C. R. Acad. Sci. Paris, Ser. I 353 (2015) 209–214

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Mathematical analysis/Functional analysis

# Functions of perturbed noncommuting self-adjoint operators

## *Fonctions d'opérateurs perturbés auto-adjoints ne commutant pas*

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ARTICLE INFO

*Article history:*
Received 7 November 2014
Accepted after revision 19 December 2014
Available online 29 January 2015

Presented by Gilles Pisier

ABSTRACT

We consider functions $f(A, B)$ of noncommuting self-adjoint operators $A$ and $B$ that can be defined in terms of double operator integrals. We prove that if $f$ belongs to the Besov class $B^1_{\infty,1}(\mathbb{R}^2)$, then we have the following Lipschitz-type estimate in the trace norm: $\|f(A_1, B_1) - f(A_2, B_2)\|_{\boldsymbol{S}_1} \le \text{const}(\|A_1 - A_2\|_{\boldsymbol{S}_1} + \|B_1 - B_2\|_{\boldsymbol{S}_1})$. However, the condition $f \in B^1_{\infty,1}(\mathbb{R}^2)$ does not imply the Lipschitz-type estimate in the operator norm.



RÉSUMÉ

Nous considérons les fonctions $f(A, B)$ d'opérateurs auto-adjoints $A$ et $B$ qui ne commutent pas. De telles fonctions peuvent être définies en termes d'intégrales doubles opératorielles. Pour $f$ dans l'espace de Besov $B^1_{\infty,1}(\mathbb{R}^2)$, nous obtenons l'estimation lipschitzienne en norme trace : $\|f(A_1, B_1) - f(A_2, B_2)\|_{\boldsymbol{S}_1} \le \text{const}(\|A_1 - A_2\|_{\boldsymbol{S}_1} + \|B_1 - B_2\|_{\boldsymbol{S}_1})$. Par ailleurs, la condition $f \in B^1_{\infty,1}(\mathbb{R}^2)$ n'implique pas l'estimation lipschitzienne en norme opératorielle.



### Version française abrégée

Il est bien connu (voir [3]) qu'une fonction lipschitzienne sur $\mathbb{R}$ n'est pas nécessairement *lipschitzienne opératorielle*, c'est-à-dire que l'inégalité $\|f(A) - f(B)\| \le \text{const}\,\|A - B\|$ pour les opérateurs auto-adjoints $A$ et $B$ peut être fausse. Dans [7] et [8], des conditions nécessaires et des conditions suffisantes sont données pour qu'une fonction $f$ soit lipschitzienne opératorielle. En particulier, il est démontré dans [7] et [8] que si $f$ appartient à l'espace de Besov $B^1_{\infty 1}(\mathbb{R})$, alors $f$ est lipschitzienne opératorielle. Il est aussi bien connu qu'une fonction $f$ est lipschitzienne opératorielle si et seulement si la condition $A - B$ appartient à la classe $\boldsymbol{S}_1$ (classe trace) implique que $f(A) - f(B) \in \boldsymbol{S}_1$ et $\|f(A) - f(B)\|_{\boldsymbol{S}_1} \le \text{const}\,\|A - B\|_{\boldsymbol{S}_1}$.

Par ailleurs, il est démontré dans [1] que, si $f$ est une fonction hölderienne d'ordre $\alpha$, $0 < \alpha < 1$, alors $\|f(A) - f(B)\| \le \text{const}\,\|A - B\|^\alpha$ pour les opérateurs auto-adjoints $A$ et $B$.

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http://dx.doi.org/10.1016/j.crma.2014.12.005
1631-073X/

Les résultats ci-dessus ont été géneralisés dans [2] au cas de fonctions d'opérateurs normaux est dans [5] au cas de fonctions de $n$-uplets d'opérateurs auto-adjoints commutants.

