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Partial differential equations

On the minimizer of a renormalized energy related to the Ginzburg–Landau model

Sur la minimisation de l'énergie renormalisée reliée au modèle de Ginzburg–Landau

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ABSTRACT

We study the configuration of vortices that minimize a renormalized energy related to the Ginzburg–Landau model. Among all the Bravais lattices, we prove that the triangular lattice minimizes this renormalized energy.

RÉSUMÉ

Nous étudions les structures des vortex qui minimisent l'énergie renormalisée reliée au modèle de Ginzburg–Landau. Parmi tous les réseaux de Bravais, nous prouvons que le réseaux triangulaire minimise cette énergie renormalisée.

1. Introduction

For type-II superconductors, A. Abrikosov [1] predicted that the triangular lattice, now called "Abrikosov lattice", would appear. There are some rigorous mathematical results related to this phenomenon, for example [2,3,5,9]. In [9], E. Sandier and S. Serfaty have proven that the vortices in minimizers of the Ginzburg–Landau energy, blown-up at a suitable scale, converges to minimizers of a "Coulombian Renormalized Energy", and in the periodic case, the triangular lattice minimizes this renormalized energy. In this paper, we consider another renormalized energy for a periodic Ginzburg–Landau energy introduced in [4] and prove that the triangular lattice is the unique minimizer of this renormalized energy among all the Bravais lattices. One can refer to [6] for a similar work that describes di-block copolymers, and [8] for a related work on the determinants of Laplacians (see Corollary 1.b of [8]).

Let $\mathcal{L} = \{\mathbb{Z}\vec{u} \oplus \mathbb{Z}\vec{v} \mid \det(\vec{u}, \vec{v}) = 1\}$. For $\Lambda \in \mathcal{L}$, we define $L = \mathbb{R}^2/\Lambda$, hence $|L| = 1$. We introduce the renormalized energy W which is defined in [4] over $\mathcal{L}$ as follows

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$$W(n,\Lambda)=\lim_{\varepsilon\to 0}\left(\pi n\log\varepsilon+\frac{1}{2}\int\limits_{L\setminus\bigcup_{i=1}^{n}B(p_i,\varepsilon)}|\nabla h|^2+h^2\right),$$

where $\{p_i\}_{i=1}^n$ are n points in L, and h satisfies

$$\begin{cases}-\Delta h+h=2\pi\sum_{i=1}^{n}\delta_{p_i} & \text{in } L\\ \text{periodic boundary conditions.}\end{cases}\tag{1}$$

In fact, this energy is a renormalized energy for the Ginzburg–Landau energy in the periodic setting. In the case of $n=1$, i.e. among the Bravais lattices, we prove Theorem 1.1.

Theorem 1.1. *The triangular lattice, modulo rotations, is the unique minimizer of W among all Bravais lattices.*

In the proof of this theorem, we use a technique which has already been used in [9] to rewrite the renormalized energy W in an explicit formula related to Jacobi's Theta Function, then by applying a result of H.L. Montgomery [7], we complete the proof.

2. Proof of Theorem 1.1

We follow the idea of [9] to rewrite the renormalized energy W in an explicit formula. When $n=1$,

$$W(\Lambda)=\lim_{\varepsilon\to 0}\left(\pi\log\varepsilon+\frac{1}{2}\int\limits_{L\setminus B(0,\varepsilon)}|\nabla h|^2+h^2\right),$$

where h satisfies

$$\begin{cases}-\Delta h+h=2\pi\delta_0 & \text{in } L\\ \text{periodic boundary conditions.}\end{cases}\tag{2}$$

Lemma 2.1. *For any $\Lambda\in\mathcal{L}$, we have:*

$$W(\Lambda)=\pi\lim_{x\to 0}\big(h(x)+\log|x|\big).$$

Proof. We have

$$\pi\log\varepsilon+\frac{1}{2}\int\limits_{L\setminus B(0,\varepsilon)}|\nabla h|^2+h^2=\pi\log\varepsilon-\frac{1}{2}\int\limits_{\partial B(0,\varepsilon)}\frac{\partial h}{\partial\nu}\cdot h,$$

where ν is the outer-pointing unit normal vector with respect to the corresponding boundary. In fact, $h(x)=-\log|x|+g(x)$, where $g(x)$ is C^1 near origin. So

$$\left.\frac{\partial h}{\partial\nu}\right|_{\partial B(0,\varepsilon)}=-\frac{1}{\varepsilon}+\left.\frac{\partial g}{\partial\nu}\right|_{\partial B(0,\varepsilon)}.$$

Therefore,

$$W(\Lambda)=\lim_{x\to 0}\big(\pi\log|x|+\pi h(x)+O\big(|x|\cdot\log|x|\big)\big)=\pi\lim_{x\to 0}\big(h(x)+\log|x|\big).\qquad\square$$

Next we prove an important lemma by following the same method in [9].

