, then the homotopy closure $h\mathcal{L}(\mathbb{F}_1, \sigma)[\beta]$ of a braid $\beta \in B_n$ with respect to the Artin operator $\sigma : \mathbb{F}_2 \to \mathbb{F}_2$ is represented by a simplicial group in $\mathrm{Ho}(\mathtt{sGr})$ that corresponds under the above equivalence to the link complement $\mathbb{R}^3 \setminus L$ of the closure $L = \hat{\beta}$ (*cf.* [5, Proposition 7.1]):

$$|\,\overline{W}\, h\mathcal{L}(\mathbb{F}_1, \sigma)[\beta]\,| \simeq \mathbb{R}^3 \setminus L\,.$$

Thus, in the case of Artin representations, the homotopy braid closure completely recovers the homotopy type of the space $\mathbb{R}^3 \setminus L$, while the categorical braid closure gives only its fundamental group.

Remark 3.8. Theorem 3.4 holds in a more general case, when various conditions on the map σ hold only up to homotopy. For example, one can require the Yang–Baxter equation (2.2) to hold only in $\mathrm{Ho}(\mathcal{C})$, the maps in Definition 2.11 to be only weak equivalences and the map $j : A \to A$ defined in (2.13) to be only an isomorphism in $\mathrm{Ho}(\mathcal{C})$. The braid group action, as well as the torsion map, are then only defined in $\mathrm{Ho}(\mathcal{C})$. This makes a precise definition of a homotopy braid closure a little tedious. We omit it here, referring the reader to [4] instead.

4. The Gelfand–MacPherson–Vilonen action

In this section, we fix a commutative ring k with unit. By a *k-category*, we mean a category enriched over the category of k-modules. Let Cat_k^* be the category of all (small) pointed k-categories, where a k-category $\mathscr{A}$ is *pointed* if there is a distinguished object $* \in \mathscr{A}$. Maps (*i.e.*, k-linear functors) between pointed k-categories are required to preserve the distinguished objects.

As in the Introduction, we consider the path category $k\langle Q\rangle$ of the quiver $Q = \big[1 \underset{a^*}{\overset{a}{\rightleftarrows}} 0\big]$ Let $T \in k\langle Q\rangle(0,0)$ be the element in the endomorphism algebra of $k\langle Q\rangle$ of the object 0 defined by $T = e_0 + aa^*$, and let $\tilde{A}$ be the k-category $\tilde{A} = k\langle Q\rangle[T^{-1}]$, which is pointed by taking the object 0 as the distinguished object.

The coproduct in Cat_k^* is given by the *fusion product*, *i.e.* the coproduct of $X, Y \in \mathsf{Cat}_k^*$ is the k-category obtained by collapsing the two distinguished objects in the disjoint union of X and Y into a single object. In particular, the n-fold coproduct of $\tilde{A} \in \mathsf{Cat}_k^*$ is the k-category $\tilde{A}^{(n)}$ defined in (1.4).

We begin with the following result mentioned in the Introduction.

Theorem 4.1 *(Gelfand, MacPherson, Vilonen). The map $\sigma : \tilde{A}^{(2)} \to \tilde{A}^{(2)}$ defined by (1.6) is a co-Cartesian Yang–Baxter operator on the object $\tilde{A}$ in the category Cat_k^*. We call σ the GMV operator.*

The GMV operator induces an action of B_n on $\tilde{A}^{(n)}$, where the generator $\sigma_i \in B_n$ acts on objects by swapping i and $i+1$, while fixing all other objects, and on morphisms by formula (1.5). The next observation is straightforward to check.

Lemma 4.2. *The GMV operator (1.6) is Reidemeister with torsion given by*

$$\chi : \tilde{A} \to \tilde{A}, \qquad a \mapsto Ta, \qquad a^* \mapsto a^* T^{-1}. \tag{4.3}$$

As explained in the Introduction, formula (1.5) for the braid group action first appeared in [19] in relation to perverse sheaves. More precisely, it was shown in [19] that any choice of 'cuts' (*i.e.*, a family Θ of n simple curves on $D \setminus \{p_1, \dots, p_n\}$, going from a chosen point near p_i to the chosen endpoint p_0 near the boundary ∂D, so that any two such curves intersect only at p_0) induces an equivalence of categories $\tilde{E}_\Theta : \mathsf{Perv}(D, \{p_1, \dots, p_n\}) \simeq \tilde{\mathscr{Q}}$ from the category of perverse sheaves on the disk D with only possible singularities at the points $\{p_1, \dots, p_n\}$, to a quiver category $\tilde{\mathscr{Q}}$ *isomorphic* to the category $\mathrm{Mod}(\tilde{A}^{(n)})$ of finite-dimensional modules over the k-category $\tilde{A}^{(n)}$.

Now, the braid group B_n acts as a mapping class group on the disk D with n marked points $\{p_1, \dots, p_n\}$, and hence acts (in a certain sense) on the category $\mathsf{Perv}(D, \{p_1, \dots, p_n\})$. If we fix a family Θ of cuts, this translates to an action of B_n on the quiver category $\tilde{\mathscr{Q}}$. In fact, it is shown in [19] that there is a *strict* action of B_n on the quiver category $\tilde{\mathscr{Q}}$ that coincides under the equivalence $\tilde{E}_\Theta$ with the natural action on the category $\mathsf{Perv}(D, \{p_1, \dots, p_n\})$ (see [19, Proposition 1.3]).

This strict B_n action on the quiver category $\tilde{\mathscr{Q}}$ is in fact induced by the action (1.5) on the k-category $\tilde{A}^{(n)}$. More precisely, the left action (1.5) induces a strict left action on the module category $\mathrm{Mod}(\tilde{A}^{(n)})$ where $\beta \in B_n$ acts by $M \mapsto (\beta^{-1})^*(M)$. This coincides under the isomorphism of categories $\mathrm{Mod}(\tilde{A}^{(n)}) \cong \tilde{\mathscr{Q}}$ with the strict B_n action on the quiver category $\tilde{\mathscr{Q}}$ constructed in [19].

Remark 4.4. In the notation of [19], a module $M \in \mathrm{Mod}(\tilde{A}^{(n)})$ corresponds to the representation of the quiver $Q^{(n)}$ which is given by a collection of vector spaces and maps $M(0) = A$, $M(i) = B_i$, $M(a_i) = q_i$, $M(a_i^*) = p_i$. The functors $(\sigma_i^{-1})^*$ on modules correspond to the operations denoted by T_i. For example, $((\sigma_i^{-1})^*M)(a_j)$ means $T_i(q_j)$ in the notation of [19].

Next, we introduce a slight modification of the GMV action. Let $\overline{Q}_n$ denote the following quiver

$$\overline{Q}_n := \quad 1 \;\underset{a_n^*}{\overset{a_1}{\rightrightarrows}}\; 0 \tag{4.5}$$

Fix an invertible element $\mu \in k^\times$, and define the k-category $A^{(n)}$ by

$$A^{(n)} = k\langle \overline{Q}_n \rangle / (a_i^* a_i = (\mu - 1)e_1)_{i=1,\dots,n}. \tag{4.6}$$

Notice that the elements $T_i = e_0 + a_i a_i^*$ are invertible in $A^{(n)}$ for all $i = 1, 2, \dots, n$. Hence, formula (1.5) still defines a braid group action on $A^{(n)}$.

The k-category $A^{(n)}$ is obtained from $\tilde{A}^{(n)}$ by applying the following two operations:

(1) taking the quotient of $\tilde{A}^{(n)}$ modulo the relations $a_i^* a_i = (\mu - 1)e_i$,
(2) collapsing the vertices $\{1, \dots, n\}$ into a single vertex 1.