Dans cette note, nous considérons les fonctions $f(A,B)$ d'opérateurs auto-adjoints $A$ et $B$ qui ne commutent pas forcément. Si une fonction $f$ sur $\mathbb{R}^2$ est un multiplicateurs de Schur, on définit $f(A,B)$ comme l'intégrale double opératorielle

$$f(A,B) = \iint f(x,y)\,\mathrm{d}E_A(x)\,\mathrm{d}E_B(y)$$

où $E_A$ et $E_B$ sont les mesures spectrales de $A$ et de $B$ (voir [1] pour des informations sur les multiplicateurs de Schur et sur les intégrales doubles opératorielles).

Nous démontrons que si $f \in B^1_{\infty 1}(\mathbb{R}^2)$, alors l'inégalité suivante du type lipschitzenne en norme $\boldsymbol{S}_1$ est vraie :

$$\big\| f(A_1,B_1) - f(A_2,B_2)\big\|_{\boldsymbol{S}_1} \le \mathrm{const}\, \|f\|_{B^1_{\infty,1}} \big(\|A_1 - A_2\|_{\boldsymbol{S}_1} + \|B_1 - B_2\|_{\boldsymbol{S}_1}\big),$$

pour tous les opérateurs $A_1$, $A_2$, $B_1$ et $B_2$ auto-adjoints tels que $A_1 - A_2 \in \boldsymbol{S}_1$ et $B_1 - B_2 \in \boldsymbol{S}_1$. Pour démontrer l'inégalité ci-dessus, nous utilisons la représentation suivante en termes d'intégrale triple opératorielle :

$$f(A_1,B) - f(A_2,B) = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \frac{f(x_1,y) - f(x_2,y)}{x_1 - x_2}\,\mathrm{d}E_{A_1}(x_1)(A_1 - A_2)\,\mathrm{d}E_{A_2}(x_2)\,\mathrm{d}E_B(y).$$

L'intégrale triple opératorielle est bien définie parce que la différence divisée appartient au produit tensoriel $L^\infty(\mathbb{R}) \otimes_{\mathrm{h}} L^\infty(\mathbb{R}) \otimes^{\mathrm{h}} L^\infty(\mathbb{R})$ (voir la définition dans la version anglaise). Plus précisément, si $f$ est une fonction bornée sur $\mathbb{R}^2$ dont la transformée de Fourier a un support dans le disque unité, nous obtenons la représentation tensorielle suivante :

$$\frac{f(x_1,y) - f(x_2,y)}{x_1 - x_2} = \sum_{j,k\in\mathbb{Z}} \frac{\sin(x_1 - j\pi)}{x_1 - j\pi} \cdot \frac{\sin(x_2 - k\pi)}{x_2 - k\pi} \cdot \frac{f(j\pi,y) - f(k\pi,y)}{j\pi - k\pi}.$$

En outre, on a :

$$\sum_{j\in\mathbb{Z}} \frac{\sin^2(x_1 - j\pi)}{(x_1 - j\pi)^2} = \sum_{k\in\mathbb{Z}} \frac{\sin^2(x_2 - k\pi)}{(x_2 - k\pi)^2} = 1, \quad x_1\, x_2 \in \mathbb{R},$$

et

$$\sup_{y\in\mathbb{R}} \left\| \left\{ \frac{f(j\pi,y) - f(k\pi,y)}{j\pi - k\pi} \right\}_{j,k\in\mathbb{Z}} \right\|_{\mathcal{B}} \le \mathrm{const}\, \|f\|_{L^\infty(\mathbb{R})},$$

où $\mathcal{B}$ est l'espace d'opérateurs bornés sur $\ell^2$. Si $j = k$, alors $(f(j\pi,y) - f(k\pi,y))(j\pi - k\pi)^{-1} \stackrel{\mathrm{def}}{=} \frac{\partial f}{\partial x}(j\pi,y)$.

