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Homogenization of the 3D Maxwell system near resonances and artificial magnetism

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Abstract

It is now well known that the homogenization of a periodic array of parallel dielectric fibers with suitably scaled high permittivity can lead to a possibly negative frequency dependent effective permeability. However this result based on a two-dimensional micro resonator problem on the section of the fibers holds merely in the case of polarized magnetic fields, reducing thus its applications to infinite cylindrical obstacles. In this Note we propose a full 3D extension of previous asymptotic analysis based on a new averaging method for the magnetic field. We evidence a vectorial spectral problem on the periodic cell which accounts for micro-resonance effects and leads to a 3D negative effective permeability tensor. This suggests that periodic bulk dielectric inclusions could be an efficient alternative to the very popular metallic split-ring structure proposed by Pendry. ***To cite this article: G. Bouchitté et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Homogénéisation du système de Maxwell 3D près des résonances et magnétisme artificiel. Il est maintenant bien connu que l'homogénéisation d'un réseau périodique de fibres diélectriques de forte permittivité peut conduire à une perméabilité effective négative sur certaines bandes de fréquences. Cependant ce résultat basé sur l'analyse d'un résonateur en dimension deux ne pouvait être justifié sans l'hypothèse de polarisation du champ magnétique et en pratique seuls des obstacles cylindriques *infinis* pouvaient être considérés. Dans cette note nous proposons une extension complète au cas 3D basée sur une nouvelle notion de moyennisation du champ magnétique. Nous obtenons un problème spectral vectoriel sur la cellule unité qui décrit l'effet de résonance et conduit à un tenseur de perméabilité effective dont les parties réelles des valeurs propres changent de signe suivant la fréquence. ***Pour citer cet article : G. Bouchitté et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Version française abrégée

On considère la diffraction d'une onde monochromatique par une structure composée d'une matrice diélectrique contenue dans un domaine borné $\Omega \subset \mathbb{R}^3$ dans laquelle sont placées périodiquement (période η) des inclusions diélec-

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triques de très forte permittivité. Celles-ci occupent un domaine $\Sigma_\eta := \bigcup_{i\in I_\eta} \eta(i+\Sigma)$ où $\Sigma \Subset Y := (-1/2, 1/2)^3$ est un ouvert connexe régulier de complémentaire $Y^* := Y \setminus \Sigma$ *simplement connexe* et où $I_\eta = \{i \in \mathbb{Z}^3\colon\ \eta(i+\Sigma) \Subset \Omega\}$. La structure diffractante est caractérisée par sa permittivité relative

$$\varepsilon_\eta(x) := \varepsilon^r/\eta^2 \quad \text{si } x \in \Sigma_\eta, \qquad \varepsilon_\eta(x) := \varepsilon^e \quad \text{si } x \in \Omega \setminus \Sigma_\eta, \qquad \varepsilon_\eta(x) := 1 \quad \text{si } x \in \mathbb{R}^3 \setminus \Omega. \tag{1}$$

Le choix du facteur η^2 dans les inclusions revient à supposer un diamètre optique constant pour chacune d'entre elles. On supposera une dépendance temporelle en $e^{-i\omega t}$, que ε^e est un réel positif (diélectrique sans perte) puis que ε^r est un complexe de partie imaginaire *strictement positive*. Le nombre d'onde k est défini par $k = \omega\sqrt{\varepsilon_0\mu_0}$, où ε_0, μ_0 sont les constantes usuelles du vide. La structure est éclairée par une onde incidente (E^i, H^i) de sorte que le champ total (E_η, H_η) vérifie :

$$\begin{cases} \operatorname{curl} E_\eta = i\omega\mu_0 H_\eta, \\ \operatorname{curl} H_\eta = -i\omega\varepsilon_0\varepsilon_\eta E_\eta, \end{cases} \tag{2}$$

avec la condition de rayonnement (8) pour le champ diffracté $(E_\eta - E^i, H_\eta - H^i)$.

