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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 545–550

COMPTES RENDUS MATHEMATIQUE

Differential Geometry

Hypercomplex structures on Courant algebroids

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Received 6 February 2009; accepted 16 February 2009

Available online 26 March 2009

Presented by Charles-Michel Marle

Abstract

Hypercomplex structures on Courant algebroids unify holomorphic symplectic structures and usual hypercomplex structures. In this Note, we prove the equivalence of two characterizations of hypercomplex structures on Courant algebroids, one in terms of Nijenhuis concomitants and the other in terms of (almost) torsionfree connections for which each of the three complex structures is parallel. ***To cite this article: M. Stiénon, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Structures hypercomplexes sur les algébroïdes de Courant. Les structures hypercomplexes sur les algébroïdes de Courant unifient les structures symplectiques holomorphes et les structures hypercomplexes usuelles. Dans cette Note, nous prouvons l'équivalence de deux caractérisations des structures hypercomplexes sur les algébroïdes de Courant, l'une en termes de concomitants de Nijenhuis et l'autre en termes de connexions (presque) sans torsion pour lesquelles les trois structures complexes sont parallèles. ***Pour citer cet article : M. Stiénon, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Nous introduisons une notion de structure hypercomplexe sur un algébroïde de Courant englobant les « structures hypercomplexes généralisées » définies dans [1] et [3]. Notre principale motivation est le cadre unifié qui en découle pour l'étude des structures hypercomplexes classiques [5] et des structures symplectiques holomorphes.

Soit $(E, \rho, \langle,\rangle, \circ)$ un algébroïde de Courant — la définition est rappelée plus loin. La relation (8) étend la notion de concomitant de Nijenhuis (noté $\mathcal{N}(F, G)$) aux paires F, G d'endomorphismes du fibré vectoriel sous-jacent $E \to M$.

Définition 1.

(i) Une *structure presque hypercomplexe* sur un algébroïde de Courant E est un triple $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ d'endomorphismes du fibré vectoriel E préservant le produit scalaire $\langle,\rangle$ et satisfaisant les relations algébriques des quaternions (7).

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.02.020

(ii) Une *structure hypercomplexe* sur un algébroïde de Courant E est une structure presque hypercomplexe $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ dont les six concomitants de Nijenhuis $\mathcal{N}(\boldsymbol{I}, \boldsymbol{I})$, $\mathcal{N}(\boldsymbol{J}, \boldsymbol{J})$, $\mathcal{N}(\boldsymbol{K}, \boldsymbol{K})$, $\mathcal{N}(\boldsymbol{I}, \boldsymbol{J})$, $\mathcal{N}(\boldsymbol{J}, \boldsymbol{K})$ et $\mathcal{N}(\boldsymbol{K}, \boldsymbol{I})$ sont nuls.

Définition 2. Soit $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ une structure presque hypercomplexe sur un algébroïde de Courant $(E \to M, \rho, \langle,\rangle, \circ)$. Une *connexion hypercomplexe* est une application $\mathbb{R}$-bilinéaire $\Gamma(E) \otimes \Gamma(E) \to \Gamma(E) : (X, Y) \mapsto \nabla_X Y$ satisfaisant (10) et (11) — où $\Delta_f(X, Y)$ est défini par la relation (9) — pour tous $f \in C^\infty(M)$ et $X, Y \in \Gamma(E)$. Sa torsion $T^\nabla : \Gamma(E) \wedge \Gamma(E) \to \Gamma(E)$ est définie par la relation (12).

Dans cette Note, nous établissons le théorème suivant :

Théorème 3. *Soit* $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ *une structure presque hypercomplexe sur un algébroïde de Courant* E. *Les assertions suivantes sont équivalentes.*

(i) $\mathcal{N}(\boldsymbol{I}, \boldsymbol{J}) = \mathcal{N}(\boldsymbol{J}, \boldsymbol{J}) = 0$.
(ii) $\mathcal{N}(\boldsymbol{I}, \boldsymbol{J}) = 0$.
(iii) *Le triple* $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ *est une structure hypercomplexe* ; *les six concomitants de Nijenhuis sont nuls.*
(iv) *Il existe une connexion hypercomplexe satisfaisant* (13) *et* (14).
(v) *Il existe une connexion hypercomplexe satisfaisant* (13) *et* (14) ; *elle est unique et donnée par la formule* (15).