Lemma 2.2. *There exists a constant $C_0\in\mathbb{R}$, such that for any $\Lambda\in\mathcal{L}$, we have*

$$W(\Lambda)=C_0+\pi\lim_{x\to 0}\left(\zeta_{\Lambda^*}(x)-\int\limits_{\mathbb{R}^2}\frac{2\pi}{1+4\pi^2|y|^{2+x}}\mathrm{d}y\right),$$

where Λ^* is the dual lattice of Λ, i.e. the set of vectors q such that $q\cdot p\in\mathbb{Z}$ for every $p\in\Lambda$, and $\zeta_{\Lambda^*}(x)=\sum_{p\in\Lambda^*}\frac{2\pi}{1+4\pi^2|p|^{2+x}}$.

Proof. We already have:

$$W(\Lambda) = \pi \lim_{x\to 0}\big(h(x) + \log|x|\big).$$

We introduce the Green function $G(x) \in L^2(\mathbb{R}^2)$, which is the solution of $-\Delta G + G = 2\pi\delta_0$ in $\mathbb{R}^2$, and by the periodic boundary conditions, we can consider the function $h(x)$ as a function in $\mathbb{R}^2$, i.e. the solution of

$$-\Delta h_\Lambda + h_\Lambda = 2\pi \sum_{p\in\Lambda} \delta_p.$$

Then we can write:

$$h_\Lambda(x) + \log|x| = G(x) + \log|x| + u_\Lambda(x),$$

where $u_\Lambda(x) = h_\Lambda(x) - G(x)$ and it depends on lattice Λ. It is well known that $h_\Lambda(x) + \log|x|$, $G(x) + \log|x|$, $u_\Lambda(x)$ are C^1 near 0. Note that $G(x) + \log|x|$ is independent of lattice Λ, so

$$W(\Lambda) = \pi \lim_{x\to 0}\big(h_\Lambda(x) + \log|x|\big) = C_0 + \pi \cdot u_\Lambda(0),$$

where $C_0 = \lim_{x\to 0} G(x) + \log|x|$.

Denote by $\varphi(x) = (2\pi)^{-1}\mathrm{e}^{-|x|^2/2}$ the Gaussian distribution in $\mathbb{R}^2$ and $\varphi_n(x) = n^2\varphi(nx)$ for any $n \in \mathbb{N}$, so $\{\varphi_n(x)\}_n$ is an approximation of the Dirac mass. Since $u_\Lambda(x)$ is C^1 near 0, we have:

$$u_\Lambda(0) = \lim_{n\to\infty} w(n,\Lambda),$$

where

$$w(n,\Lambda) = \int_{\mathbb{R}^2} \varphi_n(x) u_\Lambda(x)\mathrm{d}x = \int_{\mathbb{R}^2} \hat{\varphi}_n(\xi)\hat{u}_\Lambda(\xi)\mathrm{d}\xi.$$

We know that $\hat{\varphi}_n(\xi) = \mathrm{e}^{-2\pi^2|\xi|^2/n^2}$, and $\hat{u}_\Lambda(\xi) = \hat{h}(\xi) - \hat{G}(\xi)$, where $\hat{h}(\xi) = \frac{2\pi\sum_{p\in\Lambda^*}\delta_p(\xi)}{4\pi^2|\xi|^2+1}$ (2π comes from the fact that $|L| = 1$) and $\hat{G}(\xi) = \frac{2\pi}{4\pi^2|\xi|^2+1}$. Hence

$$w(n,\Lambda) = 2\pi\left(\sum_{p\in\Lambda^*} \frac{\mathrm{e}^{-2\pi^2|p|^2/n^2}}{4\pi^2|p|^2+1} - \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-2\pi^2|y|^2/n^2}}{4\pi^2|y|^2+1}\mathrm{d}y\right).$$

We claim that

$$\lim_{n\to\infty} w(n,\Lambda) = \lim_{x\to 0^+} v(x,\Lambda),$$

where $v(x,\Lambda) = 2\pi\big(\sum_{p\in\Lambda^*} \frac{1}{4\pi^2|p|^{2+x}+1} - \int_{\mathbb{R}^2} \frac{1}{4\pi^2|y|^{2+x}+1}\mathrm{d}y\big)$, $x > 0$.