On considère les valeurs propres $0 < \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ $(\lambda_n \to +\infty)$ du problème spectral auto-adjoint (24) sur $L^2(Y;\mathbb{R}^3)$ et une base orthonormale $\{\varphi_n\}$ de vecteurs propres associée. Le tenseur de perméabilité effective qui dépend de la fréquence (via $k = \omega/c$) est alors donné par :

$$\mu_{ij}^{\text{eff}}(\omega) = \delta_{ij} + \sum_n \frac{\varepsilon^r k^2}{\lambda_n - \varepsilon^r k^2} \left(e_j \cdot \int_Y \varphi_n \right) \left(e_i \cdot \int_Y \varphi_n \right). \tag{3}$$

A l'inverse le tenseur de permittivité effective $\varepsilon^{\text{eff}} = \varepsilon^e A^{\text{hom}}$ est indépendant de ω et associé à la matrice réelle définie positive A^{hom} obtenue via le problème local classique (11). Posons

$$\boldsymbol{\mu}(\omega, x) = \begin{cases} \mu^{\text{eff}}(\omega) & \text{for } x \in \Omega, \\ I_3 & \text{for } x \in \mathbb{R}^3 \setminus \Omega, \end{cases} \qquad \boldsymbol{\varepsilon}(x) = \begin{cases} \varepsilon^{\text{eff}} & \text{for } x \in \Omega, \\ I_3 & \text{for } x \in \mathbb{R}^3 \setminus \Omega. \end{cases} \tag{4}$$

Le problème de diffraction limite est alors décrit par la solution (E, H) du système

$$\begin{cases} \operatorname{curl} E = i\omega\mu_0\, \boldsymbol{\mu}(\omega, x) H, \\ \operatorname{curl} H = -i\omega\varepsilon_0\, \boldsymbol{\varepsilon}(x) E \end{cases} \tag{5}$$

avec $(E - E^i, H - H^i)$ vérifiant (8). Cette solution existe et est unique car ε^r admet une partie imaginaire strictement positive.

Théorème. *On a la convergence forte de* $(E_\eta, H_\eta) \to (E, H)$ *dans* $L^2_{\text{loc}}(\mathbb{R}^3 \setminus \Omega)$ *tandis que le comportement oscillant de* (E_η, H_η) *sur* Ω *est caractérisé par la limite faible à deux échelles* (E_0, H_0) *donnée par les relations* (10) *et* (16).

Notons que sur Ω, la limite faible de la suite oscillante $H_\eta \sim H_0(x, x/\eta)$ ne coincide pas avec le champs $H(x)$. Ce dernier est défini par une moyennisation « géométrique » décrite dans la Section 2 (voir (15)) qui permet de décrire le système limite à l'aide du tenseur de perméabilité effectif donné par (3). Il en résulte l'existence de bandes de fréquences interdites pour la propagation des ondes dans les directions propres associées aux valeurs propres de μ^{eff} dont la partie réelle est négative. Les effets de magnétisme artificiel induits par micro-résonance présentés dans [2] sont ainsi étendus au cas d'obstacles bornés dans $\mathbb{R}^3$.

1. Introduction

Recently in [2,4–6], a theory for artificial magnetism in two-dimensional photonic crystals has been developed for large wavelength (homogenization). The main idea was that a periodic crystal with high permittivity inclusions shows micro-resonance effects from which an effective permeability law with anomalous dispersion emerges in a explicit way. The main drawback was however that in this model we assumed magnetic parallel polarization so that merely infinite photonic crystals (invariant in one direction) could be considered. In this work we propose a full 3D generalization of previous results: the diffraction of a finite 3D-dielectric crystal is considered at a fixed wavelength and a limit analysis as the period tends to zero is performed.