1. Statement of the theorem

A Courant algebroid [4] consists of a vector bundle $\pi : E \to M$, a nondegenerate symmetric pairing $\langle,\rangle$ on the fibers of π, a bundle map $\rho : E \to TM$ called anchor and an $\mathbb{R}$-bilinear operation $\circ$ on $\Gamma(E)$ called Dorfman bracket, which, for all $f \in C^\infty(M)$ and $x, y, z \in \Gamma(E)$ satisfy the relations

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z); \tag{1}$$

$$\rho(x \circ y) = \big[\rho(x), \rho(y)\big]; \tag{2}$$

$$x \circ fy = \big(\rho(x)f\big)y + f(x \circ y); \tag{3}$$

$$x \circ y + y \circ x = 2\mathcal{D}\langle x, y\rangle; \tag{4}$$

$$\mathcal{D}f \circ x = 0; \tag{5}$$

$$\rho(x)\langle y, z\rangle = \langle x \circ y, z\rangle + \langle y, x \circ z\rangle, \tag{6}$$

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is the $\mathbb{R}$-linear map defined by $\langle \mathcal{D}f, x\rangle = \frac{1}{2}\rho(x)f$.

The symmetric part of the Dorfman bracket is given by (4). The Courant bracket is defined as the skew-symmetric part $[\![x, y]\!] = \frac{1}{2}(x \circ y - y \circ x)$ of the Dorfman bracket. Thus we have the relation $x \circ y = [\![x, y]\!] + \mathcal{D}\langle x, y\rangle$.

A standard example is due to T. Courant [2]. Given a smooth manifold M, the vector bundle $TM \oplus T^*M \to M$ carries a natural Courant algebroid structure: the anchor is the projection onto the tangent component while the pairing and Dorfman bracket are given, respectively, by

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}\big(\xi(Y) + \eta(X)\big) \quad \text{and} \quad (X + \xi) \circ (Y + \eta) = [X, Y] + (L_X\eta - \iota_Y \,\mathrm{d}\xi),$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$.

Definition 1. An *almost hypercomplex structure* on a Courant algebroid $(E, \rho, \langle,\rangle, \circ)$ is a triple $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ of endomorphisms of the vector bundle E, i.e. vector bundle maps over $\mathrm{id}_M : M \to M$, which are orthogonal transformations w.r.t. the pairing $\langle,\rangle$ and satisfy the quaternionic relations

$$\boldsymbol{I}^2 = \boldsymbol{J}^2 = \boldsymbol{K}^2 = \boldsymbol{I}\boldsymbol{J}\boldsymbol{K} = -1. \tag{7}$$

Obviously, if $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ is an almost hypercomplex structure, then so are $(\boldsymbol{K}, \boldsymbol{I}, \boldsymbol{J})$ and $(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{I})$.

Let $(E \to M, \rho, \langle,\rangle, \circ)$ be a Courant algebroid. Given two endomorphisms F and G of the vector bundle E, the relation

$$\begin{aligned}\mathcal{N}(F,G)(X,Y) = FX \circ GY - F(X \circ GY) - G(FX \circ Y) + FG(X \circ Y) \\ + GX \circ FY - G(X \circ FY) - F(GX \circ Y) + GF(X \circ Y),\end{aligned} \tag{8}$$

where $X, Y \in \Gamma(E)$, defines a (2,1)-tensor $\mathcal{N}(F,G): E \otimes E \to E$ called Nijenhuis concomitant. Obviously, $\mathcal{N}(F,G) = \mathcal{N}(G,F)$.