Par ailleurs, il se trouve que, pour la même classe de fonctions, il est impossible d'obtenir une inégalité du type lipschitzienne pour la norme d'opérateurs. Plus précisément, nous pouvons démontrer qu'il n'y a pas de nombre positif $M$ pour lequel

$$\big\| f(A_1,B_1) - f(A_2,B_2)\big\| \le M \|f\|_{L^\infty(\mathbb{R}^2)} \big(\|A_1 - A_2\| + \|B_1 - B_2\|\big),$$

chaque fois que $f$ est une fonction bornée sur $\mathbb{R}$ dont la transformée de Fourier a un support dans le disque unité et $A_1$, $A_2$, $B_1$, $B_2$ sont des opérateurs auto-adjoints de rang fini.

En outre, on peut trouver des opérateurs auto-adjoints de rang fini tels que $\|A_1 - A_2\| \le 2\pi$, $\|B_1 - B_2\| \le 2\pi$, et une fonction $f$ sur $\mathbb{R}^2$ dont la transformée de Fourier a un support dans le disque unité et telle que $\|f\|_{L^\infty(\mathbb{R})} \le 1$, pour lesquels la norme $\|f(A_1,B_1) - f(A_2,B_2)\|$ est arbitrairement grande.

## 1. Introduction

It was shown in [3] that a Lipschitz function $f$ on $\mathbb{R}$ does not have to be *operator Lipschitz*, i.e., it does not have to satisfy the inequality $\|f(A) - f(B)\| \le \mathrm{const}\,\|A - B\|$ for self-adjoint operators $A$ and $B$. In [7] and [8], it was shown that the condition $f \in B^1_{\infty,1}(\mathbb{R})$ is sufficient for $f$ to be operator Lipschitz (see [6] and [9] for information about Besov spaces $B^s_{p,q}$). It is also well known that $f$ is operator Lipschitz if and only if $f(A) - f(B)$ belongs to trace class $\boldsymbol{S}_1$ whenever $A - B \in \boldsymbol{S}_1$ and $\|f(A) - f(B)\|_{\boldsymbol{S}_1} \le \mathrm{const}\,\|A - B\|_{\boldsymbol{S}_1}$. It was shown in [1] that if $f$ is a Hölder function of order $\alpha$, $0 < \alpha < 1$, then it is *operator Hölder of order* $\alpha$, i.e., $\|f(A) - f(B)\| \le \mathrm{const}\,\|A - B\|^\alpha$ for self-adjoint operators $A$ and $B$.

Later the above results were generalized in [2] to functions of normal operators and in [5] to functions of commuting $n$-tuples of self-adjoint operators.

In this paper we are going to consider functions of noncommuting pairs of self-adjoint operators.

Let $A$ and $B$ be self-adjoint operators on Hilbert space and let $E_A$ and $E_B$ be their spectral measures. Suppose that $f$ is a function of two variables that is defined at least on $\sigma(A) \times \sigma(B)$. If $f$ is a Schur multiplier with respect to the pair $(E_A, E_B)$, we define the function $f(A,B)$ of $A$ and $B$ by

$$f(A,B) \stackrel{\text{def}}{=} \iint f(x,y)\,\mathrm{d}E_A(x)\,\mathrm{d}E_B(y) \tag{1}$$

(we refer the reader to [1] for definitions of Schur multipliers and double operator integrals). Note that this functional calculus $f \mapsto f(A,B)$ is linear, but not multiplicative.

If we consider functions of bounded operators, without loss of generality we may deal with periodic functions with a sufficiently large period. Clearly, we can rescale the problem and assume that our functions are $2\pi$-periodic in each variable.