The heterogeneous structure is placed in a *bounded* domain $\Omega \subset \mathbb{R}^3$. It consists of periodic high permittivity inclusions (period η) embedded in a lossless dielectric matrix. The inclusions occupy a subregion

$$\Sigma_\eta := \bigcup_{i\in I_\eta} \eta(i+\Sigma), \quad I_\eta = \{ i \in \mathbb{Z}^3 \colon \eta(i+\Sigma) \Subset \Omega \}.$$

Here $\Sigma \Subset Y := (-1/2, 1/2)^3$ is a regular connected domain whose complement $Y^* := Y \setminus \Sigma$ is assumed to be *simply connected*. The structure, whose relative permeability is assumed to be equal to 1, is characterized by its relative permittivity ε_η given by:

$$\varepsilon_\eta(x) := \varepsilon^r/\eta^2 \quad \text{if } x \in \Sigma_\eta, \qquad \varepsilon_\eta(x) := \varepsilon^e \quad \text{if } x \in \Omega \setminus \Sigma_\eta, \qquad \varepsilon_\eta(x) := 1 \quad \text{si } x \in \mathbb{R}^3 \setminus \Omega. \tag{6}$$

The scaling η^{-2} appearing in Σ_η implies that each inclusion has a constant (η-independent) optical diameter. We choose a time dependence in $e^{-i\omega t}$ and assume that the dielectric constant ε^e of the matrix is a positive real number whereas the parameter ε^r is a complex number whose imaginary part is positive. The wavenumber k is defined by $k = \omega\sqrt{\varepsilon_0\mu_0}$ being ε_0, μ_0 the constants of the vacuum. The structure is illuminated by an incident wave (E^i, H^i) so that the total electromagnetic field (E_η, H_η) solves the system:

$$\begin{cases} \operatorname{curl} E_\eta = i\omega\mu_0 H_\eta, \\ \operatorname{curl} H_\eta = -i\omega\varepsilon_0\varepsilon_\eta E_\eta, \end{cases} \tag{7}$$

where the diffracted field $(E_\eta - E^i, H_\eta - H^i)$ satisfies the Silver–Müller radiation condition:

$$\left(E_\eta^d, H_\eta^d\right) = O\left(\frac{1}{|x|}\right), \qquad \omega\varepsilon_0\left(\frac{x}{|x|} \wedge E_\eta^d\right) - kH_\eta^d = o\left(\frac{1}{|x|}\right), \quad \text{for } |x| \to \infty. \tag{8}$$

Our goal is to identify the asymptotic behavior of (E_η, H_η) as η tends to 0. It will be proved later that this sequence remain bounded in L^2_{loc} so that given a ball $\mathcal{B}$ such that $\Omega \Subset \mathcal{B}$, by classical energy estimates (flux of Poynting vector), we may start with the upperbound

$$\sup_\eta \int_{\mathcal{B}} \left(|H^\eta|^2 + |\varepsilon_\eta||E_\eta|^2 \right) < +\infty. \tag{9}$$

Outside of Ω, a strong convergence of (E_η, H_η) is expected thanks to the fact that all of its components are in the kernel of the Helmholtz operator $\Delta + k^2$ (hypoellipticity). In contrast the oscillatory behavior of (E_η, H_η) in Ω excludes strong compactness in $L^2(\Omega)$; we will identify their two-scale limit $(E_0(x,y), H_0(x,y))$ for $x \in \Omega$. For the notion of two-scale convergence and related topics we refer to [1].

2. Cell problems and geometric averaging

Electric field. It is easy to show that for $x \in \Omega$ the periodic field $E_0(x,\cdot)$ satisfies the equations

$$\operatorname{curl}_y E_0 = 0 \quad \text{in } Y, \qquad \operatorname{div}_y\, E_0 = 0 \quad \text{in } Y \setminus \bar{\Sigma}, \qquad E_0 = 0 \quad \text{in } \Sigma. \tag{10}$$

By the curl-free condition and letting $E(x) = \int_Y E_0(x,y)\,\mathrm{d}y$, we search a solution $E_0(x,y) = E(x) + \nabla_y \chi$ for a suitable periodic $\chi \in W^{1,2}_\sharp(Y)$. We are led to the following decomposition:

$$E_0(x,y) = \sum_{i=1}^{3} E_i(x) E^i(y), \qquad E^i(y) = e_i + \nabla_y \chi_i, \qquad \Delta\chi_i = 0 \quad \text{on } Y^*, \qquad \chi_i = -y_i \quad \text{on } \Sigma. \tag{11}$$