In fact, for any $p \in \Lambda^*$, denote by K_p the Voronoi cell centered at p, i.e. the region in $\mathbb{R}^2$ consisting of all the points closer to p than to any other point in Λ^*. Note that K_p is periodic due to the periodicity of lattice Λ^* and $|K_p| = 1$. Denote by $\mathbf{1}_{K_p}$ the characteristic function with respect to K_p, then we have

$$w(n,\Lambda) = 2\pi \int_{\mathbb{R}^2} \sum_{p\in\Lambda^*} \mathbf{1}_{K_p} \cdot \left(\frac{\mathrm{e}^{-2\pi^2|p|^2/n^2}}{4\pi^2|p|^2+1} - \frac{\mathrm{e}^{-2\pi^2|y|^2/n^2}}{4\pi^2|y|^2+1}\right)\mathrm{d}y.$$

By applying the mean value theorem to $\frac{\mathrm{e}^{-2\pi^2|p|^2/n^2}}{4\pi^2|p|^2+1} - \frac{\mathrm{e}^{-2\pi^2|y|^2/n^2}}{4\pi^2|y|^2+1}$, we get a bound for the integrand function

$$\left|\sum_{p\in\Lambda^*} \mathbf{1}_{K_p} \cdot \left(\frac{\mathrm{e}^{-2\pi^2|p|^2/n^2}}{4\pi^2|p|^2+1} - \frac{\mathrm{e}^{-2\pi^2|y|^2/n^2}}{4\pi^2|y|^2+1}\right)\right| \le C\frac{1}{|y|^3+1},$$

where the constant C is independent of n. The function at the right hand side is an integrable function over the whole plane. Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to\infty} w(n,\Lambda) = 2\pi \int_{\mathbb{R}^2} \sum_{p\in\Lambda^*} \mathbf{1}_{K_p} \cdot \left(\frac{1}{4\pi^2|p|^2+1} - \frac{1}{4\pi^2|y|^2+1}\right)\mathrm{d}y.$$

Similarly, we have

$$\lim_{x\to 0^+} v(x,\Lambda) = 2\pi \int\limits_{\mathbb{R}^2} \sum_{p\in\Lambda^*} \mathbf{1}_{K_p}\cdot\left(\frac{1}{4\pi^2|p|^2+1}-\frac{1}{4\pi^2|y|^2+1}\right)\mathrm{d}y.$$

By combining the results above, we prove the lemma. □

Now we consider the term:

$$\zeta_{\Lambda^*}(x) = \sum_{p\in\Lambda^*}\frac{2\pi}{4\pi^2|p|^{2+x}+1}.$$

Let $\zeta^0_{\Lambda^*}(x)=\sum_{p\in\Lambda^*\setminus\{0\}}\frac{2\pi}{4\pi^2|p|^{2+x}}$, we can split $\zeta_{\Lambda^*}(x)$ as follows,

$$\begin{aligned}\zeta_{\Lambda^*}(x) &= 2\pi+\zeta^0_{\Lambda^*}(x)-2\pi\sum_{p\in\Lambda^*\setminus\{0\}}\frac{1}{4\pi^2|p|^{2+x}\cdot(4\pi^2|p|^{2+x}+1)}\\ &= 2\pi+\zeta^0_{\Lambda^*}(x)-2\pi\sum_{p\in\Lambda^*\setminus\{0\}}\frac{1}{4\pi^2|p|^{2}\cdot(4\pi^2|p|^{2}+1)}+o(1).\end{aligned}$$

Note here $o(1)$ means $o(1)\to 0$ as $x\to 0$ for any fixed $\Lambda\in\mathcal{L}$, but the convergence is not uniform w.r.t. Λ.

We will consider $\zeta^0_{\Lambda^*}(x)-2\pi\sum_{p\in\Lambda^*\setminus\{0\}}\frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)}$ together.