The GMV braid action on $\tilde{A}^{(n)}$ descends to a braid action on $A^{(n)}$, which we will call the *μ-central GMV action*.

One advantage of working with the μ-central GMV action is that it fixes the set of objects of $A^{(n)}$. In particular, one can consider the induced braid action on the endomorphism algebra of any object of $A^{(n)}$. Specifically, let $A^{(n)}(1,1)$ denote the endomorphism algebra of the object '1' in $A^{(n)}$. For $i, j = 1, 2, \dots, n$, consider the elements $A_{ij} := -a_i^* a_j \in A^{(n)}(1,1)$. Then, it is easy to see that the algebra $A^{(n)}(1,1)$ has the following presentation

$$A^{(n)}(1,1) = k\langle A_{ij} \rangle / (A_{ii} = 1 - \mu).$$

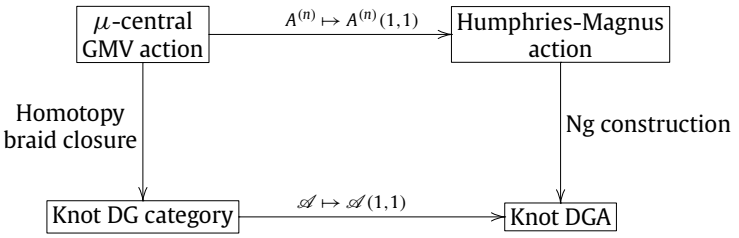
It is straightforward to compute the induced braid group action on this algebra in terms of the generators A_{ij}. However, we will write the corresponding formulas in terms of other generators a_{ij} related to A_{ij} by a simple rescaling:

$$a_{ij} := \begin{cases} A_{ij}\,, & i < j \\ -\mu^{-1} A_{ij}\,, & i > j. \end{cases} \tag{4.7}$$

The associative algebra $A^{(n)}(1,1)$ is free on these generators, and the braid group action on $A^{(n)}(1,1)$ is given by

$$\sigma_k : \begin{cases} a_{ki} \mapsto a_{k+1,i} - a_{k+1,k}\, a_{ki} & (i \neq k, k+1) \\ a_{ik} \mapsto a_{i,k+1} - a_{ik}\, a_{k,k+1} & (i \neq k, k+1) \\ a_{k+1,i} \mapsto a_{ki} & (i \neq k, k+1) \\ a_{i,k+1} \mapsto a_{ik} & (i \neq k, k+1) \\ a_{k,k+1} \mapsto -a_{k+1,k} & \\ a_{k+1,k} \mapsto -a_{k,k+1} & \\ a_{ij} \mapsto a_{ij} & (i, j \neq k, k+1). \end{cases} \tag{4.8}$$

Formulas (4.8) first appeared in [24,25] as a generalization of the classical Magnus action [32]; we therefore call (4.8) the *Humphries–Magnus braid action*. The Humphries–Magnus braid action was used by Ng in [35–39] as part of his definition of the (combinatorial) knot DGA (see, e.g., [39], Definition 3.3). Now, the main results of the present paper in relation to Ng's work can be summarized schematically by the following diagram.



(4.9)

5. The knot DG category

Let $\mathcal{C} = \mathsf{dgCat}_k^{\{0,1\}}$ be the category comprising all small DG k-categories with object set $\{0,1\}$. The morphisms of such DG categories in $\mathcal{C}$ are required to be the identity map on the object set $\{0,1\}$. The k-category $A^{(n)}$ defined in (4.6) can then be identified with the n-fold coproduct of copies of $A := A^{(1)}$ in the category $\mathcal{C}$. Moreover, the μ-central GMV action is induced by a co-Cartesian Yang–Baxter operator $\sigma : A \amalg A \to A \amalg A$ given by the same formula as in (1.6). The same calculation as in Lemma 4.2 shows that this co-Cartesian Yang–Baxter operator is Reidemeister with torsion given by formula (4.3). Moreover, the following lemma gives a σ-natural map (Definition 3.1) that can be used to color a knot.

Lemma 5.1. *For any element $\lambda \in k^\times$, the map $\theta_\lambda : A \to A$ given by $(a, a^*) \mapsto (\lambda^{-1}a, \lambda a^*)$ is σ-natural.*

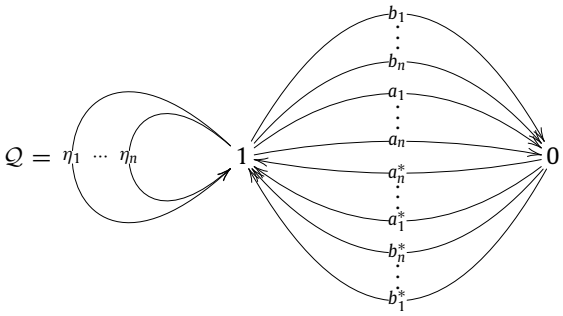
The category $\mathcal{C} = \mathsf{dgCat}_k^{\{0,1\}}$ has a model structure, in which a morphism $f : X \to Y$ is a weak equivalence (resp., fibration) if and only if for any pair of objects $a, b \in X$, the map $f : X(a,b) \to Y(a,b)$ is a quasi-isomorphism (resp., surjection) of chain complexes. This model category is cofibrantly generated; therefore the cofibrations can be characterized as retracts of relative cell complexes (see [22,4]), which in particular, include semi-free extensions by arrows in non-negative (homological) degree.

One can show that the k-category A viewed as an object of the model category $\mathcal{C} = \mathsf{dgCat}_k^{\{0,1\}}$ is pseudoflat. Therefore, the colored normalized homotopy braid closure (Definition 3.2) with respect to the GMV operator (A, σ) and the coloring $\theta = \theta_\lambda$:

$$\begin{aligned} h\bar{\mathcal{L}}_\theta(A,\sigma)[\beta] &:= \mathrm{hocoeq}\Big[\, A^{(n)} \underset{\mathrm{id}}{\overset{\Psi}{\rightrightarrows}} A^{(n)} \,\Big] \\ &= \mathrm{hocolim}\Big[\, A^{(n)} \xleftarrow{(\Psi,\mathrm{id})} A^{(n)} \amalg A^{(n)} \xrightarrow{(\mathrm{id},\mathrm{id})} A^{(n)} \,\Big] \end{aligned} \tag{5.2}$$

gives a quasi-isomorphism type in the category $\mathcal{C} = \mathsf{dgCat}_k^{\{0,1\}}$, which is a knot invariant.

We now describe this knot invariant in explicit terms. Let $\mathcal{Q}$ be the following *graded* quiver



where the degrees of arrows are assigned by

$$\deg(a_1) = \ldots = \deg(a_n) = \deg(a_1^*) = \ldots = \deg(a_n^*) = 0$$
$$\deg(b_1) = \ldots = \deg(b_n) = \deg(b_1^*) = \ldots = \deg(b_n^*) = 1$$
$$\deg(\eta_1) = \ldots = \deg(\eta_n) = 2.$$

Let $\beta \in B_n$ be a braid that closes to a knot K.

Definition 5.3. We defined the *knot DG category of K* to be the DG k-category

$$\mathscr{A}_K = k\langle\mathcal{Q}\rangle/(a_i^* a_i = (\mu - 1)e_1)_{1 \le i \le n}$$

with differential given by

$$\begin{aligned} d(b_i) &= \Psi(a_i) - a_i \\ d(b_i^*) &= \Psi(a_i^*) - a_i^* \\ d(\eta_i) &= -b_i^* a_i - \Psi(a_i^*) b_i \end{aligned} \tag{5.4}$$

where $\Psi : A^{(n)} \to A^{(n)}$ is the map defined in (3.3).