Lemma 2. *If $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ is an almost hypercomplex structure on a Courant algebroid E, then $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})(X,Y) + \mathcal{N}(\boldsymbol{I},\boldsymbol{J})(Y,X) = 0$ for all $X, Y \in \Gamma(E)$.*

Definition 3. A *hypercomplex structure* on a Courant algebroid E is an almost hypercomplex structure $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ such that the six Nijenhuis concomitants $\mathcal{N}(\boldsymbol{I},\boldsymbol{I})$, $\mathcal{N}(\boldsymbol{J},\boldsymbol{J})$, $\mathcal{N}(\boldsymbol{K},\boldsymbol{K})$, $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})$, $\mathcal{N}(\boldsymbol{J},\boldsymbol{K})$ and $\mathcal{N}(\boldsymbol{K},\boldsymbol{I})$ vanish.

Remark 1. Let $(E \to M, \rho, \langle,\rangle, \circ)$ be a Courant algebroid and let $\boldsymbol{I}$ and $\boldsymbol{J}$ be two endomorphisms of E such that: $\boldsymbol{I}^2 = \boldsymbol{J}^2 = -1$; $\boldsymbol{I}$ and $\boldsymbol{J}$ anticommute; both $\boldsymbol{I}$ and $\boldsymbol{J}$ are orthogonal w.r.t. the pairing $\langle,\rangle$; and the three Nijenhuis concomitants $\mathcal{N}(\boldsymbol{I},\boldsymbol{I})$, $\mathcal{N}(\boldsymbol{J},\boldsymbol{J})$ and $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})$ vanish. Then it is easy to check that the triple $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{I}\boldsymbol{J})$ is a hypercomplex structure on the Courant algebroid. This is the way Bredthauer originally defined hypercomplex structures in [1]. See also [3].

For any $f \in C^\infty(M)$ and $X, Y \in \Gamma(E)$, let

$$\Delta_f(X,Y) = \langle X,Y\rangle \mathcal{D}f + \langle \boldsymbol{I}X,Y\rangle \boldsymbol{I}\mathcal{D}f + \langle \boldsymbol{J}X,Y\rangle \boldsymbol{J}\mathcal{D}f + \langle \boldsymbol{K}X,Y\rangle \boldsymbol{K}\mathcal{D}f. \tag{9}$$

It is clear that

$$\Delta_f(X,\boldsymbol{I}Y) = \boldsymbol{I}\Delta_f(X,Y), \quad \Delta_f(X,\boldsymbol{J}Y) = \boldsymbol{J}\Delta_f(X,Y), \quad \Delta_f(X,\boldsymbol{K}Y) = \boldsymbol{K}\Delta_f(X,Y)$$

and

$$\Delta_f(X,Y) + \Delta_f(Y,X) = 2\langle X,Y\rangle \mathcal{D}f.$$

Definition 4. Let $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ be an almost hypercomplex structure on a Courant algebroid $(E \to M, \rho, \langle,\rangle, \circ)$. A *hypercomplex connection* is an $\mathbb{R}$-bilinear map

$$\Gamma(E) \otimes \Gamma(E) \to \Gamma(E) : (X,Y) \mapsto \nabla_X Y$$

such that, for all $f \in C^\infty(M)$ and $X, Y \in \Gamma(E)$, we have

$$\nabla_{fX} Y = f \nabla_X Y \tag{10}$$

and

$$\nabla_X(fY) = \big(\rho(X)f\big)Y + f(\nabla_X Y) - \Delta_f(X,Y). \tag{11}$$

Its torsion $T^\nabla : \Gamma(E) \wedge \Gamma(E) \to \Gamma(E)$ is given by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [\![X,Y]\!]. \tag{12}$$

Remark 2. If L is an isotropic subbundle of E stable under $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{K}$, then a hypercomplex connection on E induces a usual L-connection on L.