If $4\pi^2|p|^2>1$, we can have a series expansion of the second term. We can do this at least in a neighborhood of the triangular lattice, because the length of the edge is $\sqrt{2/\sqrt{3}}>1$.

$$\sum_{p\in\Lambda^*\setminus\{0\}}\frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)}=\sum_{p\in\Lambda^*\setminus\{0\}}\frac{1}{(4\pi^2|p|^2)^2\cdot(1+(4\pi^2|p|^2)^{-1})}=\sum_{p\in\Lambda^*\setminus\{0\}}\sum_{n=2}^{\infty}\frac{(-1)^n}{(4\pi^2|p|^2)^n}.$$

Since the summation $\sum_{p\in\Lambda^*\setminus\{0\}}\sum_{n=2}^{\infty}\frac{(-1)^n}{(4\pi^2|p|^2)^n}$ converges absolutely, we can change the order of the summation.

$$\sum_{p\in\Lambda^*\setminus\{0\}}\sum_{n=2}^{\infty}\frac{(-1)^n}{(4\pi^2|p|^2)^n}=\sum_{n=2}^{\infty}\sum_{p\in\Lambda^*\setminus\{0\}}\frac{(-1)^n}{(4\pi^2|p|^2)^n}.$$

We write $\sum_{n=2}^{\infty}\sum_{p\in\Lambda^*\setminus\{0\}}\frac{(-1)^n}{(4\pi^2|p|^2)^n}=\sum_{n=2}^{\infty}(-1)^n g_{n,\Lambda^*}$ for convenience, where $g_{n,\Lambda^*}=\sum_{p\in\Lambda^*\setminus\{0\}}\frac{1}{(4\pi^2|p|^2)^n}$.

Let $s=1+\frac{x}{2}, x>0$, then by using a result in [7], we have:

$$\frac{1}{2\pi}\cdot 4\pi^2\cdot\zeta^0_{\Lambda^*}(x)\cdot 2^s\cdot\Gamma(s)\cdot(2\pi)^{-s}=\frac{1}{s-1}-\frac{1}{s}+\int\limits_1^{+\infty}\big(\theta_{\Lambda^*}(\alpha)-1\big)\big(\alpha^s+\alpha^{1-s}\big)\frac{\mathrm{d}\alpha}{\alpha},$$

where $\theta_{\Lambda^*}(\alpha)=\sum_{p\in\Lambda^*}\mathrm{e}^{-\pi\alpha|p|^2}$.

Similarly, we have

$$\big(4\pi^2\big)^n\cdot g_{n,\Lambda^*}(x)\cdot 2^n\cdot\Gamma(n)\cdot(2\pi)^{-n}=\frac{1}{n-1}-\frac{1}{n}+\int\limits_1^{+\infty}\big(\theta_{\Lambda^*}(\alpha)-1\big)\big(\alpha^n+\alpha^{1-n}\big)\frac{\mathrm{d}\alpha}{\alpha}.$$

Therefore, we have:

$$\begin{aligned}\zeta_{\Lambda^*}(x) &= 2\pi+\zeta^0_{\Lambda^*}(x)-2\pi\sum_{p\in\Lambda^*\setminus\{0\}}\frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)}+o(1)\\ &= 2\pi+\frac{\pi^{s-1}}{2\Gamma(s)}\left(\frac{1}{s-1}-\frac{1}{s}\right)+\sum_{n=2}^{\infty}2\pi\frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}\left(\frac{1}{n-1}-\frac{1}{n}\right)\\ &\quad+2\pi\int\limits_1^{+\infty}\big(\theta_{\Lambda^*}(\alpha)-1\big)\cdot\frac{\pi^{s-1}}{4\pi\,\Gamma(s)}\cdot\big(\alpha^s+\alpha^{1-s}\big)\frac{\mathrm{d}\alpha}{\alpha}\\ &\quad+\sum_{n=2}^{\infty}2\pi\int\limits_1^{+\infty}\big(\theta_{\Lambda^*}(\alpha)-1\big)\frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}\big(\alpha^n+\alpha^{1-n}\big)\frac{\mathrm{d}\alpha}{\alpha}+o(1)\end{aligned}$$

$$= 2\pi + f(x) + c_0 + 2\pi \int_1^{+\infty} \big(\theta_{\Lambda^*}(\alpha) - 1\big) \cdot I(x,\alpha) \frac{\mathrm{d}\alpha}{\alpha} + o(1),$$

where $f(x) = \frac{\pi^{s-1}}{2\Gamma(s)}(\frac{1}{s-1} - \frac{1}{s})$, $c_0 = \sum_{n=2}^{\infty} 2\pi \frac{(-1)^{n-1}}{(4\pi)^n \Gamma(n)}(\frac{1}{n-1} - \frac{1}{n})$ and $I(x,\alpha) = \frac{\pi^{s-1}}{4\pi\Gamma(s)} \cdot (\alpha^s + \alpha^{1-s}) + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}(\alpha^n + \alpha^{1-n})$.

