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Hypocoercivity for kinetic equations with linear relaxation terms

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Abstract

This Note is devoted to a simple method for proving the hypocoercivity associated to a kinetic equation involving a linear time relaxation operator. It is based on the construction of an adapted Lyapunov functional satisfying a Gronwall-type inequality. The method clearly distinguishes the coercivity at microscopic level, which directly arises from the properties of the relaxation operator, and a spectral gap inequality at the macroscopic level for the spatial density, which is connected to the diffusion limit. It improves on previously known results. Our approach is illustrated by the linear BGK model and a relaxation operator which corresponds at macroscopic level to the linearized fast diffusion. ***To cite this article: J. Dolbeault et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Hypocoercivité pour des équations cinétiques avec termes de relaxation linéaires. Cette Note est consacrée à une méthode simple pour démontrer l'hypocoercivité associée à une équation cinétique contenant un opérateur de relaxation linéaire ; il s'agit de construire une fonctionnelle de Lyapunov adaptée vérifiant une inégalité de type Gronwall. La méthode distingue clairement la coercivité au niveau microscopique, qui provient directement des propriétés de l'opérateur de relaxation, et une inégalité de trou spectral pour la densité spatiale, qui est reliée à la limite de diffusion. Elle améliore les résultats antérieurs. Notre approche est illustrée par le modèle de BGK linéaire et par un opérateur de relaxation qui correspond, au niveau macroscopique, à la diffusion rapide linéarisée. ***Pour citer cet article : J. Dolbeault et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Le but de la théorie de l'*hypocoercivité* [10,11,8] est d'estimer des taux exponentiels de retour à un équilibre global pour des équations cinétiques dans lesquelles le terme de collision ne contrôle que le retour à un équilibre local. Nous simplifions les résultats antérieurs, [6,11,5], en distinguant l'échelle microscopique et l'échelle macroscopique, pour laquelle une propriété de trou spectral fait le lien avec les limites de diffusion, [1,2].

Considérons l'équation (1) sur $\mathbb{R}^d$, $d \geqslant 1$ où $\mathsf{T} := v \cdot \nabla_x - \nabla_x V \cdot \nabla_v$, V est un *potentiel extérieur* et où l'opérateur de relaxation linéaire L est défini par (2). La fonction F est une mesure de probabilité strictement positive ne dépendant

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que de $|v|^2/2 + V(x)$ et on notera $\mathrm{d}\mu(x,v) = F(x,v)^{-1}\,\mathrm{d}x\,\mathrm{d}v$. Norme et produit scalaire sont par défaut ceux de $L^2(\mathrm{d}\mu)$. La donnée initiale $f_0 \in L^2(\mathrm{d}\mu)$ est normalisée par $\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0\,\mathrm{d}x\,\mathrm{d}v = 1$.

Théorème 1 *(équilibre Maxwellien). Supposons que $V \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$, $d \geqslant 1$, est tel que :* 1) $\int_{\mathbb{R}^d} e^{-V}\,\mathrm{d}x = 1$, 2) *il existe une constante $\Lambda > 0$ telle que $\int_{\mathbb{R}^d} |u|^2 e^{-V}\,\mathrm{d}x \leqslant \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2 e^{-V}\,\mathrm{d}x$ pour tout $u \in H^1(e^{-V}\,\mathrm{d}x)$ vérifiant $\int_{\mathbb{R}^d} u e^{-V}\,\mathrm{d}x = 0$,* 3) *il existe des constantes $c_0 > 0$, $c_1 > 0$ et $\theta \in (0,1)$ telles que $\Delta V \leqslant \frac{\theta}{2}|\nabla_x V(x)|^2 + c_0$ et $|\nabla_x^2 V(x)| \leqslant c_1(1 + |\nabla_x V(x)|)$ $\forall x \in \mathbb{R}^d$,* 4) $\int_{\mathbb{R}^d} |\nabla_x V|^2 e^{-V}\,\mathrm{d}x < \infty$.

Si $F(x,v) := M(v)e^{-V(x)}$ avec $M(v) := (2\pi)^{-d/2}e^{-|v|^2/2}$, alors pour tout $\eta > 0$, il existe une constante positive $\lambda = \lambda(\eta)$, explicite, pour laquelle toute solution de (1) *dans $L^2(\mathrm{d}\mu)$ vérifie :*

$$\forall t \geqslant 0, \quad \|f(t) - F\|^2 \leqslant (1+\eta)\|f_0 - F\|^2 e^{-\lambda t}.$$

Dans le deuxième cas, on supposera que $F(x,v) := \omega(\frac{1}{2}|v|^2 + V(x))^{-(k+1)}$, $V(x) = (1+|x|^2)^\beta$ pour simplifier. Voir [3] pour des cas plus généraux. Ici, $\rho_F = \omega_0 V^{d/2-k-1}$, ω et ω_0 sont des constantes de normalisation.

