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Numerical Analysis

A residual based a posteriori estimator for the reaction-diffusion problem

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Abstract

A residual based a posteriori estimator for the reaction-diffusion problem is introduced. We show that the estimator gives both an upper and a lower bound to error. Numerical results are presented. ***To cite this article: M. Juntunen, R. Stenberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Un estimateur d'erreur de type résiduel pour la probleme de réaction-diffusion. Nous preséntons une estimateur a posteriori de la probleme de réaction-diffusion. Nous montrons que l'estimateur donne à la fois une borne supérieure et une borne inférieure de l'erreur. Quelques résultats numériques sont présenté. ***Pour citer cet article : M. Juntunen, R. Stenberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

We consider the finite element approximation of the reaction-diffusion problem

$$-\varepsilon^2\Delta u + u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

with the parameter $\varepsilon > 0$. For $\varepsilon \gtrsim 1$ the problem is a standard elliptic equation. We are, however, interested in the case of a "small" $\varepsilon \ll 1$. In this case, the problem is a singularly perturbed problem, and the question is how to incorporate the effect of ε into the finite element a posteriori analysis. The problem has been studied for example in [4,1]. Here we introduce and analyze an alternative a posteriori estimator. In [2], this is extended to the Brinkman equations modeling flow in porous media.

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2. The a posteriori error estimate

Let $\Omega \subset \mathbb{R}^n$ be a domain with a polygonal or a polyhedral boundary $\partial\Omega$. We assume a shape regular triangular/tetrahedral partitioning $\mathcal{C}_h$ of the domain Ω. With h_K we denote the diameter of $K \in \mathcal{C}_h$ and we let $h = \max h_K$. With $\mathcal{E}_h$ we denote the internal edges (faces in 3D) of $\mathcal{C}_h$. The constant C is a generic constant independent of the mesh size and problem parameter ε.

Defining the bilinear form

$$\mathcal{A}(u, v) = \varepsilon^2(\nabla u, \nabla v) + (u, v), \tag{2}$$

the weak form of the problem is: find $u \in V$ such that

$$\mathcal{A}(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

Defining $V_h = \{v \in H_0^1(\Omega) \mid v_{|K} \in P_k(K)\ \forall K \in \mathcal{C}_h\}$, the finite element method is: find $u_h \in V_h$ such that

$$\mathcal{A}(u_h, v) = (f, v) \quad \forall v \in V_h. \tag{4}$$

The natural energy norm is

$$\|v\|_\varepsilon^2 = \varepsilon^2\|\nabla v\|_0^2 + \|v\|_0^2, \tag{5}$$

and the finite element solution is the best approximation with respect to this norm

$$\|u - u_h\|_\varepsilon = \inf_{v \in V_h} \|u - v\|_\varepsilon. \tag{6}$$

In general, the problem has a boundary layer of the form $e^{-d/\varepsilon}$, where d is the distance from the boundary. Hence, even for a smooth load f, a uniform mesh will only lead to the following estimate:

$$\|u - u_h\|_\varepsilon \leqslant C\sqrt{h} \tag{7}$$

uniformly valid with respect to ε. For a smooth solution the estimate obtained is

$$\|u - u_h\|_\varepsilon \leqslant C\left(\varepsilon h^k + h^{k+1}\right). \tag{8}$$

To improve the convergence, adaptive mesh refinement is natural. Here, we introduce a novel residual based a posteriori estimator. The elementwise estimator is defined as

$$E_K(u_h)^2 = \frac{h_K^2}{\varepsilon^2 + h_K^2}\left\|\varepsilon^2\Delta u_h - u_h + f\right\|_{0,K}^2 + \frac{h_K}{\varepsilon^2 + h_K^2}\left\|\left[\!\left[\varepsilon^2\partial_n u_h\right]\!\right]\right\|_{0,\partial K\cap\mathcal{E}_h}^2 \tag{9}$$

and the global estimator is

$$\eta = \left(\sum_{K \in \mathcal{C}_h} E_K(u_h)^2\right)^{1/2}. \tag{10}$$

Above $[\![\cdot]\!]$ denotes the jump and ∂_n denotes the normal derivative.

If $\varepsilon \gtrsim 1$, the elementwise estimator recovers the usual estimator for second order elliptic equations

$$E_K(u_h)^2 \approx h_K^2\left\|\varepsilon^2\Delta u_h - u_h + f\right\|_{0,K}^2 + h_K\left\|\left[\!\left[\varepsilon^2\partial_n u_h\right]\!\right]\right\|_{0,\partial K\cap\mathcal{E}_h}^2.$$

On the other hand, in the limit $\varepsilon \to 0$ (or $\varepsilon \ll h$), when the FE solution is the L^2-projection of the loading, we have $E_K(u_h)^2 \approx \|-u_h + f\|_{0,K}^2$.