For any α fixed, we have

$$\begin{aligned} & I(x,\alpha) \\ &= \left(\frac{\pi^{s-1}}{4\pi\Gamma(s)} \cdot \alpha^s + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}\alpha^n\right) + \left(\frac{\pi^{s-1}}{4\pi\Gamma(s)}\alpha^{1-s} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(4\pi)^n\Gamma(n)}\alpha^{1-n}\right) \\ &= \frac{\alpha}{4\pi}\left(\frac{(\pi\alpha)^{s-1}}{\Gamma(s)} + \mathrm{e}^{-\frac{\alpha}{4\pi}} - 1\right) + \frac{1}{4\pi}\left(\frac{\pi^{s-1}}{\Gamma(s)}\alpha^{1-s} + \mathrm{e}^{-\frac{1}{4\pi\alpha}} - 1\right). \end{aligned}$$

$\Gamma(s)$ is convex in $[1,2]$, and $\Gamma(1) = \Gamma(2) = 1$, so for $s \in [1,2]$, $\Gamma(s) \le 1$, while $(\pi\alpha)^{s-1} \ge 1$, for $\alpha \ge 1$, $s \in [1,2]$. Hence

$$\frac{(\pi\alpha)^{s-1}}{\Gamma(s)} - 1 \ge 0.$$

Similarly, we have $\frac{\pi^{s-1}}{\Gamma(s)}\alpha^{1-s} \ge \alpha^{1-s}$, and the fact that $1 - \mathrm{e}^{-\frac{1}{4\pi\alpha}} < \frac{1}{4\pi\alpha}$ implies that $\frac{\pi^{s-1}}{\Gamma(s)}\alpha^{1-s} + \mathrm{e}^{-\frac{1}{4\pi\alpha}} - 1 > 0$ for $\alpha \ge 1$, $s \in [1,2]$.

By combining the results above, we have $I(x,\alpha) > 0$ for $\alpha \ge 1$, $s \in [1,2]$.

Next we will prove that

$$\zeta_{\Lambda^*}(x) = 2\pi + f(x) + c_0 + 2\pi \int_1^{+\infty} \big(\theta_{\Lambda^*}(\alpha) - 1\big) I(x,\alpha) \frac{\mathrm{d}\alpha}{\alpha} + o(1)$$

is true not just for lattices in the neighborhood of a triangular lattice, but for all Bravais lattices with area 1. We claim that both

$$f_1(\Lambda) = \zeta^0_{\Lambda^*}(x) - 2\pi \sum_{p \in \Lambda^* \setminus \{0\}} \frac{1}{4\pi^2|p|^2 \cdot (4\pi^2|p|^2 + 1)}$$

and

$$f_2(\Lambda) = f(x) + c_0 + 2\pi \int_1^{+\infty} \big(\theta_{\Lambda^*}(\alpha) - 1\big) I(x,\alpha) \frac{\mathrm{d}\alpha}{\alpha}$$

are analytic w.r.t. lattices. It means that if we denote by $\vec{u} = (a,0), a > 0$, $\vec{v} = (b,c) = (b, 1/a)$ the vectors that generate lattice Λ^*, the two functions are analytic w.r.t. $\vec{u}, \vec{v}$, i.e. a, b. If $p = m\vec{u} + n\vec{v} = (ma + nb, nc)$, then $|p|^2 = (ma + nb)^2 + n^2c^2$. For $\sum_{p \in \Lambda^* \setminus \{0\}} \frac{1}{4\pi^2|p|^2 \cdot (4\pi^2|p|^2+1)}$, at $(a_0, b_0, c_0), a_0 > 0$, we have

$$\begin{aligned} & \sum_{p \in \Lambda^* \setminus \{0\}} \frac{1}{4\pi^2|p|^2 \cdot (4\pi^2|p|^2 + 1)} \\ &= \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{4\pi^2[(ma + nb)^2 + n^2c^2] \cdot [4\pi^2((ma + nb)^2 + n^2c^2) + 1]} \\ &= \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{4\pi^2[(ma_0 + nb_0)^2 + n^2c_0^2 + R(a - a_0, b - b_0, c - c_0)]} \\ &\quad \cdot \frac{1}{4\pi^2[(ma_0 + nb_0)^2 + n^2c_0^2 + R(a - a_0, b - b_0, c - c_0)] + 1} \\ &= \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{[4\pi^2(m^2a_0^2 + 2a_0b_0mn + n^2(b_0^2 + c_0^2))] \cdot [4\pi^2(m^2a_0^2 + 2a_0b_0mn + n^2(b_0^2 + c_0^2)) + 1]} \\ &\quad \cdot \frac{1}{1 + \frac{R(a-a_0,b-b_0,c-c_0)}{m^2a_0^2 + 2a_0b_0mn + n^2(b_0^2+c_0^2)}} \cdot \frac{1}{1 + \frac{4\pi^2 R(a-a_0,b-b_0,c-c_0)}{4\pi^2(m^2a_0^2+2a_0b_0mn+n^2(b_0^2+c_0^2))+1}}. \end{aligned}$$