If $f$ is a trigonometric polynomial of degree $N$, we can represent $f$ in the form

$$f(x,y) = \sum_{j=-N}^{N} e^{\mathrm{i}jx}\left(\sum_{k=-N}^{N} \hat{f}(j,k)e^{\mathrm{i}ky}\right).$$

Thus $f$ belongs to the projective tensor product $L^\infty \hat{\otimes} L^\infty$ and

$$\|f\|_{L^\infty \hat{\otimes} L^\infty} \le \sum_{j=-N}^{N} \sup_y \left|\sum_{k=-N}^{N} \hat{f}(j,k)e^{\mathrm{i}ky}\right| \le (1+2N)\|f\|_{L^\infty}.$$

It follows that every periodic function $f$ of class $B^1_{\infty 1}(\mathbb{R}^2)$ belongs to $L^\infty \hat{\otimes} L^\infty$ and the operator $f(A,B)$ is well defined by (1).

## 2. Lipschitz-type estimates in the trace norm

In this paper we use triple operator integrals to estimate functions of perturbed noncommuting operators in trace norm. Let $E_1$, $E_2$, and $E_3$ be spectral measures on measurable spaces $(\mathcal{X}_1, \mathfrak{M}_1)$, $(\mathcal{X}_2, \mathfrak{M}_2)$, and $(\mathcal{X}_3, \mathfrak{M}_3)$. We say that a function $\Psi$ on $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ belongs to the *Haagerup tensor product of the spaces* $L^\infty(E_j)$, $j = 1,2,3$, (notationally, $\Psi \in L^\infty(E_1) \otimes_{\mathrm{h}} L^\infty(E_2) \otimes_{\mathrm{h}} L^\infty(E_3)$) if $\Psi$ admits a representation

$$\Psi(x_1,x_2,x_3) = \sum_{j,k\ge 0} \alpha_j(x_1)\beta_{jk}(x_2)\gamma_k(x_3); \tag{2}$$

here $\{\alpha_j\}_{j\ge0}$, $\{\gamma_k\}_{k\ge0} \in L^\infty(\ell^2)$ and $\{\beta_{jk}\}_{j,k\ge0} \in L^\infty(\mathcal{B})$, where $\mathcal{B}$ is the space of infinite matrices that induce bounded linear operators on $\ell^2$. We refer the reader to [11] for Haagerup tensor products. It is well known (see [4]) that for a function $\Psi$ satisfying (2) and for bounded linear operators $T$ and $R$, one can define the triple operator integral

$$W = \int\limits_{\mathcal{X}_1}\int\limits_{\mathcal{X}_2}\int\limits_{\mathcal{X}_3} \Psi(x_1,x_2,x_3)\,\mathrm{d}E_1(x_1)T\,\mathrm{d}E_2(x_2)R\,\mathrm{d}E_3(x_3) \tag{3}$$

and

$$\|W\| \le \|\Psi\|_{L^\infty \otimes_{\mathrm{h}} L^\infty \otimes_{\mathrm{h}} L^\infty}\|T\| \cdot \|R\|, \tag{4}$$

where

$$\|\Psi\|_{L^\infty \otimes_{\mathrm{h}} L^\infty \otimes_{\mathrm{h}} L^\infty} \stackrel{\text{def}}{=} \inf\left\{\left\|\{\alpha_j\}_{j\ge0}\right\|_{L^\infty(\ell^2)}\left\|\{\beta_{jk}\}_{j,k\ge0}\right\|_{L^\infty(\mathcal{B})}\left\|\{\gamma_k\}_{k\ge0}\right\|_{L^\infty(\ell^2)}\right\},$$

the infimum being taken over all representations of $\Psi$ in the form (2). Note that this extends the definition of triple operator integrals given in [10] for projective tensor products of $L^\infty$ spaces.

We would like to define triple operator integrals of the form (3) in the case when one of the operators $T$ or $R$ is of trace class and find conditions on $\Psi$ under which the operator $W$ must be in $\boldsymbol{S}_1$.

