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Partial Differential Equations/Harmonic Analysis

Boundedness of the gradient of a solution to the Neumann–Laplace problem in a convex domain

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Abstract

It is shown that solutions of the Neumann problem for the Poisson equation in an arbitrary convex n-dimensional domain are uniformly Lipschitz. Applications of this result to some aspects of regularity of solutions to the Neumann problem on convex polyhedra are given. ***To cite this article: V. Maz'ya, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Bornitude du gradient d'une solution du problème de Neumann pour le Laplacien dans un domaine convexe. On démontre que les solutions du problème de Neumann pour l'équation de Poisson dans un domaine convexe arbitraire de dimension n sont uniformément Lipschitz. Les applications de ce résultat à quelques aspects de régularité de solutions du problème de Neumann sur les polyèdres convexes sont données. ***Pour citer cet article : V. Maz'ya, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let Ω be a bounded convex domain in $\mathbb{R}^n$ and let $W^{l,p}(\Omega)$ stand for the Sobolev space of functions in $L^p(\Omega)$ with distributional derivatives of order l in $L^p(\Omega)$. By $L^p_\perp(\Omega)$ and $W^{l,p}_\perp(\Omega)$ we denote the subspaces of functions v in $L^p(\Omega)$ and $W^{l,p}(\Omega)$ subject to $\int_\Omega v\,\mathrm{d}x = 0$.

Let $f \in L^2_\perp(\Omega)$ and let u be the unique function in $W^{1,2}(\Omega)$, also orthogonal to 1 in $L^2(\Omega)$, and satisfying the Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where ν is the unit outward normal vector to $\partial\Omega$ and the problem (1) is understood in the variational sense. It is well known that the inverse mapping $L^2_\perp(\Omega) \ni f \to u \in W^{2,2}_\perp(\Omega)$ is continuous [3,4,10,12,14–17,20,23,24]. As shown

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in [2] (see also [11] for a different proof, and [1,7–9] for the Dirichlet problem), the operator $L^p_\perp(\Omega) \ni f \to u \in W^{2,p}_\perp(\Omega)$ is also continuous if $1 < p < 2$. One cannot guarantee the continuity of this mapping for any $p \in (2, \infty)$ without additional information about the domain. The situation is the same as in the case of the Dirichlet problem (see [4,7–9]), which, moreover, possesses the following useful property: if Ω is convex, the gradient of the solution is uniformly bounded provided the right-hand side of the equation is good enough. This property can be easily checked by using a simple barrier. Other approaches to similar results were exploited in [18] and [13] for different equations and systems but only for the Dirichlet boundary conditions. In this respect, other boundary value problems are in a nonsatisfactory state. For instance, it was unknown up to now whether solutions of (1) with a smooth f are uniformly Lipschitz under the only condition of convexity of Ω.

The main result of the present Note is *the boundedness of* $|\nabla u|$ *for the solution* u *of the Neumann problem* (1) *in any convex domain* $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$. A direct consequence of this fact is the sharp lower estimate $\Lambda \geqslant n - 1$ for the first nonzero eigenvalue Λ of the Neumann problem for the Beltrami operator on a convex subdomain of a unit sphere. It was obtained by a different argument for manifolds of positive Ricci curvature by J.F. Escobar in [6], where the case of equality was settled as well. This estimate answered a question raised by M. Dauge [5], and it leads, in combination with known techniques of the theory of elliptic boundary value problems in domains with piecewise smooth boundaries (see [5,22]), to estimates for solutions of the problem (1) in various function spaces. Two examples are given at the end of this article.

2. Main result

In what follows, we need a constant C_Ω in the relative isoperimetric inequality $s(\Omega \cap \partial g) \geqslant C_\Omega |g|^{1-1/n}$, where g is an arbitrary open set in Ω such that $|g| \leqslant |\Omega|/2$ and $\Omega \cap \partial g$ is a smooth (not necessarily compact) submanifold of Ω. By s we denote the $(n-1)$-dimensional area and by $|g|$ the n-dimensional Lebesgue measure. The Poincaré–Gagliardo–Nirenberg inequality

$$\inf_{t \in \mathbb{R}} \|v - t\|_{L^{n/(n-1)}(\Omega)} \leqslant \text{const.} \|\nabla v\|_{L^1(\Omega)}, \quad \forall v \in W^{1,1}(\Omega), \tag{2}$$

where const. $\leqslant C_\Omega^{-1}$ stems the above isoperimetric inequality (see Theorem 2.2.3 [21]).

Theorem. *Let* $f \in L^q_\perp(\Omega)$ *with a certain* $q > n$. *Then there exists is a constant* c *depending only on* n *and* q *such that the solution* $u \in W^{1,2}_\perp(\Omega)$ *of the problem* (1) *satisfies the estimate*

$$\|\nabla u\|_{L^\infty(\Omega)} \leqslant c(n, q) C_\Omega^{-1} |\Omega|^{(q-n)/qn} \|f\|_{L^q(\Omega)}. \tag{3}$$

The argument leading to (3) is based on the inequality

$$\int_\Omega \Psi'\big(|\nabla u|\big)\Big(\big(|\nabla u|\big)_{x_j} u_{x_j} f + \big(|\nabla u|\big)_{x_i} u_{x_j} u_{x_i x_j}\Big)\,\mathrm{d}x \leqslant \int_\Omega \Psi\big(|\nabla u|\big) f^2\,\mathrm{d}x \tag{4}$$

with a properly chosen Ψ. The proof will be published elsewhere.