Our main results regarding the knot DG category $\mathscr{A} = \mathscr{A}_K$ can be encapsulated into the following two theorems.

Theorem 5.5. *The knot DG category $\mathscr{A}$ represents the quasi-isomorphism type of the homotopy coequalizer (5.2). Hence, the quasi-isomorphism type of the knot DG category $\mathscr{A}$ is independent of the choice of a braid β that closes to a given knot K.*

Theorem 5.6. *Let the base commutative ring be $k = \mathbb{Z}[\mu^{\pm 1}, \lambda^{\pm 1}]$. Then, the quasi-isomorphism type of the endomorphism DG algebra $\mathscr{A}(1,1)$ coincides with the quasi-isomorphism type of the knot DGA constructed in [37].*

Theorem 5.6 gives an alternative proof of one of the main results in [35,37] that states that the underlying quasi-isomorphism type of the combinatorial knot DGA is a knot invariant.[10]

Proof of Theorem 5.6. Define the following morphisms in $\mathscr{A}$, which are elements of the endomorphism DG algebra $\mathscr{A}(1,1)$ of different homological degrees:

$$\begin{aligned} A_{ij} &= -a_i^* a_j \in \mathscr{A}(1,1)_0 \\ B_{ij} &= b_i^* a_j \in \mathscr{A}(1,1)_1 \\ C_{ij} &= a_i^* b_j \in \mathscr{A}(1,1)_1 \\ D_{ij} &= b_i^* b_j \in \mathscr{A}(1,1)_2 \\ e_i &= -\eta_i \in \mathscr{A}(1,1)_2. \end{aligned} \tag{5.7}$$

Then, the DG algebra $\mathscr{A}(1,1)$ is freely generated by the elements (5.7), modulo the relations $A_{ii} = 1 - \mu$. The differentials of these elements can be easily computed by the Leibniz rule, using formulas (5.4):

$$\begin{aligned} d(A_{ij}) &= 0 \\ d(B_{ij}) &= \Psi(a_i^*) a_j - a_i^* a_j \\ d(C_{ij}) &= a_i^* \Psi(a_j) - a_i^* a_j \\ d(D_{ij}) &= (\Psi(a_i^*) - a_i^*) b_j + b_i^* (\Psi(a_j) - a_j) \\ d(e_i) &= b_i^* a_i + \Psi(a_i^*) b_i. \end{aligned} \tag{5.8}$$

This explicit description allows one to identify $\mathscr{A}(1,1)$ with the combinatorial knot DGA as defined in [37, Definition 2.6] (see also [39]). See [4] for details of this calculation. □

To prove Theorem 5.5, one has to calculate the homotopy pushout (5.2). As shown in [4], it suffices for this to resolve the right-pointing arrow by a strong cofibration (*i.e.* a cofibration whose domain is cofibrant), and then take the ordinary pushout of the resulting diagram. Thus, we need to find a semi-free resolution $p : B \xrightarrow{\sim} A$ and then construct an appropriate cylinder object $\mathrm{Cyl}(B)$ on B. The right-pointing arrow in the pushout diagram in (5.2) will then be resolved by taking the n-fold coproduct $\mathrm{Cyl}(B)^{(n)}$ of this cylinder object.

To construct a semi-free resolution of A, we consider the *graded* quiver

$$\tilde{Q} = \left[\xi \circlearrowright 1 \underset{a^*}{\overset{a}{\rightleftarrows}} 0 \right] \tag{5.9}$$

where $\deg(a) = \deg(a^*) = 0$ and $\deg(\xi) = 1$. Define $B \in \mathcal{C}$ to be the semi-free DG category $B := k\langle \tilde{Q} \rangle$ with differential given by $d\xi = a^* a - (\mu - 1) e_1$. Then, one can show (see [4]) that the canonical map

$$p : B \to A, \qquad a \mapsto a, \quad a^* \mapsto a^*, \qquad \xi \mapsto 0$$

is a quasi-isomorphism. Thus, B can be used as a cofibrant replacement for A.

[10] Note, however, that the results in [35,37] are slightly stronger as they refer to the invariance of the stable tame isomorphism type rather than the quasi-isomorphism type of the corresponding knot DGA.

Next, to define a cylinder on B, we will use the construction of canonical cylinder objects for the semi-free DG algebras given in [3]. This construction will play an important role in our calculations, so we review it in some detail.

Let R be a DG algebra whose underlying graded algebra is free over a graded k-module V. We write $R = (T(V), d)$. Let $\mathrm{Cyl}(R)$ be the graded algebra defined by $\mathrm{Cyl}(R) := T(V \oplus V' \oplus sV)$, where $sV = V[1]$ is the graded vector space obtained by shifting the (homological) degree of V up by 1. The inclusion of V into the two copies V and V' in $\mathrm{Cyl}(R)$ induces two maps of graded algebra $i : R \to \mathrm{Cyl}(R)$ and $i' : R \to \mathrm{Cyl}(R)$.

We say that a map $S : R \to \mathrm{Cyl}(R)$ of graded k-modules of degree -1 is a (i, i')-*derivation* if, for all homogeneous elements $a, b \in R$, we have

$$S(ab) = S(a) \cdot i'(b) + (-1)^{|a|} i(a) \cdot S(b).$$

It is easy to see that there exists a unique (i, i')-derivation $S : R \to \mathrm{Cyl}(R)$ such that $S(v) = sv \in sV \subset \mathrm{Cyl}(R)$ for all $v \in V$. This derivation S allows us to define a differential on $\mathrm{Cyl}(R)$. Indeed, there exists a unique derivation $d_{\mathrm{Cyl}} : \mathrm{Cyl}(R) \to \mathrm{Cyl}(R)$ of degree -1 satisfying

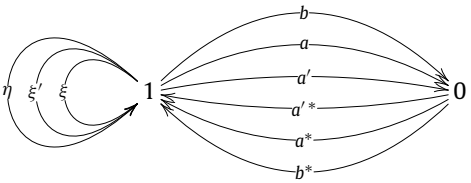
$$\begin{aligned} &(1) \quad d_{\mathrm{Cyl}} \circ i = i \circ d \\ &(2) \quad d_{\mathrm{Cyl}} \circ i' = i' \circ d \\ &(3) \quad d_{\mathrm{Cyl}} \circ S = i - i' - S \circ d. \end{aligned} \tag{5.10}$$

It follows from (5.10) that $d_{\mathrm{Cyl}}^2 = 0$, which is easy to check on generators of $\mathrm{Cyl}(R)$. Hence, d_{Cyl} makes $\mathrm{Cyl}(R)$ into a DG algebra $\mathrm{Cyl}(R) = (T(V \oplus V' \oplus sV),\, d_{\mathrm{Cyl}})$.

Next, we define a map $\pi : \mathrm{Cyl}(R) \to R$ by sending the two copies of V in $R = T(V \oplus V' \oplus sV)$ identically onto $V \subset T(V) = R$ and sV to zero. It is straightforward to check that π is a map of DG algebras and, in fact, a quasi-isomorphism from $\mathrm{Cyl}(R)$ onto R. Thus, together with i and i', the map π fits in the diagram $R \amalg R \overset{(i,i')}{\hookrightarrow} \mathrm{Cyl}(R) \overset{\pi}{\twoheadrightarrow} R$, which shows that $\mathrm{Cyl}(R)$ is a cylinder object on R. We emphasize that this cylinder object is *canonically* attached to the semi-free DG algebra R. We call it the *Baues–Lemaire cylinder* on R.