The purpose of this Note is to establish the following result:

Theorem 5. *Let $(\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ be an almost hypercomplex structure on a Courant algebroid E. The following assertions are equivalent*:

(i) $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})=\mathcal{N}(\boldsymbol{J},\boldsymbol{J})=0$.
(ii) $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})=0$.
(iii) *The triple* $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K})$ *is a hypercomplex structure, i.e. all six Nijenhuis concomitants vanish.*
(iv) *There exists a hypercomplex connection* ∇ *satisfying*

$$\nabla\boldsymbol{I}=\nabla\boldsymbol{J}=\nabla\boldsymbol{K}=0 \tag{13}$$

and, for all $X,Y\in\Gamma(E)$,

$$T^{\nabla}(X,Y)=\boldsymbol{I}\mathcal{D}\langle X,\boldsymbol{I}Y\rangle+\boldsymbol{J}\mathcal{D}\langle X,\boldsymbol{J}Y\rangle+\boldsymbol{K}\mathcal{D}\langle X,\boldsymbol{K}Y\rangle. \tag{14}$$

(v) *There exists a hypercomplex connection satisfying* (13) *and* (14)*; it is unique and given by*

$$\nabla_X Y=-\frac{1}{2}\boldsymbol{K}\big(\boldsymbol{J}Y\circ\boldsymbol{I}X-\boldsymbol{J}(Y\circ\boldsymbol{I}X)-\boldsymbol{I}(\boldsymbol{J}Y\circ X)+\boldsymbol{J}\boldsymbol{I}(Y\circ X)\big).$$

2. Proof of the theorem

The remainder of this note is devoted to the proof of this theorem. Straightforward computations lead to the first two lemmas below, of which the former is a generalization of Theorem 1.1 in [7].

Lemma 6. *Given an almost hypercomplex structure* $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K})$*, the relation*

$$\nabla_X Y=-\frac{1}{2}\boldsymbol{K}\big(\boldsymbol{J}Y\circ\boldsymbol{I}X-\boldsymbol{J}(Y\circ\boldsymbol{I}X)-\boldsymbol{I}(\boldsymbol{J}Y\circ X)+\boldsymbol{J}\boldsymbol{I}(Y\circ X)\big) \tag{15}$$

defines a hypercomplex connection. Permuting $\boldsymbol{I}$*,* $\boldsymbol{J}$ *and* $\boldsymbol{K}$ *cyclically in* (15)*, we obtain two other hypercomplex connections*:

$$\nabla'_X Y=-\frac{1}{2}\boldsymbol{I}\big(\boldsymbol{K}Y\circ\boldsymbol{J}X-\boldsymbol{K}(Y\circ\boldsymbol{J}X)-\boldsymbol{J}(\boldsymbol{K}Y\circ X)+\boldsymbol{K}\boldsymbol{J}(Y\circ X)\big), \tag{16}$$

$$\nabla''_X Y=-\frac{1}{2}\boldsymbol{J}\big(\boldsymbol{I}Y\circ\boldsymbol{K}X-\boldsymbol{I}(Y\circ\boldsymbol{K}X)-\boldsymbol{K}(\boldsymbol{I}Y\circ X)+\boldsymbol{I}\boldsymbol{K}(Y\circ X)\big). \tag{17}$$

Lemma 7. *Given an almost hypercomplex structure* $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K})$*, the hypercomplex connection* (15) *satisfies*

$$\nabla_X\boldsymbol{J}=0, \tag{18}$$

$$(\nabla_X\boldsymbol{I})Y=\frac{1}{2}\boldsymbol{K}\mathcal{N}(\boldsymbol{I},\boldsymbol{J})(X,\boldsymbol{I}Y)+\frac{1}{2}\boldsymbol{J}\mathcal{N}(\boldsymbol{I},\boldsymbol{J})(X,Y), \tag{19}$$

and

$$X\circ Y+\frac{1}{2}\boldsymbol{K}\mathcal{N}(\boldsymbol{I},\boldsymbol{J})(X,Y)=\nabla_X Y-\nabla_Y X+\mathcal{D}\langle X,Y\rangle-\big(\boldsymbol{I}\mathcal{D}\langle X,\boldsymbol{I}Y\rangle+\boldsymbol{J}\mathcal{D}\langle X,\boldsymbol{J}Y\rangle+\boldsymbol{K}\mathcal{D}\langle X,\boldsymbol{K}Y\rangle\big). \tag{20}$$

