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Numerical Analysis

A mixed PDE/Monte-Carlo method for stochastic volatility models

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Abstract

We propose a pricing method for derivatives modeled by a set of stochastic differential equations with the objective of reducing the computing time. The speed up observed in our numerical implementation can be as large as 50. The method is based on a joint use of Monte-Carlo simulations and PDE or analytical formulas. The method is tested in the framework of the Heston stochastic volatility model with and without barriers. ***To cite this article: G. Loeper, O. Pironneau, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Pricing d'option financière avec volatilité stochastique par une métode mixte EDP / Monte-Carlo. Nous proposons dans cette note une méthode pour accélérer les calculs d'options financières modélis'ees par un système d'équations différentielles stochastiques. La méthode consiste à intégrer un groupe d'équation par une méthode de Monte-Carlo et les autres par une méthode déterministe, EDP ou formules de Black–Scholes. La méthode est présentée avec une justification euristique seulement sur le modl'ele de Heston puis testée numériquement et comparée à une solution Monte-Carlo classique du modlèle de Heston. Les simulations numériques montrent qu'on peut obtenir un facteur d'accérération allant jusqu'a 50. ***Pour citer cet article : G. Loeper, O. Pironneau, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

In this Note, we propose a pricing method for options (see [2,5] or [1] for instance) under complex diffusion processes. The goal of the work is to take advantage of the flexibility of Monte-Carlo simulations, and when it is possible the accuracy and rapidity of analytical formulas or partial differential equations. We present two examples to illustrate our method, one mixed Monte-Carlo/analytic solver for vanilla options, and one mixed Monte-Carlo/PDE solver for barrier options.

The diffusion process that we have chosen for our examples is the Heston stochastic volatility model [3]: under a risk neutral probability, the risky stock S_t and the volatility σ_t follow the following diffusion process:

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$$\frac{\mathrm{d}S}{S} = r\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t^1, \tag{1}$$
$$\mathrm{d}v_t = k(\theta - v_t)\,\mathrm{d}t + \delta\sqrt{v_t}\,\mathrm{d}W_t^2, \tag{2}$$

with $v_t = \sigma_t^2$, and $\mathbb{E}(\mathrm{d}W_t^1 \cdot \mathrm{d}W_t^2) = \rho\,\mathrm{d}t$, $\mathbb{E}(\cdot)$ denoting the usual expectation.

The pair W^1, W^2 is a two-dimensional correlated Brownian motion, the correlation between the two components being equal to ρ. It is usually observed in financial markets that options with low strikes have a higher implied volatility than at the money or high strikes options, this phenomenon can be rendered by a negative value of ρ, it is known as the smile.

2. The algorithm

The time is discretized into N steps of length δt, and denoting by T the maturity of the option, we have $T = N\delta t$. For a put option with pay off $P = \mathbb{E}(K - S_T)^+$, full Monte-Carlo simulation (see [4]) consists in a time loop starting at S_0, $v_0 = \sigma_0^2$ of

$$\begin{aligned} S_{i+1} &= S_i(1 + r\delta t + \sigma_i\sqrt{\delta t})\big(N^1(0,1)\rho + N^2(0,1)\sqrt{1-\rho^2}\big), \\ v_{i+1} &= v_i + k(\theta - v_i)\delta t + \sigma_i\sqrt{\delta t}N^2(0,1)\delta \quad \text{with } \sigma_i = \sqrt{v_i} \end{aligned} \tag{3}$$

where $N^j(0,1)$, $j = 1,2$ are realizations of two independent Gaussian variables, and then set $P_M = \frac{1}{M}\sum(K - S_N^m)^+$ where $\{S_N^m\}_{m=1}^M$ are M realizations of S_N.

The method is slow, so instead we propose to keep equation (3b) for v_{i+1} but use a PDE for P_M, result of an Îto calculus on (3sa) for each realization $\sigma^m = \{\sqrt{v_i^m}\}_{i=1}^N$. In other words for each m, we solve analytically or numerically the PDE with respect to the variables t and S *conditionally to the volatility realization* σ^m.