**Definition.** A function $\Psi$ is said to belong to the tensor product $L^\infty(E_1) \otimes_{\mathrm{h}} L^\infty(E_2) \otimes^{\mathrm{h}} L^\infty(E_3)$ if it admits a representation

$$\Psi(x_1,x_2,x_3) = \sum_{j,k\ge 0} \alpha_j(x_1)\beta_k(x_2)\gamma_{jk}(x_3), \quad x_j \in \mathcal{X}_j, \tag{5}$$

with $\{\alpha_j\}_{j\ge0}$, $\{\beta_k\}_{k\ge0} \in L^\infty(\ell^2)$ and $\{\gamma_{jk}\}_{j,k\ge0} \in L^\infty(\mathcal{B})$. For a bounded linear operator $R$ and for a trace class operator $T$, we define the triple operator integral

$$W = \int\limits_{\mathcal{X}_1}\int\limits_{\mathcal{X}_2}\int\limits_{\mathcal{X}_3} \Psi(x_1,x_2,x_3)\,\mathrm{d}E_1(x_1)T\,\mathrm{d}E_2(x_2)R\,\mathrm{d}E_3(x_3)$$

as the following continuous linear functional on the class of compact operators:

$$Q \mapsto \operatorname{trace}\Big(\Big(\int\limits_{\mathcal{X}_1}\int\limits_{\mathcal{X}_2}\int\limits_{\mathcal{X}_3} \Psi(x_1,x_2,x_3)\,\mathrm{d}E_2(x_2)R\,\mathrm{d}E_3(x_3)Q\,\mathrm{d}E_1(x_1)\Big)T\Big). \tag{6}$$

The fact that the linear functional (6) is continuous on the class of compact operators is a consequence of inequality (4), which also implies the following estimate:

$$\|W\|_{\boldsymbol{S}_1} \le \|\Psi\|_{L^\infty \otimes_{\mathrm h} L^\infty \otimes^{\mathrm h} L^\infty}\|T\|_{\boldsymbol{S}_1}\|R\|,$$

where $\|\Psi\|_{L^\infty \otimes_{\mathrm h} L^\infty \otimes^{\mathrm h} L^\infty}$ is the infimum of $\|\{\alpha_j\}_{j\ge 0}\|_{L^\infty(\ell^2)}\|\{\beta_k\}_{k\ge 0}\|_{L^\infty(\ell^2)}\|\{\gamma_{jk}\}_{j,k\ge 0}\|_{L^\infty(\mathcal{B})}$ over all representations in (5).

Similarly, suppose that $\Psi \in L^\infty(E_1) \otimes^{\mathrm h} L^\infty(E_2) \otimes_{\mathrm h} L^\infty(E_3)$, i.e., $\Psi$ admits a representation

$$\Psi(x_1,x_2,x_3) = \sum_{j,k\ge 0} \alpha_{jk}(x_1)\beta_j(x_2)\gamma_k(x_3)$$

where $\{\beta_j\}_{j\ge 0}$, $\{\gamma_k\}_{k\ge 0} \in L^\infty(\ell^2)$, $\{\alpha_{jk}\}_{j,k\ge 0} \in L^\infty(\mathcal{B})$, $T$ is a bounded linear operator, and $R \in \boldsymbol{S}_1$. Then the continuous linear functional

$$Q \mapsto \operatorname{trace}\Big(\Big(\int\limits_{\mathcal{X}_1}\int\limits_{\mathcal{X}_2}\int\limits_{\mathcal{X}_3} \Psi(x_1,x_2,x_3)\,dE_3(x_3)Q\,\mathrm{d}E_1(x_1)T\,\mathrm{d}E_2(x_2)\Big)R\Big)$$

on the class of compact operators determines a trace class operator

$$W \stackrel{\text{def}}{=} \int\limits_{\mathcal{X}_1}\int\limits_{\mathcal{X}_2}\int\limits_{\mathcal{X}_3} \Psi(x_1,x_2,x_3)\,\mathrm{d}E_1(x_1)T\,\mathrm{d}E_2(x_2)R\,\mathrm{d}E_3(x_3).$$

Moreover,

$$\|W\|_{\boldsymbol{S}_1} \le \|\Psi\|_{L^\infty \otimes^{\mathrm h} L^\infty \otimes_{\mathrm h} L^\infty}\|T\| \cdot \|R\|_{\boldsymbol{S}_1}.$$

Note that the above definitions of triple operator integrals extend the definition given in [10] in terms of the projective tensor product of the $L^\infty$ spaces.