Note that $E_0(x,y) = E(x)$ for $x \notin \Omega$. Further we define a real positive symmetric tensor A^{hom} (depending only on the shape of Σ) and the effective permittivity tensor ε^{eff}:

$$A^{\text{hom}}_{i,j} := \int_Y E^i \cdot E^j \,\mathrm{d}y = \int_{Y^*} (e_i + \nabla_y \chi_i) \cdot (e_j + \nabla_y \chi_j)\,\mathrm{d}y, \qquad \varepsilon^{\text{eff}} := \varepsilon^e A^{\text{hom}}. \tag{12}$$

Magnetic field. By (9), the sequence of rescaled displacement vectors $J_\eta := \eta\varepsilon_\eta E_\eta$ is bounded in $L^2(\Omega)$. We may assume that it two-scale converges to some field $J_0(x,y)$. Then $J_0(x,\cdot) = 0$ on Y^* and by using equations (7), we derive the following system for the periodic fields $H_0(x,\cdot)$ and $J_0(x,\cdot)$:

$$\operatorname{curl}_y H_0 + i\omega\varepsilon_0 J_0 = 0 \quad \text{in } Y, \qquad \operatorname{div}_y H_0 = 0 \quad \text{in } Y, \tag{13}$$

$$\operatorname{curl}_y J_0 + i\varepsilon^r \omega\mu_0 H_0 = 0 \quad \text{in } \Sigma, \qquad J_0 = 0 \quad \text{in } Y^*. \tag{14}$$

From (13), we deduce that $H_0(x,\cdot)$ belongs to the Sobolev space $W^{1,2}_\sharp(Y;\mathbb{C}^3)$. It is not the case for the divergence free field $J_0(x,\cdot)$ (supported in Σ) which may have a tangential jump across $\partial\Sigma$. The analysis of the full system is a delicate task which relies on the simple connectedness of Y^*.

Geometric averaging. To every vector field u in the space $X = \{u \in W^{1,2}_\sharp(Y;\mathbb{C}^3)\colon \operatorname{curl} u = 0 \text{ on } Y^*\}$, we associate the *circulation vector* denoted $\oint u \in \mathbb{C}^3$ which is characterized by the identity

$$\int_Y u\cdot\varphi\,\mathrm{d}y = \left(\int_Y \varphi\,\mathrm{d}y\right)\cdot\left(\oint u\right) \quad \text{if } \varphi \text{ is periodic, } \operatorname{div}\varphi = 0 \text{ and } \varphi = 0 \text{ on } \Sigma. \tag{15}$$

When u is smooth, the components of $\oint u$ represents the circulation of u along any curve in Y^* connecting opposite points on the faces of ∂Y. In general we have $\oint u \neq \int_Y u\,\mathrm{d}y$ (however equality holds if $\operatorname{curl} u = 0$ on all Y). As Y^* is simply connected, any $u \in X$ can be written on Y^* in the form $u = z + \nabla_y w$ where $z = \oint u$ and $w \in W^{1,2}_\sharp(Y^*)$. This representation is unique (up to an additive constant for w).

Lemma 1. *For $i \in \{1,2,3\}$ there is a unique solution $H^i(y)$ to* (13)–(14) *with $\oint H^i = e_i$.*

By Lemma 1, the space of solutions to (13)–(14) is of dimension 3 and $H_0(x,y)$ can be decomposed as

$$H_0(x,y) = \sum_{i=1}^3 H_i(x)H^i(y) \quad \text{for } x \in \Omega, \qquad H_0(x,y) = H(x) \quad \text{for } x \in \mathcal{B}\setminus\Omega. \tag{16}$$

The macroscopic field $H(x) = (H_i(x)) \in L^2(\mathcal{B};\mathbb{C}^3)$ is related to the weak limit $[H_0](x) := \int_Y H_0(x,\cdot)$ of (H_η) in $L^2(\mathcal{B};\mathbb{C}^3)$ by the tensorial relation

$$[H_0](x) = \mu^{\text{eff}} H(x), \quad \mu^{\text{eff}}_{i,j} := \int_Y \left(H^j\cdot e_i\right)\mathrm{d}y. \tag{17}$$