2.1. Derivation of the "conditional" Black & Scholes PDE

The main task is to obtain the Black & Scholes PDE conditional to the knowledge of the volatility realization (past and future). Having simulated a trajectory of the volatility, let us first consider the following process:

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \rho\sigma_t S_t\mu_t\,\mathrm{d}t + \sigma_t\sqrt{1-\rho^2}\,\mathrm{d}\tilde{W}_t^1, \tag{4}$$
$$\mu_t = \frac{W^2(t_{i+1}) - W^2(t_i)}{\delta t} - \frac{1}{2}\rho\sigma_t S_t \quad \text{for } t \in [t_i, t_{i+1}). \tag{5}$$

By Ito's formula we have

$$\mathrm{d}\log(S) = r\,\mathrm{d}t + \rho\sigma_t\mu_t\,\mathrm{d}t + \sqrt{1-\rho^2}\sigma_t\,\mathrm{d}\tilde{W}_t^1 - \frac{1}{2}\big(1-\rho^2\big)\sigma_t^2\,\mathrm{d}t.$$

Thus we get

$$\begin{aligned} S(t_{i+1}) &= S(t_i)\exp\left(r\delta t + \sigma_{t_i}\big(\sqrt{1-\rho^2}\big(\tilde{W}_{t_{i+1}}^1 - W_{t_i}^1\big) + \rho\big(W_{t_{i+1}}^2 - W_{t_i}^2\big)\big) - \frac{1}{2}\sigma_{t_i}^2\delta t\right) \\ &= S(t_i)\exp\left(r\delta t + \sigma_{t_i}\big(W_{t_{i+1}}^1 - W_{t_i}^1\big) - \frac{1}{2}\sigma_{t_i}^2\delta t\right). \end{aligned}$$

We recognize here a discretization of the stochastic integral

$$\exp\left(rt + \int_0^t \sigma_t\,\mathrm{d}W_t^1 - \frac{1}{2}\int_0^t \sigma^2\,\mathrm{d}t\right),$$

where σ_t solves (2), hence we can conclude the following

Proposition 2.1. *As $\delta t \to 0$, the process S_t converges to the solution of the stochastic differential equations* (1), (2).

Table 1
Precision versus ρ.

ρ	-0.5	0	0.5	0.9
Heston MC	11.135	10.399	9.587	8.960
Heston MC+BS	11.102	10.391	9.718	8.977
Speed-up	42	44	42	42

Table 2
Variations from sample to sample for various values of M' and M.

Sample	MC+BS			MC		
	$M' = 100$	$M' = 1000$	$M' = 10\,000$	$M = 3000$	$M = 30\,000$	$M = 300\,000$
P^1	10.475	11.129	11.100	11.564	11.481	11.169
P^2	10.436	11.377	11.120	11.6978	11.409	11.249
P^3	11.025	11.528	11.113	11.734	11.383	11.143
P^4	11.205	11.002	11.113	11.565	11.482	11.169
P^5	11.527	11.360	11.150	11.085	11.519	11.208
$P = \frac{1}{5}\sum P^i$	10.934	11.279	11.119	11.529	11.454	11.187
$(\frac{1}{5}\sum(P^i - P)^2)^{\frac{1}{2}}$	0.422	0.188	0.0168	0.232	0.0507	0.0370

The Black & Scholes PDE corresponding to the diffusion (4) reads

$$\partial_t u + rS\partial_S u + \frac{1}{2}\sqrt{1-\rho^2}\sigma_t^2 S^2 \partial_{SS} u + \rho\sigma_t\mu_t S\partial_S u = ru. \tag{6}$$

3. Vanilla grid pricing by the mixed Monte-Carlo/analytic method

For vanilla options with a terminal condition for (6) given by $u(T,S) = \phi(S)$, (6) has an analytical solution:

$$\bar\sigma^2 = \frac{1}{T}\int_0^T \sigma_t^2\,\mathrm{d}t, \qquad M = \frac{\rho}{T}\sum_i \sigma_{t_i}\big(W_{t_{i+1}}^2 - W_{t_i}^2\big),$$

$$S(x) = S_0 \exp\left((r+M)T - \frac{1}{2}\bar\sigma^2 T + \sqrt{1-\rho^2}\bar\sigma x\right),$$

$$u(0,S_0) = e^{-rT}\int_{\mathbb{R}^+} \phi\big(S(x)\big)\frac{\exp(-x^2/2T)}{\sqrt{2\pi T}}\,\mathrm{d}x.$$

We have implemented this algorithm, and compared it against a plain Monte-Carlo simulation for different values of the parameter ρ and different values of the discretization parameters N, δS, M', M where δS is the size of the (uniform) interval used in connection with the finite element method discretization of (6) and M' is the number of Monte-Carlo trials in the new algorithm. Recall that N is the number of time steps (same in (6)) and M is the number of realizations in the pure Monte-Carlo method for the Heston model.