We would like to obtain a sufficient condition on a function $f$ on $\mathbb{R}^2$ under which

$$\big\|f(A_1,B_1) - f(A_2,B_2)\big\|_{\boldsymbol{S}_1} \le \text{const}\big(\|A_1 - A_2\|_{\boldsymbol{S}_1} + \|B_1 - B_2\|_{\boldsymbol{S}_1}\big),$$

whenever $(A_1,B_1)$ and $(A_2,B_2)$ are pairs of (not necessarily commuting) self-adjoint operators such that $A_1 - A_2 \in \boldsymbol{S}_1$ and $B_1 - B_2 \in \boldsymbol{S}_1$. Clearly, it suffices to verify the following inequalities:

$$\big\|f(A_1,B) - f(A_2,B)\big\| \le \text{const}\,\|A_1 - A_2\|_{\boldsymbol{S}_1} \quad \text{and} \quad \big\|f(A,B_1) - f(A,B_2)\big\| \le \text{const}\,\|B_1 - B_2\|_{\boldsymbol{S}_1}.$$

**Theorem 2.1.** *Let $f$ be a bounded function on $\mathbb{R}^2$ whose Fourier transform is supported in the ball $\{\xi \in \mathbb{R}^2 : \|\xi\| \le 1\}$. Then*

$$\frac{f(x_1,y) - f(x_2,y)}{x_1 - x_2} = \sum_{j,k\in\mathbb{Z}} \frac{\sin(x_1 - j\pi)}{x_1 - j\pi} \cdot \frac{\sin(x_2 - k\pi)}{x_2 - k\pi} \cdot \frac{f(j\pi,y) - f(k\pi,y)}{j\pi - k\pi}.$$

*Moreover,*

$$\sum_{j\in\mathbb{Z}} \frac{\sin^2(x_1 - j\pi)}{(x_1 - j\pi)^2} = \sum_{k\in\mathbb{Z}} \frac{\sin^2(x_2 - k\pi)}{(x_2 - k\pi)^2} = 1, \quad x_1, x_2 \in \mathbb{R},$$

*and*

$$\sup_{y\in\mathbb{R}} \left\| \left\{ \frac{f(j\pi,y) - f(k\pi,y)}{j\pi - k\pi} \right\}_{j,k\in\mathbb{Z}} \right\|_{\mathcal{B}} \le \text{const}\,\|f\|_{L^\infty(\mathbb{R})}.$$

Note that if $j = k$, we assume that $(f(j\pi,y) - f(k\pi,y))(j\pi - k\pi)^{-1} = \frac{\partial f}{\partial x}(j\pi,y)$.

Theorem 2.1 implies the following result:

**Theorem 2.2.** *Let $f$ be a bounded function on $\mathbb{R}^2$ whose Fourier transform is supported in the ball $\{\xi \in \mathbb{R}^2 : \|\xi\| \le \sigma\}$. Then the function $\Psi$ defined by*

$$\Psi(x_1,x_2,y) = \frac{f(x_1,y) - f(x_2,y)}{x_1 - x_2}, \quad x_1, x_2, y \in \mathbb{R},$$

*belongs to the tensor product* $L^\infty(\mathbb{R})\otimes_{\rm h} L^\infty(\mathbb{R})\otimes^{\rm h} L^\infty(\mathbb{R})$,

$$\|\Psi\|_{L^\infty\otimes_{\rm h}L^\infty\otimes^{\rm h}L^\infty}\le \mathrm{const}\,\sigma\,\|f\|_{L^\infty(\mathbb{R}^2)}$$

*and*

$$f(A_1,B)-f(A_2,B)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\frac{f(x_1,y)-f(x_2,y)}{x_1-x_2}\,\mathrm{d}E_{A_1}(x_1)(A_1-A_2)\,\mathrm{d}E_{A_2}(x_2)\,\mathrm{d}E_B(y), \tag{7}$$

*whenever* $A_1$, $A_2$, *and* $B$ *are self-adjoint operators such that* $A_1-A_2\in\boldsymbol{S}_1$.