For our analysis we will need a saturation assumption. The partitioning $\mathcal{C}_h$ is refined into $\mathcal{C}_{h/2}$ by dividing each triangle/tetrahedron K into four/eight elements with mesh size $h_K/2$. By $u_{h/2} \in V_{h/2}$ we denote the finite element solution on the refined mesh.

Assumption 2.1. There exists a positive constant $\beta < 1$ such that

$$\|u - u_{h/2}\|_\varepsilon \leqslant \beta\|u - u_h\|_\varepsilon. \tag{11}$$

The main result is the following theorem:

Theorem 2.2. *Let Assumption* 2.1 *hold. Then there exists* $C > 0$ *such that*

$$\|u - u_h\|_\varepsilon \leqslant C\eta. \tag{12}$$

Proof. By the triangle inequality the saturation assumption gives

$$\|u - u_h\|_\varepsilon \leqslant \frac{C}{1-\beta}\big(\|u_{h/2} - u_h\|_\varepsilon\big). \tag{13}$$

Next, with $v = (u_{h/2} - u_h)/\|u_{h/2} - u_h\|_\varepsilon$, we have

$$\|u_{h/2} - u_h\|_\varepsilon = \mathcal{A}(u_{h/2} - u_h, v) \tag{14}$$

and $\|v\|_\varepsilon = 1$. Let $\tilde{v} \in V_h$ be the Lagrange interpolant of v. Since both v and $\tilde{v}$ are in the finite element spaces, scaling arguments give

$$\bigg(\sum_{K\in\mathcal{C}_{h/2}} \left(\frac{\varepsilon + h_K}{h_K}\right)^2 \|v - \tilde{v}\|_{0,K}^2\bigg)^{1/2} \leqslant C\bigg(\sum_{K\in\mathcal{C}_{h/2}} \big(\varepsilon^2\|\nabla v\|_{0,K}^2 + \|v\|_{0,K}^2\big)\bigg)^{1/2} = C\|v\|_\varepsilon = C \tag{15}$$

and

$$\begin{aligned}\bigg(\sum_{K\in\mathcal{C}_{h/2}} \frac{\varepsilon^2 + h_K^2}{h_K}\|v - \tilde{v}\|_{0,\partial K}^2\bigg)^{1/2} &\leqslant C\bigg(\sum_{K\in\mathcal{C}_{h/2}} \frac{\varepsilon^2 + h_K^2}{h_K} h_K^{-1}\|v - \tilde{v}\|_{0,K}^2\bigg)^{1/2} \\ = C\bigg(\sum_{K\in\mathcal{C}_{h/2}} \left(\frac{\varepsilon^2}{h_K^2} + 1\right)\|v - \tilde{v}\|_{0,K}^2\bigg)^{1/2} &\leqslant C\bigg(\sum_{K\in\mathcal{C}_{h/2}} \big(\varepsilon^2\|\nabla v\|_{0,K}^2 + \|v\|_{0,K}^2\big)\bigg)^{1/2} = C\|v\|_\varepsilon = C.\end{aligned} \tag{16}$$

Since it holds $\mathcal{A}(u_{h/2} - u_h, \tilde{v}) = 0$, we have

$$\mathcal{A}(u_{h/2} - u_h, v) = \mathcal{A}(u_{h/2} - u_h, v - \tilde{v}). \tag{17}$$

Using the fact that $u_{h/2}$ satisfies

$$\mathcal{A}(u_{h/2}, v - \tilde{v}) = (f, v - \tilde{v}) \tag{18}$$

and integrating by parts, we get

$$\begin{aligned}\mathcal{A}(u_{h/2} - u_h, v - \tilde{v}) &= (f, v - \tilde{v}) - \varepsilon^2\big(\nabla u_h, \nabla(v - \tilde{v})\big) - (u_h, v - \tilde{v}) \\ &= \sum_{K\in\mathcal{C}_{h/2}} \big\{\big(\varepsilon^2\Delta u_h - u_h + f, v - \tilde{v}\big)_K + \varepsilon^2\langle\partial_n u_h, v - \tilde{v}\rangle_{\partial K\cap\mathcal{E}_{h/2}}\big\}.\end{aligned} \tag{19}$$

Using Schwartz inequality and the estimates (15)–(16) we then obtain

$$\mathcal{A}(u_{h/2} - u_h, v - \tilde{v}) \leqslant C\eta. \qquad \square \tag{20}$$

The a posteriori upper bound η is also a lower bound to the error. In this sense the estimator is sharp. The proof of the following theorem uses classical techniques, see [3]:

Theorem 2.3. *Let* $f_h \in V_h$ *be an approximation of the load* f. *Then there exist* $C > 0$ *such that*

$$\eta^2 \leqslant C\bigg\{\|u - u_h\|_\varepsilon^2 + \sum_{K\in\mathcal{C}_h}\left(\frac{h_K^2}{\varepsilon^2 + h_K^2}\|f - f_h\|_{0,K}^2\right)\bigg\}. \tag{21}$$