The tensor μ^{eff} is symmetric and will be written explicitly by means of a suitable spectral problem (see (25)). Eventually applying (15) to $u = H^i$ and $\varphi = E^j \wedge z$ with E^j given in (11) ($z \in \mathbb{R}^3$), we infer

$$\int_Y \left(H^i \wedge E^j\right)\mathrm{d}y = e^i \wedge e^j, \quad \text{for every } i,j \in \{1,2,3\}. \tag{18}$$

3. The homogenization result

Recalling (12) and (17), we introduce the tensors valued functions

$$\boldsymbol{\mu}(\omega,x) = \begin{cases} \mu^{\text{eff}}(\omega) & \text{for } x \in \Omega, \\ I_3 & \text{for } x \in \mathbb{R}^3\setminus\Omega, \end{cases} \qquad \boldsymbol{\varepsilon}(x) = \begin{cases} \varepsilon^{\text{eff}} & \text{for } x \in \Omega, \\ I_3 & \text{for } x \in \mathbb{R}^3\setminus\Omega. \end{cases} \tag{19}$$

The limit diffraction problem as $\eta \to 0$ consists in finding $(E,H) \in L^2_{\text{loc}}(\mathbb{R}^3;\mathbb{C}^3)$ such that

$$\begin{cases} \operatorname{curl} E = i\omega\mu_0\,\boldsymbol{\mu}(\omega,x)H, \\ \operatorname{curl} H = -i\omega\varepsilon_0\,\boldsymbol{\varepsilon}(x)E \end{cases} \tag{20}$$

with $(E - E^i, H - H^i)$ satisfying (8). The uniqueness of (E,H) is ensured under the assumption that ε^r has a positive imaginary part.

Theorem. *Let (E_η, H_η) be the solution to* (7)*. Then $(E_\eta,H_\eta) \to (E,H)$ in $L^2_{\text{loc}}(\mathbb{R}^3\setminus\Omega)$. On the other hand the oscillatory behavior of (E_η,H_η) in the diffractive structure Ω is characterized by its two scale limit (E_0,H_0) given by* (11) *and* (16).

Let us emphasize that in Ω, the weak limit of $H_\eta \sim H_0(x, x/\eta)$ does not coincide with our macroscopic field H as it was observed in the polarized case studied in [2] where the field H_0 had the form $H_0 = u_0(x, y_1, y_2)e_3$ with $u_0(x,\cdot) = u(x)$ on Y^* and $H(x) = u(x)e_3$ (the constancy of $u_0(x,\cdot)$ corresponds to the curl free condition of H_0 on Y^*). Our definition $H(x) := \oint H_0(x,\cdot)$ covers this case and appears to be the right extension for dealing with general 3D geometries.

Remark 1. The limit equation in (20) is to be understood in the distributional sense. It can be written as a transmission problem composed of the homogenized system $\operatorname{curl} E = i\omega\mu_0\mu^{\text{eff}}(\omega)H$, $\operatorname{curl} H = -i\omega\varepsilon_0\varepsilon^{\text{eff}}E$, in Ω, the Maxwell system in the vacuum (Helmholtz equation) in $\mathbb{R}^3 \setminus \Omega$ and the following transmission conditions on $\partial\Omega$: $(n \wedge E)^+ = (n \wedge E)^-$, $(n \wedge H)^+ = (n \wedge H)^-$ *(continuity of the tangential traces)* whereas $(E \cdot n)^+ = (\varepsilon^{\text{eff}} E \cdot n)^-$, $(H \cdot n)^+ = (\mu^{\text{eff}} H \cdot n)^-$.