The above construction can be naturally extended to semi-free DG categories, *i.e.* DG categories whose underlying graded category is freely generated by a set of arrows. In our present situation, the underlying graded category of the DG category B is freely generated by the graded quiver (5.9). Hence, the Baues–Lemaire construction of the cylinder on $R = T(V)$ can be carried over to $B = k\langle \tilde{Q} \rangle$.

Specifically, let $c\tilde{Q} = \tilde{Q} \amalg \tilde{Q}' \amalg (\tilde{Q}[1])$ be the graded quiver

$c\tilde{Q} =$ 

(5.11)

which has three copies $\{a, a^*, \xi\}$, $\{a', a'^*, \xi'\}$ and $\{b, b^*, \eta\}$ of the generating arrows of $\tilde{Q}$, with $\{b, b^*, \eta\}$ having homological degree shifted up by 1. Thus,

$$\begin{aligned} &\deg(a) = \deg(a') = \deg(a^*) = \deg(a'^*) = 0 \\ &\deg(\xi) = \deg(\xi') = 1 \\ &\deg(b) = \deg(b^*) = 1 \\ &\deg(\eta) = 2. \end{aligned}$$

Then, we define $\mathrm{Cyl}(B)$ to be the graded k-category $\mathrm{Cyl}(B) := k\langle c\tilde{Q} \rangle$, with differential $d = d_{\mathrm{Cyl}}$ given by the Baues–Lemaire formulas (5.10):

$$\begin{aligned} d(\xi) &= a^* a - (\mu - 1)e_1 \\ d(\xi') &= a'^* a' - (\mu - 1)e_1 \\ d(b) &= a - a' \\ d(b^*) &= a^* - a'^* \\ d(\eta) &= \xi - \xi' - b^* a' - a^* b. \end{aligned} \tag{5.12}$$

For example, by Equation (3) in (5.10), we have

$$\begin{aligned} d(\eta) &= d(S(\xi)) = i(\xi) - i'(\xi) - S(d(\xi)) \\ &= \xi - \xi' - S(a^*a - (\mu-1)e_1) \\ &= \xi - \xi' - b^*a' - a^*b. \end{aligned}$$

The following proposition implies that $\mathrm{Cyl}(B)$ is indeed a cylinder object on B.

Proposition 5.13. *The canonical map $\pi : \mathrm{Cyl}(B) \to B$ defined by*

$$\pi(a) = \pi(a') = a, \qquad \pi(a^*) = \pi(a'^*) = a, \qquad \pi(\xi) = \pi(\xi') = \xi,$$
$$\pi(b) = 0, \qquad \pi(b^*) = 0, \qquad \pi(\eta) = 0$$

is a quasi-isomorphism.

Now, as explained in the Introduction, the homotopy pushout (5.2) can be computed as the ordinary pushout of the following diagram

$$A^{(n)} \xleftarrow{(\Psi, \mathrm{id}^{(n)}) \circ (p^{(n)}, p^{(n)})} B^{(n)} \amalg B^{(n)} \overset{(i^{(n)}, i'^{(n)})}{\hookrightarrow} \mathrm{Cyl}(B)^{(n)}. \tag{5.14}$$

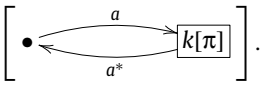
A straightforward calculation shows that the result is the knot DG category presented in Definition 5.3.

6. The knot category

Let K be a knot, and let $\mathscr{A}_K$ be the knot DG category of K presented in Definition 5.3.

Definition 6.1. We call the 0-th homology of $\mathscr{A}_K$ the *knot k-category* of K and denote it by $A_K := H_0(\mathscr{A}_K)$. This is a k-category whose isomorphism class is a knot invariant.

Let $\pi = \pi_1(\mathbb{R}^3 \setminus K)$ be the knot group of K. Consider the group algebra $k[\pi]$ as a k-category with one object 0. Similarly, the ring k can itself be considered a k-category with one object 1, which we denote by $\mathbf{1}_{\{1\}}$. Let $k[\pi]^+ = k[\pi] \amalg \mathbf{1}_{\{1\}}$ be the disjoint union of these two k-categories. Thus, $k[\pi]^+ \in \mathsf{Cat}_k^{\{0,1\}}$ is a k-category with object set $\{0, 1\}$. Now, let $k[\pi]^+\langle a, a^*\rangle$ be the free extension in $\mathsf{Cat}_k^{\{0,1\}}$ of $k[\pi]^+$ by the arrows a, a^* where a goes from the vertex 1 to the vertex 0, while a^* goes in the opposite direction, *i.e.* from 0 to 1. We will denote this k-category schematically by

$k[\pi]^+\langle a, a^*\rangle =$ 

Then, we have the following description of the knot k-category in terms of the peripheral pair $(\pi, (m, l))$, where $m, l \in \pi$ are respectively a meridian and a longitude of the knot K.

Theorem 6.2. *The knot k-category A_K can be described as*

$$A_K \cong k[\pi]^+\langle a, a^*\rangle / J$$

where J is the ideal generated by the following elements

(1) $aa^* + e_0 - m$
(2) $a^*a + e_1 - \mu e_1$
(3) $\lambda a - la\ ,\ \lambda a^* - a^*l.$

Remark 6.3. A peripheral pair (m, l) is well defined up to inner automorphisms of π. Suppose that $(m', l') = (\gamma m \gamma^{-1}, \gamma l \gamma^{-1})$ is another such pair, then letting $a' = \gamma a$ and $(a^*)' = a^*\gamma^{-1}$, we reduce the defining relations (1)–(3) of the k-category A_K to the same form written in terms of $a', (a^*)', l', m'$. Hence, up to isomorphism, this k-category is independent of the choice of the peripheral pair.

To prove Theorem 6.2, we notice that the braid group B_n acts on the elements $T_i \in A^{(n)}(0, 0)$ the same way as it acts on the generators $x_i \in \mathbb{F}_n$ in the Artin representation. This implies that, after taking the categorical braid closure, there is a map ϕ from $k[\pi]$ to the endomorphism algebra $A_K(0, 0)$ of the knot k-category at 0, taking x_i to T_i. Define $\tilde{\phi} : k[\pi]^+\langle a, a^*\rangle \to A_K$ by extending the map ϕ, so that $a \mapsto a_1$, and $a^* \mapsto a_1^*$. Then, one can show that if $m = T_1 \in \pi$ and $l \in \pi$ is the corresponding

longitude, then the map $\tilde{\phi}$ sends the ideal J defined in Theorem 6.2 to zero, and hence descends to a map from the quotient $k[\pi]^+\langle a, a^*\rangle / J$ to A_K, which can be shown to be an isomorphism. (See [4] for details.)

Recall (see Theorem 5.6) that the endomorphism DG algebra of the object 1 in the knot DG category is quasi-isomorphic to the knot DGA. Therefore, in particular, the endomorphism algebra of the object 1 of the knot k-category recovers the 0th homology of the knot DGA. Thus, Theorem 6.2 implies the following.

Theorem 6.4 *([37]). The 0th homology of the knot DGA is isomorphic to the tensor algebra over k freely generated by elements* $[\gamma]$, *where* $\gamma \in \pi_1(\mathbb{R}^3 \setminus L)$, *modulo the relations*

(1) $[e] = 1 - \mu$, *where e is the identity element;*
(2) $[\gamma_1\gamma_2] - [\gamma_1 m \gamma_2] - [\gamma_1][\gamma_2] = 0$ *for* $\gamma \in \pi_1(\mathbb{R}^3 \setminus L)$;
(3) $[\gamma l] = [l\gamma] = \lambda[\gamma]$ *for* $\gamma_1, \gamma_2 \in \pi_1(\mathbb{R}^3 \setminus L)$.