Corollary 8. *Let* $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K})$ *be an almost hypercomplex structure on a Courant algebroid* E*. If* $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})=0$*, then the hypercomplex connection* (15) *satisfies* (13) *and* (14)*.*

Proof. We always have $\nabla\boldsymbol{J}=0$ by (18). Since $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})=0$, (19) implies that $\nabla\boldsymbol{I}=0$. And it follows from $\boldsymbol{K}=\boldsymbol{I}\boldsymbol{J}$ that

$$\nabla_X\boldsymbol{K}=(\nabla_X\boldsymbol{I})\circ\boldsymbol{J}+\boldsymbol{I}\circ(\nabla_X\boldsymbol{J})=0.$$

Thus (13) is proved and (14) follows immediately from (20) and the relation $x\circ y=[\![x,y]\!]+\mathcal{D}\langle x,y\rangle$. □

Lemma 9. *Given an almost hypercomplex structure* $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K})$*, there exists at most one hypercomplex connection satisfying* (13) *and* (14)*.*

Proof. Assume there exist two such hypercomplex connections ∇^1, ∇^2. Let

$$\Xi(X,Y)=\nabla^2_X Y-\nabla^1_X Y.$$

It follows from (13) that

$$\Xi(X,\boldsymbol{I}Y)=\boldsymbol{I}\Xi(X,Y),\quad \Xi(X,\boldsymbol{J}Y)=\boldsymbol{J}\Xi(X,Y),\quad \Xi(X,\boldsymbol{K}Y)=\boldsymbol{K}\Xi(X,Y)$$

and from (14) that $\Xi(X,Y)=\Xi(Y,X)$. Therefore

$$\begin{aligned}\boldsymbol{K}\Xi(X,X)&=\boldsymbol{I}\boldsymbol{J}\Xi(X,X)=\boldsymbol{I}\Xi(X,\boldsymbol{J}X)=\boldsymbol{I}\Xi(\boldsymbol{J}X,X)=\Xi(\boldsymbol{J}X,\boldsymbol{I}X)=\Xi(\boldsymbol{I}X,\boldsymbol{J}X)\\&=\boldsymbol{J}\Xi(\boldsymbol{I}X,X)=\boldsymbol{J}\Xi(X,\boldsymbol{I}X)=\boldsymbol{J}\boldsymbol{I}\Xi(X,X)=-\boldsymbol{K}\Xi(X,X).\end{aligned}$$

Hence $\Xi(X,X)=0$ for all $X\in\Gamma(E)$ and, consequently,

$$\Xi(X,Y)=\frac{1}{2}\big(\Xi(X+Y,X+Y)-\Xi(X,X)-\Xi(Y,Y)\big)=0$$

for all $X,Y\in\Gamma(E)$. □

Lemma 10. *Given an almost hypercomplex structure* $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K})$*, if there exists a hypercomplex connection satisfying* (13) *and* (14)*, then* $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})=0$.

Proof. From (14), it follows that

$$X\circ Y=\nabla_X Y-\nabla_Y X+\mathcal{D}\langle X,Y\rangle-\big(\boldsymbol{I}\mathcal{D}\langle X,\boldsymbol{I}Y\rangle+\boldsymbol{J}\mathcal{D}\langle X,\boldsymbol{J}Y\rangle+\boldsymbol{K}\mathcal{D}\langle X,\boldsymbol{K}Y\rangle\big).$$

This relation can be used to evaluate each of the terms of $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})$. It follows from (13), the quaternionic relations (7), and the orthogonality of the endomorphisms $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{K}$ w.r.t. the pairing that $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})$ vanishes. □

Together, Lemma 10, Corollary 9 and Lemma 8 imply the following:

Proposition 11. *Given an almost hypercomplex structure* $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K})$ *on a Courant algebroid E, there exists a hypercomplex connection satisfying* (13) *and* (14) *if and only if* $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})=0$. *And in that case, it coincides with all three hypercomplex connections given by* (15), (16) *and* (17).