Sketch of proofs. In a first step, (E_η, H_η) is assumed to be bounded in $L^2(\mathcal{B})$ so that possibly after extracting subsequences, it has a two-scale limit $(E_0(x,y), H_0(x,y))$ characterized as in Section 2 in terms of macroscopic fields $(E(x), H(x))$. The first equation in (20) is then a direct consequence of (17). To close the system, we consider $\Psi = (\Psi_j) \in \mathcal{D}(\mathcal{B}; \mathbb{C}^3)$ a smooth test function and set $\Psi_\eta(x) := \sum_{j=1}^{j=3} E^j(x/\eta)\Psi_j(x)$. As $\int_Y E^j = e_j$, we have $\Psi_\eta \rightharpoonup \Psi$. Further we establish the following convergences:

$$\lim_{\eta\to 0} \int_{\mathcal{B}} \operatorname{curl} H_\eta \cdot \Psi_\eta \, dx = \int_{\mathcal{B}} \operatorname{curl} H \cdot \Psi \, dx, \tag{21}$$

$$\lim_{\eta\to 0} \int_{\mathcal{B}} \varepsilon_\eta E_\eta \cdot \Psi_\eta \, dx = \int_{\mathcal{B}} \boldsymbol{\varepsilon}(x) E \cdot \Psi \, dx. \tag{22}$$

To obtain (21), we integrate by parts and pass to the limit exploiting that by (16), $H_\eta \rightharpoonup\rightharpoonup \sum_i H_i(x)H^i(y)$, $\operatorname{curl}\Psi_\eta \rightharpoonup\rightharpoonup \sum_j E^j(y) \wedge \nabla_x \Psi_j(x)$, where the last two-scale convergence is strong. The limit is computed with the help of relations (18). The derivation of (22) using the characterization (11) of E_0 and (12) is straightforward since Ψ_η vanishes on Σ_η whereas ε_η is piecewise constant on $\mathcal{B} \setminus \Sigma_\eta$. After multiplying the second equation in (7) by Ψ_η and integrating over $\mathcal{B}$, we infer from (21)–(22) the second equation of (20). As usual the uniform convergence of (E_η, H_η) on compacts subsets of $\mathbb{R}^3 \setminus \Omega$ implies that the radiation condition (8) is preserved as $\eta \to 0$.

In a second step, the upperbound (9) is established a posteriori by a contradiction argument exploiting the uniqueness for the limit diffraction problem. □

4. Spectral problem in the unit cell

It turns out that the periodic fields $H^i(y)$ solutions of (13)–(14) with normalization $\oint H^i = e_i$ can be written as $H^i = e_i + u_i$ where u_i is characterized by the variational equation

$$b_0(u_i, v) - k^2\varepsilon^r \int u_i \cdot \bar{v} \, dy = k^2\varepsilon^r \int e_i \cdot \bar{v} \, dy, \quad \forall v \in X_0. \tag{23}$$

Here X_0 is the Hilbert space $\{u \in W^{1,2}_\sharp(Y; \mathbb{C}^3)\colon \operatorname{curl} u = 0 \text{ on } Y^*,\ \oint u = 0\}$ (note that constant functions are ruled out) equipped with the scalar product: $b_0(u,v) := \int_Y (\operatorname{curl} u \cdot \overline{\operatorname{curl} v} + \operatorname{div} u \cdot \overline{\operatorname{div} v}) \, dy$.

To solve (23), we introduce the eigenvalue problem in $L^2(Y; \mathbb{R}^3)$

$$b_0(\varphi, v) = \lambda \int \varphi \cdot v \, dy, \quad \forall v \in X_0 \cap L^2(Y; \mathbb{R}^3). \tag{24}$$

By the compact embedding of $W^{1,2}_\sharp(Y)$ in $L^2(Y)$, the operator B_0 on $L^2(Y; \mathbb{R}^3)$ associated with b_0 has a compact self adjoint resolvent. Therefore there exists a sequence of real eigenvalues $0 < \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ ($\lambda_n \to +\infty$) and we denote by $\{\varphi_n, n \in \mathbb{N}\}$ an orthonormal basis of $L^2(Y; \mathbb{R}^3)$ made of eigenfunctions in X_0. Accordingly the solution u_i to (23) is given by $u_i = \sum_{n\in\mathbb{N}} c_{i,n}\varphi_n$ where $c_{i,n} = \frac{\varepsilon^r k^2}{\lambda_n - \varepsilon^r k^2} \int_Y (e_i \cdot \varphi_n) \, dy$. The tensor μ^{eff} defined in (17) can be therefore rewritten as an absolutely convergent series

$$\mu_{ij}^{\text{eff}}(\omega) = \delta_{ij} + \sum_{n\in\mathbb{N}} \frac{\varepsilon^r k^2}{\lambda_n - \varepsilon^r k^2} \left(e_j \cdot \int_Y \varphi_n\right)\left(e_i \cdot \int_Y \varphi_n\right). \tag{25}$$