In a similar way we can prove the following fact:

**Theorem 2.3.** *Under the same hypotheses on* $f$, *the function* $\Psi$ *defined by*

$$\Psi(x,y_1,y_2)=\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2},\quad x,y_1,y_2\in\mathbb{R},$$

*belongs to the tensor product* $L^\infty(\mathbb{R})\otimes^{\rm h} L^\infty(\mathbb{R})\otimes_{\rm h} L^\infty(\mathbb{R})$, $\|\Psi\|_{L^\infty\otimes^{\rm h}L^\infty\otimes_{\rm h}L^\infty}\le \mathrm{const}\,\sigma\,\|f\|_{L^\infty(\mathbb{R}^2)}$, *and*

$$f(A,B_1)-f(A,B_2)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}}\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2}\,\mathrm{d}E_{A}(x)\,\mathrm{d}E_{B_1}(x_1)(B_1-B_2)\,\mathrm{d}E_{B_2}(y_2), \tag{8}$$

*whenever* $A$, $B_1$, *and* $B_2$ *are self-adjoint operators such that* $B_1-B_2\in\boldsymbol{S}_1$.

Theorems 2.2 and 2.3 imply the main result of this section:

**Theorem 2.4.** *Let* $f\in B^1_{\infty,1}(\mathbb{R}^2)$. *Then*

$$\big\|f(A_1,B_1)-f(A_2,B_2)\big\|_{\boldsymbol{S}_1}\le \mathrm{const}\,\|f\|_{B^1_{\infty,1}}\big(\|A_1-A_2\|_{\boldsymbol{S}_1}+\|B_1-B_2\|_{\boldsymbol{S}_1}\big), \tag{9}$$

*whenever* $(A_1,B_1)$ *and* $(A_2,B_2)$ *are pairs of self-adjoint operators,* $A_1-A_2\in\boldsymbol{S}_1$ *and* $B_1-B_2\in\boldsymbol{S}_1$.

We have defined functions $f(A,B)$ for $f$ in $B^1_{\infty,1}(\mathbb{R}^2)$ only for bounded self-adjoint operators $A$ and $B$. However, formulae (7) and (8) allow us to define the difference $f(A_1,B_1)-f(A_2,B_2)$ in the case when $f\in B^1_{\infty,1}(\mathbb{R}^2)$ and the self-adjoint operators $A_1,A_2,B_1,B_2$ are possibly unbounded once we know that the pair $(A_2,B_2)$ is a trace class perturbation of the pair $(A_1,B_1)$. Moreover, inequality (9) also holds for such operators.

## 3. Lipschitz-type estimates in the operator norm

The main result of this section shows that unlike in the case of commuting pairs of self-adjoint operators, the condition $f\in B^1_{\infty,1}(\mathbb{R}^2)$ does not imply Lipschitz-type estimates in the operator norm.

**Theorem 3.1.** *There is no positive number* $M$ *such that*

$$\big\|f(A_1,B_1)-f(A_2,B_2)\big\|\le M\|f\|_{L^\infty(\mathbb{R}^2)}\big(\|A_1-A_2\|+\|B_1-B_2\|\big)$$

*for all bounded functions* $f$ *on* $\mathbb{R}^2$ *with Fourier transform supported in* $[-1,1]^2$ *and for all finite rank self-adjoint operators* $A_1,A_2,B_1,B_2$.