Proposition 12. *Let* $(\boldsymbol{I},\boldsymbol{J},\boldsymbol{K})$ *be an almost hypercomplex structure on a Courant algebroid. The following assertions are equivalent*:

(i) $\mathcal{N}(\boldsymbol{I},\boldsymbol{I})=\mathcal{N}(\boldsymbol{J},\boldsymbol{J})=0$;
(ii) $\mathcal{N}(\boldsymbol{I},\boldsymbol{J})=0$;
(iii) $\mathcal{N}(\boldsymbol{I},\boldsymbol{I})=\mathcal{N}(\boldsymbol{J},\boldsymbol{J})=\mathcal{N}(\boldsymbol{K},\boldsymbol{K})=\mathcal{N}(\boldsymbol{I},\boldsymbol{J})=\mathcal{N}(\boldsymbol{J},\boldsymbol{K})=\mathcal{N}(\boldsymbol{K},\boldsymbol{I})=0$.

Proof. (i) ⇒ (ii) The proof is a lengthy computation similar to that of [6, Theorem 3.1]. It is omitted. (ii) ⇒ (iii) For any pair of elements P, Q in $\{\boldsymbol{I},\boldsymbol{J},\boldsymbol{K}\}$, we can evaluate the Nijenhuis concomitant

$$\begin{aligned}\mathcal{N}(P,Q)(X,Y)&=PX\circ QY-P(X\circ QY)-Q(PX\circ Y)+PQ(X\circ Y)\\&\quad+QX\circ PY-Q(X\circ PY)-P(QX\circ Y)+QP(X\circ Y)\end{aligned}\tag{21}$$

by successively making use of: *primo* relation (20) to get rid of all the Dorfman brackets in the r.h.s. of (21); *secondo* (13) and the quaternionic relations (7) to cancel all terms involving ∇; *terzo* (7) and the orthogonality of $\boldsymbol{I}$, $\boldsymbol{J}$ and $\boldsymbol{K}$ w.r.t. the pairing to cancel all remaining terms. (iii) ⇒ (i) This is trivial. □

Theorem 5 immediately follows from Propositions 11 and 12.

Example 1. Let i, j, k be three almost complex structures on a smooth manifold X. The triple

$$\boldsymbol{I}=\begin{pmatrix}-i&0\\0&i^*\end{pmatrix},\qquad \boldsymbol{J}=\begin{pmatrix}-j&0\\0&j^*\end{pmatrix},\qquad \boldsymbol{K}=\begin{pmatrix}-k&0\\0&k^*\end{pmatrix}$$

is a hypercomplex structure on $TX \oplus T^*X$ if and only if the triple i, j, k is hypercomplex in the classical sense (see [5]).

Example 2. Let j be an almost complex structure on a smooth manifold X and let ω_1 and ω_2 be two nondegenerate 2-forms on X. The triple

$$\boldsymbol{I} = \begin{pmatrix} 0 & \omega_2^{-1} \\ -\omega_2 & 0 \end{pmatrix}, \qquad \boldsymbol{J} = \begin{pmatrix} -j & 0 \\ 0 & j^* \end{pmatrix}, \qquad \boldsymbol{K} = \begin{pmatrix} 0 & \omega_1^{-1} \\ -\omega_1 & 0 \end{pmatrix}$$

is hypercomplex on $TX \oplus T^*X$ if and only if $\omega_1 + i\omega_2 \in \Omega^2_{\mathbb{C}}(X)$ is a holomorphic symplectic structure on X. Theorem 5 has interesting consequences in this case, which we will discuss somewhere else.

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