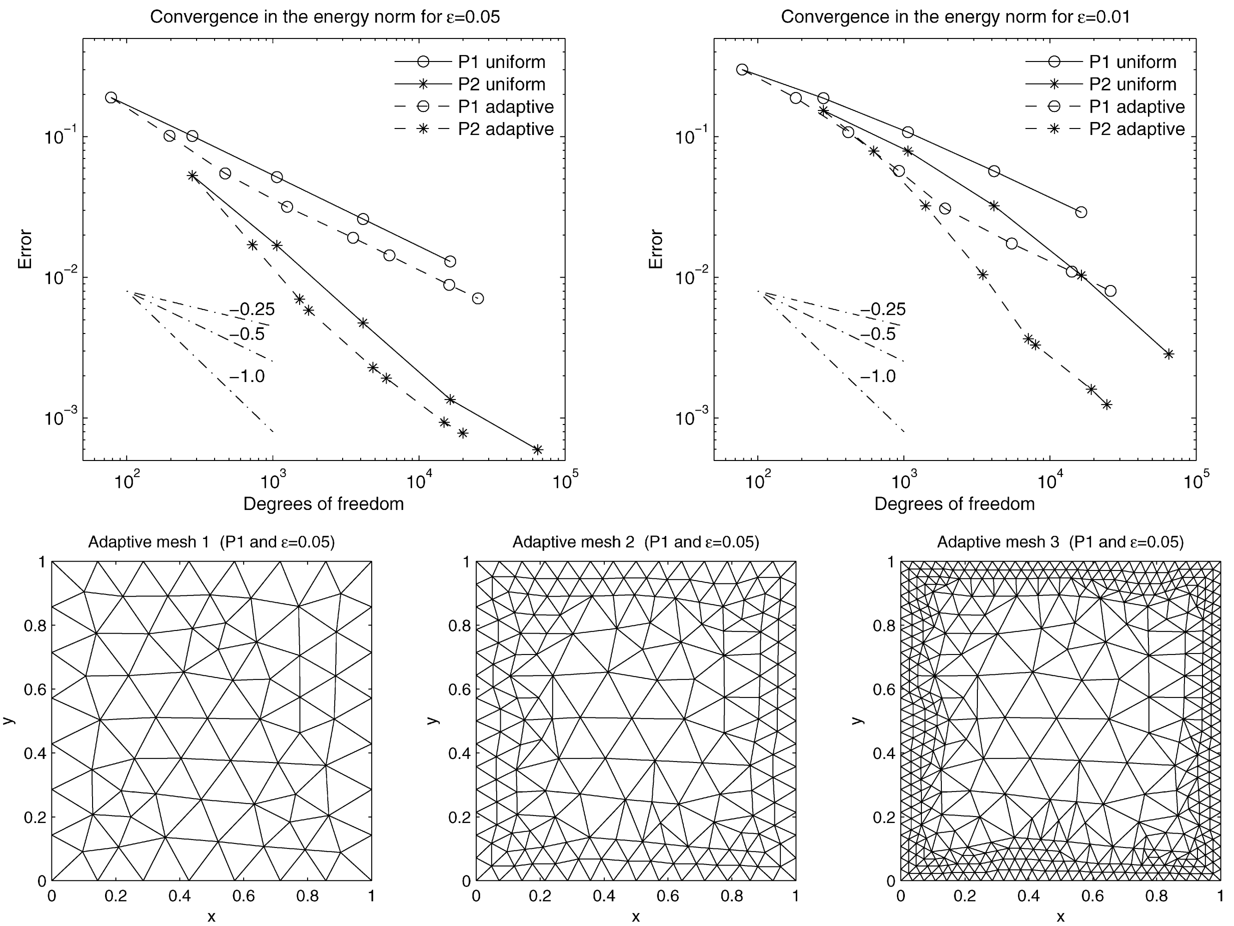

Fig. 1. Upper panels: Convergence for uniform and adaptive meshes for parameter values $\varepsilon = 0.05$ and $\varepsilon = 0.01$. Lower panels: First three meshes of the adaptive scheme using linear elements and parameter value $\epsilon = 0.05$.

3. Numerical results

For the computations we choose the unit square $\Omega = (0,1) \times (0,1)$ and a unit load $f = 1$. For the number of degrees of freedom N, the uniform estimate (7) and the asymptotic estimate (8) become

$$\|u - u_h\|_\varepsilon \leqslant CN^{-0.25} \quad \text{and} \quad \|u - u_h\|_\varepsilon \leqslant C\big(\varepsilon N^{-k/2} + N^{-(k+1)/2}\big), \tag{22}$$

respectively. In Fig. 1 this behavior is seen for linear and quadratic elements ($k = 1, 2$).

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