Proof. The endomorphism algebra of the object 1 of the k-category $k[\pi]^+\langle a, a^*\rangle$ is freely generated by the elements $[\gamma] := -a\gamma a^*$. The ideal J of Theorem 6.2 defines the relations in this endomorphism algebra, which are simply the three relations given in the theorem. □

Remark 6.5. Theorem 6.2 also shows that the endomorphism algebra of the knot category at the vertex 0 is given by $A_K(0,0) = k[\pi]/\langle (m-1)(m-\mu),\ (m-1)(l-\lambda)\rangle$.

7. The fully noncommutative link DG category

Recall that, in Section 4, we have 'simplified' the GMV k-category $\tilde{A}^{(n)}$ by performing the following two operations on the underlying quiver:

(1) we have collapsed the vertices $1, \dots, n$ to a single vertex 1,
(2) we have set the elements $e_i + a_i^* a_i$ to be equal to a central element $\mu \in k^\times$.

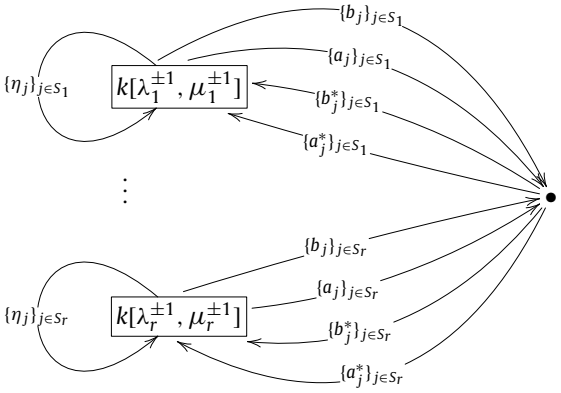
In this section, we will work with the original GMV category $\tilde{A}^{(n)}$ and the associated braid action. We will show that the corresponding homotopy braid closure is related to the "fully noncommutative knot DGA" introduced in [13,39].

Let $\tilde{A}^{(n)}$ be the k-category (1.4) with the GMV braid action defined as in (1.5). Consider the elements $\mu_i = a_i^* a_i + e_i \in \tilde{A}^{(n)}(i,i)$, which are now no longer central.

Suppose we are given a braid $\beta \in B_n$ which closes to a link L with r component $L = L_1 \cup \ldots \cup L_r$. For each $1 \le i \le r$, let S_i be the set of strands $S_i \subset \{1, \dots, n\}$ that closes to the component L_i. Note that these are precisely the orbits of the cyclic group generated by β acting on the set $\{1, \dots, n\}$ by permutations.

Now, for each $1 \le i \le r$, identify all the vertices j in $A^{(n)}$ that are in the set S_i to a single vertex i, and identify all the elements μ_j, for $j \in S_i$, to a single element μ_i. Let $\overline{A^{(n)}}$ be the resulting k-category. Then, the action map $\beta : A^{(n)} \to A^{(n)}$ induces $\overline{\beta} : \overline{A^{(n)}} \to \overline{A^{(n)}}$. We can use this induced braid action to define the fully noncommutative link DG category.

Definition 7.1. The *fully noncommutative link DG category* of L is the DG category $\tilde{\mathscr{A}}_L$, whose underlying graded k-category is defined to be the quotient of



(7.2)

modulo the relations $e_i + a_j^* a_j = \mu_i$ for all $j \in S_i$, $1 \le i \le r$, where the degrees of the generators are given by

$$\deg(a_j) = \deg(a_j^*) = 0, \qquad \deg(b_j) = \deg(b_j^*) = 1, \qquad \deg(\eta_j) = 2.$$

To define the differential, choose a strand $j_i \in S_i$, one for each $1 \le i \le r$, and for each $j \in S_i$, set

$$d(b_j) = \begin{cases} \overline{\beta}(a_j)\lambda_i^{-1}\mu_i^{-w_i} - a_j & (j = j_i) \\ \overline{\beta}(a_j) - a_j & (j \neq j_i) \end{cases}$$
$$d(b_j^*) = \begin{cases} \lambda_i\mu_i^{w_i}\overline{\beta}(a_j^*) - a_j & (j = j_i) \\ \overline{\beta}(a_j^*) - a_j & (j \neq j_i) \end{cases} \tag{7.3}$$
$$d(\eta_j) = \begin{cases} -b_j^* a_j - \lambda_i\mu_i^{w_i}\overline{\beta}(a_j^*)b_j & (j = j_i) \\ -b_j^* a_j - \overline{\beta}(a_j^*)b_j & (j \neq j_i). \end{cases}$$

Theorem 7.4. *Let R_0 be the k-category with r objects, given by the disjoint union of k-algebras*

$$R_0 = k[\lambda_1^{\pm 1}, \mu_1^{\pm 1}] \amalg \ldots \amalg k[\lambda_r^{\pm 1}, \mu_r^{\pm 1}].$$

Then, the quasi-isomorphism type of the pair $(R_0, \tilde{\mathscr{A}})$ consisting of the k-category R_0, together with the canonical map from R_0 to the fully noncommutative link DG category $\tilde{\mathscr{A}}$, is a link invariant. Moreover, if we collapse the objects $\{1, \ldots, r\}$ to a single object 1, then the endomorphism DG algebra at this collapsed vertex coincides with the fully noncommutative knot DGA constructed in [13]. (*Here, we take the base commutative ring k to be $\mathbb{Z}$.*)

The first part of the above theorem is proved by interpreting the fully noncommutative link DG category as a homotopy braid closure in a suitable model category. The second part follows from the first by a direct calculation similar to the one in Section 5 (see also the beginning of Section 8). The identification of the fully noncommutative link DG category with a homotopy braid closure is completely parallel to the μ-central case discussed above. The crucial difference, however, is that one should work in a different model category (see [4] for details).

The above theorem identifies the *quasi-isomorphism* type of the pair $(R_0, \tilde{\mathscr{A}})$; however, if we are only interested in the underlying *quasi-equivalence* type, then we have the following theorem.

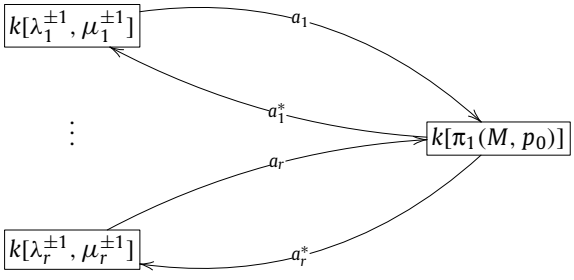
Theorem 7.5. *The quasi-equivalence type of the link DG category $\tilde{\mathscr{A}}$ is given by the (normalized) homotopy closure of the braid $\beta \in B_n$ with respect to the GMV operator, taken in the category dgCat_k^* of pointed DG categories with model structure defined in [48].*

Notice that, in this theorem, no coloring is needed. The extra parameters λ_i are formed in the process of taking the homotopy braid closure. This is not "visible" if we, like in the μ-central case, work with a more rigid model structure, where the weak equivalences are quasi-isomorphisms (*cf.* also Remark 7.9). "Normalizing" is also not necessary in Theorem 7.5, as it only changes the parameter $\lambda_i \mapsto \lambda_i \mu_i^{w_i}$ in R_0.

Definition 7.6. The *fully noncommutative link k-category* of a link L is defined to be $\tilde{A}_L := H_0(\tilde{\mathscr{A}}_L)$, the 0th homology of the fully noncommutative link DG category of L.