The parameters are: $S0 = 100$, $K = 90$, $r = 0.05$, $\sigma_0 = 0.6$, $\theta = 0.36$, $k = 5$, $\delta = 0.2$, $T = 0.5$. To observe the precision with respect to ρ we have taken a large number of Monte-Carlo samples: $M = 3 \times 10^5$ and $M' = 10^4$. Similarly the number of time steps is 300 with 400 mesh points and $S_{max} = 600$ (i.e. $\delta S = 1.5$) – see Table 1.

To study the precision we let M and M' vary. Table 2 shows the results for 5 samples and the corresponding mean value for P_N and variance. Note that one needs many more samples for pure MC than for the mixed strategy MC+BS. Hence MC+BS is, in fact, faster by a factor much greater than 50.

4. Barrier pricing by the mixed Monte-Carlo/PDE method

Assume that the contract becomes void if the asset S_t goes out of the interval S_m, S_M. The algorithm is the same but the Black–Scholes formula cannot be used; however the PDE is solved only in the interval S_m, S_M, with the two

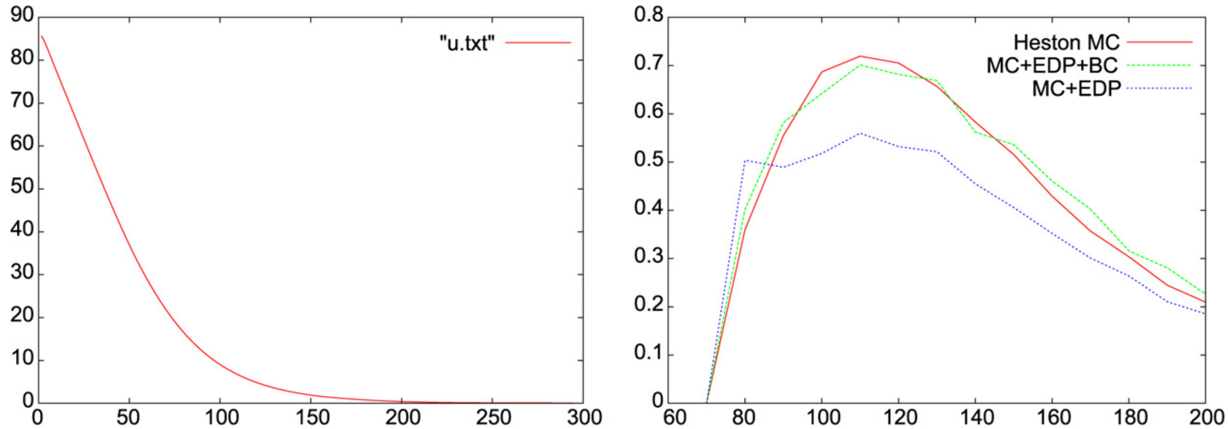

Fig. 1. On the left, the put versus S_0 (the value of the underlying asset today) computed by the mixed Monte-Carlo/EDP method, which is faster than Heston Monte-Carlo but also gives the value of the put for all S_0; the parameters are the same as before with $\rho = -0.5$, $M' = 1000$. On the right the same with a barrier at 70. For a selected set of values for S_0, the results plotted are obtained by 3 different methods: Heston Monte-Carlo, mixed Monte-Carlo/EDP method with and without the Brownian bridge correction. Notice that the correction dramatically improves the precision.

boundary conditions $u(t, S_m) = u(t, S_M) = 0$ for all $t \in (0, T)$. While this PDE is numerically easier than the previous one, the problem is harder for a pure Monte-Carlo solution of Heston's model because one needs to retain only the feasible samples. The cost for the pure Monte-Carlo algorithm is about double the case without constraints while the mixed algorithm is now roughly 5 times slower than pure Monte-Carlo. However, the mixed method gives P_N for all values of S_0, as shown on Fig. 1 for the case without barriers.