3. Neumann problem in a convex polyhedron

The following assertion essentially stemming from the above theorem is a particular case of Escobar's result in [6] mentioned in the Introduction:

Corollary. *Let* ω *be a convex subdomain of the unit sphere* S^{n-1}. *The first positive eigenvalue* Λ *of the Beltrami operator on* ω *with zero Neumann data on* $\partial\omega$ *is not less than* $n - 1$.

Proof. Let $\lambda(\lambda + n - 2) = \Lambda$ and $\lambda > 0$. In the convex domain $\Omega = \{x \in \mathbb{R}^n\colon\ 0 < |x| < 1, x|x|^{-1} \in \omega\}$, we define the function $u(x) = |x|^\lambda \Phi(x/|x|)\eta(|x|)$, where Φ is an eigenfunction corresponding to Λ and η is a smooth cut-off function on $[0, \infty)$, equal to one on $[0, 1/2]$ and vanishing outside $[0, 1]$. Let N be an integer satisfying $4N > n - 1 \geqslant 4(N - 1)$ and let $j = 0, 1, \ldots, N$. We set $q_j = 2(n-1)/(n-1-4j)$ if $0 \leqslant j < (n-1)/4$, q_j is arbitrary if

$j = (n-1)/4$, and $q_N = \infty$. Iterating the estimate $\|\Phi\|_{L^{q_{j+1}}(\omega)} \leqslant c\Lambda \|\Phi\|_{L^{q_j}(\omega)}$ obtained in Theorems 5 and 6 [19], we see that $\Phi \in L^\infty(\omega)$.

The function u satisfies the problem (1) with $f(x) = -\Phi(x/|x|)[\Delta, \eta(|x|)]|x|^\lambda$. Since $\Phi \in L^\infty(\omega)$, it follows that $f \in L^\infty(\Omega)$ and by Theorem, $|\nabla u| \in L^\infty(\Omega)$, which is possible only if $\lambda \geqslant 1$, i.e. $\Lambda \geqslant n-1$. The proof is complete.

Two applications of the above estimate for Λ will be formulated.

Let Ω be a convex bounded 3-dimensional polyhedron. By the techniques, well-known nowadays (see [5,22]), one can show the unique solvability of the variational Neumann problem in $W_\perp^{1,p}(\Omega)$ for every $p \in (1,\infty)$. By definition of this problem, its solution is subject to the integral identity

$$\int_\Omega \nabla u \cdot \nabla \eta \, \mathrm{d}x = f(\eta),$$

where $f \in (W^{1,p'}(\Omega))^*$, $f(1) = 0$ and η is an arbitrary function in $W^{1,p'}(\Omega)$.

Let us turn to the second application of Corollary. We continue to deal with the polyhedron Ω in $\mathbb{R}^3$. Let $\{\mathcal{O}\}$ be the collection of all vertices and let $\{U_\mathcal{O}\}$ be an open finite covering of $\overline{\Omega}$ such that $\mathcal{O}$ is the only vertex in $U_\mathcal{O}$. Let also $\{E\}$ be the collection of all edges and let α_E denote the opening of the dihedral angle with edge E, $0 < \alpha_E < \pi$. The notation $r_E(x)$ stands for the distance between $x \in U_\mathcal{O}$ and the edge E such that $\mathcal{O} \in \overline{E}$.

With every vertex $\mathcal{O}$ and edge E we associate real numbers $\beta_\mathcal{O}$ and δ_E, and we introduce the weighted L^p-norm

$$\|v\|_{L^p(\Omega;\{\beta_\mathcal{O}\},\{\delta_E\})} := \left(\sum_{\{\mathcal{O}\}} \int_{U_\mathcal{O}} |x-\mathcal{O}|^{p\beta_\mathcal{O}} \prod_{\{E:\mathcal{O}\in\overline{E}\}} \left(\frac{r_E(x)}{|x-\mathcal{O}|} \right)^{p\delta_E} \left| v(x) \right|^p \mathrm{d}x \right)^{1/p},$$

where $1 < p < \infty$. Under the conditions $3/p' > \beta_\mathcal{O} > -2 + 3/p'$ and $2/p' > \delta_E > -\min\{2, \pi/\alpha_E\} + 2/p'$ the inclusion $f \in L^p(\Omega; \{\beta_\mathcal{O}\}, \{\delta_E\})$ implies the unique solvability of (1) in the class of functions with all derivatives of the second order in $L^p(\Omega; \{\beta_\mathcal{O}\}, \{\delta_E\})$. This fact follows from Corollary and a result in Section 7.5 [22].

An important particular case when all $\beta_\mathcal{O}$ and δ_E vanish, i.e. when we deal with a standard Sobolev space $W^{2,p}(\Omega)$, is also included here. To be more precise, if $1 < p < \min\{3, 2\alpha_E/(2\alpha_E - \pi)_+\}$ for all edges E, then the inverse operator of the problem (1): $L_\perp^p(\Omega) \ni f \to u \in W_\perp^{2,p}(\Omega)$ is continuous whatever the convex polyhedron $\Omega \subset \mathbb{R}^3$ may be. The above bounds for p are sharp for the class of all convex polyhedra. □

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