The k-category $\tilde{A}_L$ can be expressed in terms of the link group, together with meridians and longitudes chosen in each link component. To be precise, let $M = \mathbb{R}^3 \setminus L$ be the link complement. For $1 \le i \le r$, let $\partial_i M \subset M$ denote the torus boundary of M corresponding to the link component L_i. Choose basepoints $p_i \in \partial_i M$, and $p_0 \in M$. Then, there are canonical meridian and longitude elements $\mu_i, \lambda_i \in \pi_1(\partial_i M, p_i)$, which identify the group algebra $k[\pi_1(\partial_i M, p_i)]$ as $k[\lambda_i^{\pm 1}, \mu_i^{\pm 1}]$. By choosing a path a_i in M from p_i to p_0, one can define a map $\phi_i : \pi_1(\partial_i M, p_i) \to \pi_1(M, p_0)$. Let m_i and l_i be the images of μ_i and λ_i under ϕ_i, respectively. Then, we have the following description of the fully noncommutative link category.

Theorem 7.7. *The fully noncommutative link k-category $\tilde{A}_L$ is the quotient of the k-category*



(7.8)

modulo the ideal of relations

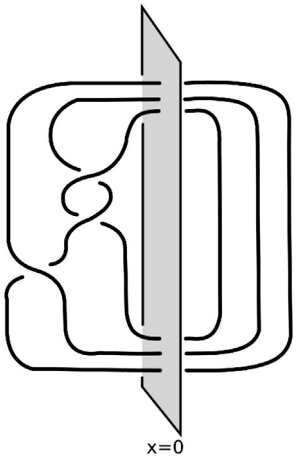
(1) $a_i a_i^* + e_0 - m_i$
(2) $a_i^* a_i + e_i - \mu_i$
(3) $a_i \lambda_i - l_i a_i$, $\lambda_i a_i^* - a_i^* l_i$.

Remark 7.9. While the fully noncommutative link DG category is the homotopy braid closure of the GMV action, it is *not* true that its 0-th homology, *i.e.* the fully noncommutative link k-category, is the categorical braid closure of the GMV action. The categorical braid closure can be obtained as a specialization of the fully noncommutative link k-category when all parameters λ_i are set to be 1. This discrepancy is due to the fact that, in Tabuada's model structure on dgCat_k, the weak equivalences are quasi-equivalences, which, by definition, induce equivalences (not isomorphisms) of k-categories at the level of 0-th homology. Then, the homotopy colimits of diagrams in dgCat_k with respect to Tabuada's model structure induce, at the level of 0th homology, not strict colimits, but rather 2-colimits, which can be viewed, in part, as homotopy colimits. Thus, the fully noncommutative link k-category is already a homotopy braid closure, rather than a strict categorical braid closure.

As mentioned in the introduction, the fully noncommutative link k-category is closely related to perverse sheaves. To be precise, let $\mathcal{S}$ be the stratification on $\mathbb{R}^3$ with two strata $(L, \mathbb{R}^3 \backslash L)$, where L is a link in $\mathbb{R}^3$. Following the degree conventions of [30], we let p be the perversity of $\mathcal{S}$ given by $p(1) = 0$ and $p(3) = -1$. (The values at other integers do not matter.) Then, we have

Theorem 7.10. *Suppose that k is a field. The category $\mathsf{Perv}^p(\mathbb{R}^3, L)$ of p-perverse sheaves of k-vector spaces on $\mathbb{R}^3$ constructible with respect to the stratification $\mathcal{S}$ is equivalent to the category of finite-dimensional left modules over the fully noncommutative link category $\tilde{A}_L$.*

Sketch of proof. Suppose that a braid $\beta \in B_n$ is placed in the region $\{x < 0\}$, and closes to the link L by letting the two ends of the braid pass through the hyperplane $\{x = 0\}$ and close in the region $\{x > 0\}$, as in the following diagram.



Let U, V be open subsets of $\mathbb{R}^3$ defined by $U = \{x < \varepsilon\}$ and $V = \{x > -\varepsilon\}$ for some small $\varepsilon > 0$. Then, both the pairs $(U, U \cap L)$ and $(V, V \cap L)$ are diffeomorphic to the pair $(\mathring{D} \times \mathring{I}, \{p_1, \ldots, p_n\} \times \mathring{I})$, where $\mathring{D}$ denotes the interior of the disk and $\mathring{I}$ denotes the open interval $(0, 1)$. The pair $(U \cap V, U \cap V \cap L)$ is diffeomorphic to the pair $(\mathring{D} \times \mathring{I}, \{p_1, \ldots, p_n, p'_1, \ldots, p'_n\} \times \mathring{I})$. Therefore, the category $\mathsf{Perv}^p(U, U \cap L)$ can be identified with the category $\mathsf{Perv}(D, \{p_1, \ldots, p_n\})$ with middle perversity, which, under a suitable choice of 'cuts', is equivalent to the category $\mathrm{Mod}(\tilde{A}^{(n)})$ of finite-dimensional modules over the k-category $\tilde{A}^{(n)}$.

Similar statements are true for the pairs $(V, V \cap L)$ and $(U \cap V, U \cap V \cap L)$. One can show then that the following diagram of restriction functors

$$\mathsf{Perv}^p(U, U \cap L) \to \mathsf{Perv}^p(U \cap V, U \cap V \cap L) \leftarrow \mathsf{Perv}^p(V, V \cap L) \tag{7.11}$$

is equivalent to the following diagram of functors

$$\mathrm{Mod}(A^{(n)}) \xrightarrow{(\beta^*, \mathrm{id})} \mathrm{Mod}(A^{(2n)}) \xleftarrow{(\mathrm{id}, \mathrm{id})} \mathrm{Mod}(A^{(n)}). \tag{7.12}$$

Since perverse sheaves form a stack (see [30, Propositions 10.2.7 and 10.2.9]), the category $\mathsf{Perv}^p(\mathbb{R}^3, L)$ is equivalent to the 2-limit of the diagram (7.11), and hence of the diagram (7.12). This implies the desired result. For details, see [4]. □

When combined with Theorem 7.7, Theorem 7.10 gives a description of the category $\mathsf{Perv}(\mathbb{R}^3, L)$ of perverse sheaves in terms of linear algebra data, similar in spirit to the original description of the category $\mathsf{Perv}(D, \{p_1, p_2, \dots, p_n\})$ given in [19].

8. Generalizations and further questions

In the GMV braid action, the group B_n acts on the generators $T_i = a_i a_i^* + e_0$ via the Artin representation (1.1). Thus, regarding the free group $\mathbb{F}_n$ as a category with a single object, we can regard the GMV action as an extension of the Artin action. In [52], Wada constructed several examples of braid group actions on $\mathbb{F}_n$ generalizing the classical Artin representation. Like the Artin representation, Wada's braid group actions are local and homogeneous, *i.e.* generated by a single cocartensian Yang–Baxter operator on $\mathbb{F}_1$. It is therefore natural to ask whether they admit extensions similar to the GMV extension.

Consider, for example, the following co-Cartesian Yang–Baxter operator constructed in [52]:

$$\sigma : \mathbb{F}_2 \to \mathbb{F}_2 \qquad x_1 \mapsto x_1^N x_2 x_1^{-N}, \qquad x_2 \mapsto x_1 \,,$$

where N is an arbitrary (fixed) integer.