4.1. Brownian bridge

However, in order to be more accurate, one should not forget the additional noise corresponding to the Brownian bridge of W^2 between two time steps: indeed, we have replaced the Brownian of the volatility W^2 by an affine function between t_i and t_{i+1}. This is only true at times t_i and t_{i+1}. Between two time steps, the Brownian W^2 can be represented as

$$W^2 = W^2_{t_i} + \frac{t - t_i}{t_{i+1} - t_i}(W_{t_{i+1}} - W_{t_i}) + B_t,$$
$$B_t = \left(\frac{t_{i+1} - t}{\Delta t}\right)^{1/2} W^3_t,$$

where W^3 is a standard Brownian motion independent from W^1, W^2 on each $[t_i, t_{i+1}]$.

Hence, the diffusion for the spot should be replaced by

$$\tilde{S}_t = S_t\left(1 + \sigma_t \rho \left(\frac{t_{i+1} - t}{\Delta t}\right)^{1/2} W^3_t\right). \tag{7}$$

Then we have to derive a suitable PDE: considering a new variable Y, for which

$$Y_t = \sqrt{\frac{t_{i+1} - t}{\Delta t}} W^3_t,$$
$$\mathrm{d}Y_t = \sqrt{\frac{t_{i+1} - t}{\Delta t}}\, \mathrm{d}W^3_t - \frac{1}{\sqrt{(t_{i+1} - t)\Delta t}} W^3_t.$$

Then new B&S PDE system is

$$\partial_t u + rS\partial_S u + \frac{1}{2}\sqrt{1-\rho^2}\sigma^2 S^2 \partial_{SS} u + \rho\sigma_t\mu_t S\partial_S u$$
$$+ \sigma^2 S^2 \rho^2 \left(\frac{t_{i+1} - t}{\Delta t}\partial_{YY} u - \frac{1}{t_{i+1} - t} Y\partial_Y u\right) = ru \quad \text{for } S + Y > B, \tag{8}$$

$$u|_{S+Y=B} = 0, \tag{9}$$
$$u(T,S,Y) = \phi(S). \tag{10}$$

4.2. Numerical approximation by barrier shifting

The law of the maximum for Brownian motion: We consider a Brownian motion W_t, t between 0 and T. We consider $M_T = \max\{W_t, t \in [0,T]\}$. By the reflection principle, we have for $\alpha \leqslant \beta$

$$P(M_T \geqslant \beta, W_T \leqslant \alpha) = P(W_T \geqslant 2\beta - \alpha).$$

Then

$$\begin{aligned} P(M_T \geqslant \beta) &= P(M_T \geqslant \beta, W_T \leqslant \beta) + P(M_T \geqslant \beta, W_T \geqslant \beta) \\ &= P(W_T \geqslant 2\beta - \beta) + P(W_T \geqslant \beta) \\ &= 2P(W_T \geqslant \beta). \end{aligned}$$

Moreover, we have

$$\begin{aligned} P(M_T \geqslant \beta | W_T = \alpha) &= \lim_{\mathrm{d}\alpha \to 0} \frac{P(M_T \geqslant \beta, W_T \leqslant \alpha + \mathrm{d}\alpha) - P(M_T \geqslant \beta, W_T \leqslant \alpha)}{P(W_T \in [\alpha + \mathrm{d}\alpha])} \\ &= \exp\left(-\frac{2\beta(\beta - \alpha)}{T}\right). \end{aligned}$$

We know that

$$B_t = \left(\frac{t_{i+1} - t}{\Delta t}\right)^{1/2} W_t^3$$

is the Brownian bridge between t_i and t_{i+1}. We can approximate its minimum by its expectation, following the considerations above, and thus

$$\mathbb{E}\left(\inf\left\{B_t, t \in [t_i, t_{i+1}]\right\}\right) = -\sqrt{\Delta t/2}.$$

We can therefore approximate the system (8) by the 1-d system, with a barrier shifted upward by a quantity $(\sqrt{\Delta t/2})\sigma\rho B$.

5. Conclusion

As it turns out, the convergence is quite good! Moreover, the computation time is a great deal improved, at least when the Black–Scholes formula can be used. One could argue that there exists closed analytic formulas for the Heston model, but our algorithm adapts with no modification for any diffusion process for the volatility. Moreover, this can be seen as a useful tool when calibrating a stochastic volatility model on the implied volatility surface.

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