**Construction.** Let $\{f_j\}_{1\le j\le N}$ and $\{g_k\}_{1\le k\le N}$ be orthonormal systems. Consider the rank one projections $P_j$ and $Q_j$ defined by

$$P_ju=(u,f_j)f_j\quad\text{and}\quad Q_ju=(u,g_j)g_j,\quad 1\le j\le N.$$

We define the self-adjoint operators $A_1$, $A_2$, and $B=B_1=B_2$ by

$$A_1=\sum_{j=1}^N 4j\pi P_j,\quad A_2=\sum_{j=1}^N(4j+2)\pi P_j,\quad\text{and}\quad B=\sum_{k=1}^N 4k\pi Q_k.$$

Clearly, $\|A_1-A_2\|\le 2\pi$. Put

$$\varphi(x,y)=4\cdot\frac{1-\cos x}{x^2}\cdot\frac{1-\cos y}{y^2},\quad x,y\in\mathbb{R}.$$

Given a matrix $\{\tau_{jk}\}_{1\le j,k\le N}$, we define the function $f$ by

$$f(x,y)=\sum_{1\le j,k\le N}\tau_{jk}\varphi(x-4\pi j,\,y-4\pi k).$$

It is easy to see that

$$\|f\|_{L^\infty(\mathbb{R}^2)}\le \text{const}\max_{1\le j,k\le N}|\tau_{jk}|,\quad f(4j\pi,4k\pi)=\tau_{jk},\quad f\big((4j+2)\pi,4k\pi\big)=0,\quad 1\le j,k\le N,$$

and the Fourier transform of $f$ is supported in $[-1,1]\times[-1,1]$. One can easily verify that

$$f(A_1,B)=\sum_{1\le j,k\le N}\tau_{jk}P_jQ_k,\quad \text{while } f(A_2,B)=\mathbf{0}.$$

It can easily be shown that the supremum of $\|f(A_1,B)\|$ over all $P_j$ and $Q_k$ is the Schur multiplier norm of the matrix $\{\tau_{jk}\}_{1\le j,k\le N}$. This norm can be made arbitrarily large as $N\to\infty$, while assumption $|\tau_{jk}|\le 1$, $1\le j,k\le N$, can still hold.

**Remark.** The above construction shows that there exist a function $f$ on $\mathbb{R}^2$ whose Fourier transform is supported in $[-1,1]^2$ such that $\|f\|_{L^\infty(\mathbb{R})}\le 1$ and self-adjoint operators of finite rank $A_1$, $A_2$, $B_1$, $B_2$ such that $\|A_1-A_2\|\le 2\pi$ and $\|B_1-B_2\|\le 2\pi$, but $\|f(A_1,B_1)-f(A_2,B_2)\|$ is greater than any given positive number. In particular, the fact that $f$ is a Hölder function of order $\alpha\in(0,1)$ does not imply the estimate $\|f(A_1,B_1)-f(A_2,B_2)\|\le\text{const}(\|A_1-A_2\|^\alpha+\|B_1-B_2\|^\alpha)$.

We conclude the paper with a theorem that can be deduced from the results of this section.

**Theorem 3.2.** *There are spectral measure $E_1$, $E_2$ and $E_3$ on Borel subsets of $\mathbb{R}$, a function $\Psi$ in the Haagerup tensor product $L^\infty(E_1)\otimes_{\rm h}L^\infty(E_2)\otimes_{\rm h}L^\infty(E_3)$ and an operator $Q$ in $\boldsymbol{S}_1$ such that*

$$\iiint \Psi(x_1,x_2,x_2)\,\mathrm{d}E_1(x_1)\,\mathrm{d}E_2(x_2)Q\,\mathrm{d}E_3(x_3)\notin \boldsymbol{S}_1.$$

## Acknowledgements

The research of the first author is partially supported by RFBR grant 14-01-00198, the research of the second author is partially supported by NSF grant DMS 126562, the research of the third author is partially supported by NSF grant DMS 1300924.

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