This action does admit an extension similar to the GMV action. Indeed,

$$\sigma_k : \begin{cases} a_i \mapsto a_i & (i \neq k, k+1) \\ a_k \mapsto T_k^N a_{k+1} & \\ a_{k+1} \mapsto a_k & \\ a_i^* \mapsto a_i^* & (i \neq k, k+1) \\ a_k^* \mapsto a_{k+1}^* T_k^{-N} & \\ a_{k+1}^* \mapsto a_k^*. & \end{cases} \tag{8.1}$$

Note that, for $N = 1$, this is the original GMV action (1.5). Moreover, using a result of [8], one can show that the actions (8.1) are non-equivalent to each other for different N's; thus, for $N \neq 1$, (8.1) is a genuine generalization of the GMV action.

The elements $T_i^{\pm N}$ can be written in an alternative form involving the conjugate elements $\mu_i = e_i + a_i^* a_i \in \tilde{A}^{(n)}(i,i)$. (We recall that μ_i are no longer central elements in $\tilde{A}^{(n)}(i,i)$.) Indeed, by induction, one can show that

$$T_i^N = e_0 + a_i \, [N]_{\mu_i} \, a_i^* \qquad \text{for all } N \in \mathbb{Z}$$

where $[N]_\mu \in k$ are the "quantum integers" defined by

$$[N]_{\mu_i} = \frac{\mu_i^N - 1}{\mu_i - 1} := \begin{cases} e_i + \mu_i + \mu_i^2 + \ldots + \mu_i^{N-1} & \text{if } N > 0 \\ 0 & \text{if } N = 0 \\ -\mu_i^{-1} - \mu_i^{-2} - \ldots - \mu_i^N & \text{if } N < 0. \end{cases}$$

As in Section 4, we set $A_{ij} = -a_i^* a_j$ for all i, j. Then, we have the following formulas defining the braid group action on the restriction of the k-category $\tilde{A}$ to the vertices $\{1, \dots, r\}$:

$$\sigma_k : \begin{cases} A_{ki} \mapsto A_{k+1,i} - A_{k+1,k} \, [-N]_{\mu_k} \, A_{ki} & i \neq k, k+1 \\ A_{ik} \mapsto A_{i,k+1} - A_{ik} \, [N]_{\mu_k} \, A_{k,k+1} & i \neq k, k+1 \\ A_{k+1,i} \mapsto A_{ki} & i \neq k, k+1 \\ A_{i,k+1} \mapsto A_{ik} & i \neq k, k+1 \\ A_{k,k+1} \mapsto A_{k+1,k} \, \mu_k^{-N} & \\ A_{k+1,k} \mapsto \mu_k^N \, A_{k,k+1} & \\ A_{ij} \mapsto A_{ij} & i, j \neq k, k+1 \\ \mu_k \mapsto \mu_{k+1} & \\ \mu_{k+1} \mapsto \mu_k & \\ \mu_i \mapsto \mu_i & i \neq k, k+1. \end{cases}$$

Now, for $i \neq j$, define

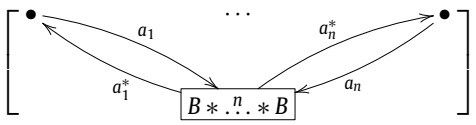
$$a_{ij} = \begin{cases} A_{ij} \, [N]_{\mu_j}, & i < j \\ A_{ij} \, [-N]_{\mu_j}, & i > j. \end{cases} \tag{8.2}$$

Then, the above action becomes

$$\sigma_k:\begin{cases} a_{ki}\mapsto a_{k+1,i}-a_{k+1,k}\,a_{ki} & i\neq k,k+1\\ a_{ik}\mapsto a_{i,k+1}-a_{ik}\,a_{k,k+1} & i<k\\ a_{ik}\mapsto a_{i,k+1}-a_{ik}\,\mu_k^N\,a_{k,k+1}\,\mu_{k+1}^{-N} & i>k+1\\ a_{k+1,i}\mapsto a_{ki} & i\neq k,k+1\\ a_{i,k+1}\mapsto a_{ik} & i\neq k,k+1\\ a_{k,k+1}\mapsto -a_{k+1,k} & \\ a_{k+1,k}\mapsto -\mu_k^N\,a_{k,k+1}\,\mu_{k+1}^{-N} & \\ a_{ij}\mapsto a_{ij} & i,j\neq k,k+1\\ \mu_k\mapsto \mu_{k+1} & \\ \mu_{k+1}\mapsto \mu_k & \\ \mu_i\mapsto \mu_i & i\neq k,k+1. \end{cases}$$

For $N=1$, this coincides with the 'fully noncommutative' action defined in [13] (see also [39, Appendix]).

In a different direction, one can also construct a large family of GMV-type braid actions by extending the family of generalized Artin actions found in [8]. Specifically, let $B\in \mathsf{Alg}_k$ be an associative algebra over k, and let $x,y\in B^\times$ be a pair of *invertible* and *commuting* elements. Let $\hat{A}^{(n)}$ be the k-category given by

$\hat{A}^{(n)} =$ 

which can be interpreted as an n-fold coproduct in the category Cat_k^* of (small) pointed k-categories. Then, one can check by a direct calculation that the following assignments define a braid group action on $\hat{A}^{(n)}$.

$$\sigma_k:\begin{cases} a_k\mapsto x_k a_{k+1}\\ a_k^*\mapsto a_{k+1}^* x_k^{-1}\\ b_k\mapsto x_k b_{k+1} x_k^{-1}\\ a_{k+1}\mapsto y_k a_k\\ a_{k+1}^*\mapsto a_k^* y_k^{-1}\\ b_{k+1}\mapsto y_k b_k y_k^{-1}\\ a_i\mapsto a_i & (i\neq k,k+1)\\ a_i^*\mapsto a_i^* & (i\neq k,k+1)\\ b_i\mapsto b_i & (i\neq k,k+1) \end{cases} \tag{8.3}$$

where b_i is the element $b\in B$ put in the i-th copy of B in $B^{(n)}:=B*.^n.*B$. Notice that, when $B=k[H]$ is the group algebra of a group H, and when $x=h\in H$ and $y=h^{-1}$, the braid action on $H^{(n)}\subset k[H]^{(n)}$ at the vertex 0 coincides with the action defined in [8].

Consider any ideal $I\subset \hat{A}(0,0)=B\langle aa^*\rangle$, and let $\hat{A}/I$ be the k-category obtained by quotienting $\hat{A}$ by the ideal generated by I. Then, for any element $f\in I$, we have

$$\sigma_k(f_k)=x_k f_{k+1} x_k^{-1}\qquad \sigma_k(f_{k+1})=y_k f_k y_k^{-1}\qquad \sigma_k(f_i=f_i \text{ if } i\neq k,k+1).$$

Therefore, the co-Cartesian Yang–Baxter operator corresponding to the above braid group action descends to the quotient $\hat{A}/I$. If we take $B=k[T^{\pm}]$, $x=T$ and $y=1$, and consider the ideal I generated by the element aa^*+1-T, then the resulting quotient $\hat{A}/I$, together with its corresponding co-Cartesian Yang–Baxter operator σ, is equivalent to the k-category $\tilde{A}$, together with the GMV operator, constructed in Section 4.

Theorem 8.4. *The braid group actions (8.1) and (8.3) are generated by Reidemeister operators in the category* dgCat_k^* *of pointed DG categories on objects* $\tilde{A}$ *and* $\hat{A}/I$*, respectively. These objects are pseudoflat with respect to Tabuada's model structure on* dgCat_k^**. Thus, the homotopy braid closure with respect to these operators gives link invariants that generalize the fully noncommutative link DG category* $\mathscr{A}_L$*.*

We conclude the paper with a few questions and remarks.