Remark 2. The eigenfunctions associated with $w_p = \nabla(\exp(2i\pi p\cdot y))$, $p \in \mathbb{Z}^3$ (and $\lambda = 4\pi^2|p|^2$) have zero average and do not contribute in the series (25). All the other eigenvectors are divergence free as being orthogonal to $\{w_p,\ p \in \mathbb{Z}^3\}$. Writing them in the form $\varphi = \operatorname{curl}\Psi - z$ where $\Psi \in W_\sharp^{1,2}(Y)$, $\operatorname{div}\Psi = 0$ and $z = \oint \operatorname{curl}\Psi$, we may then substitute (24) with a spectral problem on Σ which fits better to numerical treatments (see [3]) (in which the unknown $f = \operatorname{curl}\varphi$ is searched in the space $\{f \in (L^2(\Sigma))^3\colon\ \operatorname{div} f = 0,\ f\cdot n = 0 \text{ on } \partial\Sigma\}$).

Remark 3. We may rewrite (25) without repeating the eigenvalues. Denoting V_λ the (finite dimensional) eigenspace associated with λ and P_{V_λ} the orthogonal projector in $L^2(Y;\mathbb{R}^3)$, we have

$$\mu_{ij}^{\text{eff}}(\omega) = \delta_{ij} + \sum_{\lambda\in\Lambda_0} \frac{\varepsilon^r k^2}{\lambda - \varepsilon^r k^2}(M_\lambda)_{i,j}, \quad (M_\lambda)_{i,j} := \big(P_{V_\lambda}(e_i)\,|\,P_{V_\lambda}(e_j)\big). \tag{26}$$

Here Λ_0 denotes the eigenvalues λ such that the real symmetric nonnegative matrix M_λ is not zero. Note that M_λ is of rank one for λ simple and to have M_λ of full rank, we need that λ has multiplicity $\geqslant 3$ (this is the case if the inclusion Σ presents enough symmetries).

5. Resonances and physical issues

Owing to (26), the real part of the eigenvalues of tensor $\mu^{\text{eff}}(\omega)$ will change of sign for particular frequencies. This corresponds to internal resonances at the period scale η (although the wavelength is large compared to η): the magnetic field shows very large components on some eigenspace of (24) and the associated oscillations produce a loop of displacement current inducing a microscopic magnetic moment. All the microscopic moments add up to a collective macroscopic moment, resulting in artificial magnetism. As, by (12), the effective permittivity remains real positive, the diffracted field will be exponentially damped (at least in some directions). More precisely assume for simplicity that ε^r is real and denote $\mu^{\pm}(\omega)$ the maximal (resp. minimal) eigenvalue of the real tensor μ^{eff}. Given λ an eigenvalue for (24) such that $M_\lambda \neq 0$ and letting $\omega_\lambda = \sqrt{\lambda}(\varepsilon_0\mu_0\varepsilon^r)^{-1/2}$, we see that $\mu^-(\omega) \searrow -\infty$ as $\omega \searrow \omega_\lambda$ whereas the same holds for $\mu^+(\omega)$ if in addition M_λ has full rank. In the latter case, we obtain the existence of a band-gap (of forbidden frequencies) on which the effective permeability tensor is negative. This effect predicted by Pendry [7] was proved rigorously in [2,5] for infinite parallel dielectric rods and in [8] for a metallic split-ring structure in a 2D-setting. In the case where M_λ is not of full rank, the nearest eigenvalues have to be considered and it may happen that the magnetic field has a non vanishing component in the kernel of M_λ. A similar situation has been noticed in [9] in the context of elastic waves in a composite.

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