We obtain a series expansion of the formula above by expanding the function $\frac{1}{1+x}$ at 0 and rearranging the terms since that the coefficients converge absolutely. Take a function composition with $c = 1/a$, we obtain that

$$\sum_{p\in\Lambda^*\setminus\{0\}} \frac{1}{4\pi^2|p|^2\cdot(4\pi^2|p|^2+1)}$$

is an analytic w.r.t. lattice.

Similarly, the function $\zeta^0_{\Lambda^*}(x)$ is an analytic w.r.t. lattice.

For the function $f_2(\Lambda)$, $f(x)+c_0$ is independent of the lattice, so we only need to prove that $2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha)-1)I(x,\alpha)\frac{d\alpha}{\alpha}$ is an analytic w.r.t. lattice. The series is a positive series, it converges absolutely. The function $\theta_{\Lambda^*}(\alpha)-1$ is a positive series and converges absolutely for any α, and each term in the series is analytic, so we rewrite the function $2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha)-1)I(x,\alpha)\frac{d\alpha}{\alpha}$ in the form of a series w.r.t. lattice. Therefore, the function $f(x)+c_0+2\pi\int_1^{+\infty}(\theta_{\Lambda^*}(\alpha)-1)I(x,\alpha)\frac{d\alpha}{\alpha}$ is an analytic w.r.t. lattice.

Now we know that the functions $f_1(\Lambda)$ and $f_2(\Lambda)$ are analytic, and $f_1=f_2$ in the neighborhood of a triangular lattice, so $f_1\equiv f_2$ for all lattices with fixed area 1.

We use a result due to Montgomery.

Theorem 2.1. *(See [7].) For any $\alpha>0$,*

$$\theta_f(\alpha)\ge\theta_h(\alpha),$$

where $f(\mathbf{u})=f(u_1,u_2)=au_1^2+bu_1u_2+cu_2^2$ is a positive definite binary quadratic form with real coefficient and discriminant $b^2-4ac=-1$, and $h(\mathbf{u})=\frac{1}{\sqrt{3}}(u_1^2+u_1u_2+u_2^2)$. If there is an $\alpha>0$ such that $\theta_f(\alpha)=\theta_h(\alpha)$, then f and h are equivalent forms and $\theta_f(\alpha)\equiv\theta_h(\alpha)$.

From the theorem above, we know that the minimum of the Jacobi Theta function θ over $\mathcal{L}$ (recall that $\mathcal{L}$ is the set of all Bravais lattices with area 1) is uniquely achieved by Λ_0^*, $\Lambda_0=\sqrt{\frac{2}{\sqrt{3}}}(\mathbb{Z}(1,0)\oplus\mathbb{Z}(1/2,\sqrt{3}/2))$. Denote by Λ a Bravais lattice, then apply Lebesgue's dominated convergence theorem, we have:

$$\begin{aligned} W(\Lambda)-W(\Lambda_0) &= \pi\lim_{x\to0}\big(\zeta_{\Lambda^*}(x)-\zeta_{\Lambda_0^*}(x)\big)=\pi\lim_{x\to0}2\pi\int_1^{+\infty}(\theta_{\Lambda^*}-\theta_{\Lambda_0^*})I(x,\alpha)\frac{d\alpha}{\alpha}\\ &=2\pi^2\int_1^{+\infty}(\theta_{\Lambda^*}-\theta_{\Lambda_0^*})I(0,\alpha)\frac{d\alpha}{\alpha}. \end{aligned}$$

By using Theorem 1 of [7] and the fact that $I(0,\alpha)>0$, we have $W(\Lambda)\ge W(\Lambda_0)$ for all lattices $\Lambda\in\mathcal{L}$, and the equality holds if and only if $\Lambda=\Lambda_0$. Therefore the triangular lattice is the unique minimizer of energy $W(\Lambda)$.

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