1. *Constructible sheaves and contact homology.* Recently, some interesting work has been done on the geometric side of contact homology relating it to constructible sheaves (see [15,40,47,45,46]). It would be interesting to understand our Theorem 7.10 in this geometric context and more generally, to clarify the meaning of our construction from Floer-theoretic and constructible sheaves point of view.

In more detail, the relation between Legendrian contact homology and constructible sheaves is based on a theorem of Nadler and Zaslow [34] (see also [33]) that, for any real analytic manifold M, establishes an equivalence between the derived category $D_c(M)$ of constructible sheaves on M and the derived Fukaya category $D\mathsf{Fuk}(T^*M)$ of the cotangent bundle T^*M of M. This equivalence of triangulated categories is induced by a quasi-equivalence of A_∞-categories $\mu : \mathsf{Sh}_c(M) \to \mathrm{Tw}\mathsf{Fuk}(T^*M)$, where $\mathsf{Sh}_c(M)$ is a DG category defined as the DG quotient of the (naive) DG category of constructible sheaves on M modulo acyclic complexes and $\mathrm{Tw}\mathsf{Fuk}(T^*M)$ is the A_∞-category of twisted complexes in the Fukaya category $\mathsf{Fuk}(T^*M)$. The functor μ can be viewed as a categorification of the classical characteristic cycle construction and is called the *microlocalization functor*.

Now, for any conical Lagrangian submanifold $\tilde{\Lambda} \subseteq T^*M$, the restriction of the microlocalization functor to the subcategory $\mathsf{Sh}_c(M)_{\tilde{\Lambda}} \subseteq \mathsf{Sh}_c(M)$ of constructible sheaves with singular support in $\tilde{\Lambda}$ gives a quasi-equivalence $\mu : \mathsf{Sh}_c(M)_{\tilde{\Lambda}} \xrightarrow{\sim} \mathrm{Tw}\mathsf{Fuk}(T^*M)_{\tilde{\Lambda}}$ onto the full subcategory $\mathrm{Tw}\mathsf{Fuk}(T^*M)_{\tilde{\Lambda}}$ of the twisted Fukaya category consisting of Lagrangians whose boundary at infinity lies in the boundary of $\tilde{\Lambda}$. Such a submanifold $\tilde{\Lambda}$ is determined by its intersection $\Lambda := \tilde{\Lambda} \cap ST^*M$ with the unit cotangent bundle of M; the bundle ST^*M has a natural contact structure, and Λ is a Legendrian submanifold of ST^*M. It turns out that the Legendrian contact homology (LHC) of the pair (ST^*M, Λ) is related to the Fukaya category $\mathrm{Tw}\mathsf{Fuk}(T^*M)_{\tilde{\Lambda}}$ and hence, via the microlocalization functor, to the sheaf category $\mathsf{Sh}_c(M)_{\tilde{\Lambda}}$. More precisely, it is expected that the complexes of constructible sheaves in $\mathsf{Sh}_c(M)_{\tilde{\Lambda}}$ determine augmentations of the Legendrian DGA of (ST^*M, Λ) via a geometric symplectic filling construction.

In the case of one-dimensional Legendrians, this relation has been worked out in detail in [40,47]. Specifically, if $M = \mathbb{R}^2$, then $ST^*\mathbb{R}^2 \cong \mathbb{R}^2 \times S^1$ contains an open contact submanifold $\mathbb{R}^3 \subset \mathbb{R}^2 \times S^1$. Hence, any Legendrian link $L \subset \mathbb{R}^3$ can be considered as a Legendrian submanifold in $ST^*\mathbb{R}^2$. In [40], for a Legendrian link $L \subset \mathbb{R}^3$, the authors construct a (unital) A_∞-category $\mathsf{Aug}_+(L)$, whose objects are augmentations of the Chekanov–Eliashberg DG algebra of L, and show that there is an A_∞-equivalence $\mathsf{Aug}_+(L) \simeq \mathcal{C}_1(L)$, where $\mathcal{C}_1(L)$ is the full subcategory of $\mathsf{Sh}_c(\mathbb{R}^2)_{\tilde{L}}$ consisting of sheaves of 'microlocal rank one along the link L'.

A possible extension of this equivalence to higher dimensions (specifically, to the case of knot contact homology and knot DGA in $\mathbb{R}^3$) has been recently proposed by V. Shende et al. (see, e.g., [45, Section 4], [15, Section 6.6], [46, Section 6.5]). In this case, $M = \mathbb{R}^3$ and the Legendrian $\Lambda \subset ST^*M$ is given by the unit conormal bundle $\Lambda_L := ST^*_L\mathbb{R}^3$ associated with a link $L \subset \mathbb{R}^3$. It is interesting that the support condition defining the subcategory $\mathsf{Sh}_c(\mathbb{R}^3)_{\tilde{\Lambda}} \subset \mathsf{Sh}_c(\mathbb{R}^3)$ coincides with the constructibility condition in our Theorem 7.10, and some geometric arguments suggest that there is a relation between this sheaf category and knot contact homology (see [15, Section 6.6]). Whether this geometric relation can be used to prove the result of Theorem 7.10 is not clear to us at the moment: *a priori*, the equivalence of categories in Theorem 7.10 originates from a different direction. In fact, there are three approaches to knot contact homology:

(1) combinatorial knot contact homology,
(2) Legendrian contact homology of the pair $\Lambda_L \subset ST^*\mathbb{R}^3$,
(3) constructible sheaves on $\mathbb{R}^3$ with singular support in $\tilde{\Lambda}_L$.

The papers [13,14] establish an equivalence between (1) and (2) by identifying the generators of the combinatorial knot DGA with Reeb cords and defining the differentials in terms of pseudoholomorphic disks. The geometric approach of [15, 45,46] relates (2) and (3) via the geometry of symplectic fillings. Our result, Theorem 7.10, establishes the relation between (1) and (3) by appealing to the classical description of perverse sheaves on the disk in terms of nearby and vanishing cycle functors [19] and using an algebraic 'gluing' construction (homotopy braid closure). It would be interesting to see whether these approaches actually 'agree'; in particular, can one prove Theorem 7.10 using the approach of [15,45,46]?

2. *Categorification of the link DG category.* There seems to be a natural way to categorify the DG category $\tilde{\mathscr{A}}$, using the notion of 'perverse schobers' introduced in [28] (see also [29]). First, one can construct a (higher) category $\mathscr{C}$ of $(\infty, 2)$-categories that includes the category dgCat_k as an object (see [49,17]). In $\mathscr{C}$, one can find an object $\mathscr{A}^{(n)}$ such that the category of 2-representations of $\mathscr{A}^{(n)}$, *i.e.* an appropriately defined internal hom $\underline{\mathrm{Hom}}_{\mathscr{C}}(\mathscr{A}^{(n)}, \mathsf{dgCat}_k)$, is equivalent to the (higher) category of perverse schobers on the disk with n marked points. Then, there should exist a B_n-action on $\mathscr{A}^{(n)}$ for all $n \geq 1$ that would allow us to take the homotopy braid closure. The result should be an object in $\mathscr{C}$ (*i.e.*, an $(\infty, 2)$-category $\mathbb{A}$), whose category $\underline{\mathrm{Hom}}_{\mathscr{C}}(\mathbb{A}, \mathsf{dgCat}_k)$ of 2-representations is equivalent to a category of 'perverse schobers on $\mathbb{R}^3$ singular along a link'.

3. *Yang–Baxter operators related to coherent sheaves.* Many interesting examples of braid group actions related to coherent sheaves have been constructed in the literature (see, e.g., [43,42,1] and references therein). It would be interesting to look at these examples in relation to the examples studied in the present paper and clarify the relations between the corresponding link invariants.

Acknowledgements

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