Théorème 2. *Soit $d \geqslant 1$, $k > d/2 + 1$. Il existe une constante $\beta_0 > 1$ telle que, pour tout $\beta \in (\min\{1, (d-4)/(2k-d-2)\}, \beta_0)$, il existe deux constantes strictement positives C et λ, explicites, pour laquelle, avec F et V définis ci-dessus, toute solution de* (1) *dans $L^2(\mathrm{d}\mu)$ vérifie aussi l'estimation du Théorème* 1.

Sur $L^2(\mathrm{d}\mu)$, on définit les opérateurs $\mathsf{b}f := \Pi(vf)$, $\mathsf{a}f := \mathsf{b}(\mathsf{T}f)$, $\hat{\mathsf{a}}f := -\Pi(\nabla_x f)$, $\mathsf{A} := (1 + \hat{\mathsf{a}}\cdot\mathsf{a}\Pi)^{-1}\hat{\mathsf{a}}\cdot\mathsf{b}$ et $H(f) := \frac{1}{2}\|f\|^2 + \varepsilon\langle \mathsf{A}f, f\rangle$. Si f est une solution de l'Éq. (1), alors $D(f-F) := \frac{\mathrm{d}}{\mathrm{d}t}H(f-F)$ est donné par (3). Quitte à remplacer f par $f - F$, il s'agit de montrer que $D(f) + \lambda H(f) \geqslant 0$. Par l'inégalité de trou spectral, $-\varepsilon\langle \mathsf{AT}\Pi f, f\rangle \leqslant -\varepsilon\frac{\Lambda}{1+\Lambda}\|\Pi f\|^2$ contrôle les termes faisant intervenir $\|\Pi f\|^2$ à l'ordre ε. Comme $2\|\mathsf{A}f\|^2 + \|\mathsf{TA}f\|^2 \leqslant \|(1-\Pi)f\|^2$, le terme $\langle f, \mathsf{L}f\rangle = -\|(1-\Pi)f\|^2$ permet de contrôler tous les autres termes, et en particulier $\|(\mathsf{AT}(1-\Pi))f\|^2$ par dualité : si $(\mathsf{AT}(1-\Pi))^* f = (\hat{\mathsf{a}}\cdot\mathsf{a}(1-\Pi))^* g$ avec $g = (1 + \hat{\mathsf{a}}\cdot\mathsf{a}\Pi)^{-1}f$, alors $u := \rho(g)/\rho_F$ est donnée par (4) avec $m_F(x) := \int_{\mathbb{R}^d} |v|^2 F(x,v)\,\mathrm{d}v$. Ceci revient à établir une estimation H^2 pour la solution de (4). Dans le cas Maxwellien, il faut d'abord établir une inégalité de Poincaré améliorée : il existe une constante $\kappa > 0$ telle que, pour tout $u \in H^1(e^{-V}\,\mathrm{d}x)$ vérifiant $\int_{\mathbb{R}^d} u e^{-V}\,\mathrm{d}x = 0$, $\kappa\int_{\mathbb{R}^d} |\nabla_x V|^2 |u|^2\,\mathrm{d}x \leqslant \|\nabla_x u\|_0^2$, d'où l'on déduit que $\int_{\mathbb{R}^d} |\nabla_x V|^2 |\nabla_x u|^2 e^{-V}\,\mathrm{d}x$ et $\|\nabla_x^2 u\|_0^2$ sont contrôlés par $\|f\|^2$. Dans le cas de la diffusion rapide, il suffit de multiplier (4) par $V^{1-1/\beta}u$ et par $V\Delta u$ pour contrôler $\|(\mathsf{AT}(1-\Pi))^* f\|^2$ par $\|f\|^2$.

1. Introduction

A fundamental question which goes back to the early days of kinetic theory is to estimate the *rate of relaxation* of the solutions towards a global equilibrium. This is not an easy issue since the collision term responsible for the relaxation acts, in most of the cases, only on the velocity space. Rates of convergence have been investigated in many papers for the so-called homogeneous kinetic equations, but understanding how the transport operator interacts with collisions to produce a global relaxation is a different and much more recent story. The point is to understand how the spatial density evolves towards a density corresponding to a distribution function which is simultaneously in the kernels of the collision and transport operators, a property of the stationary solutions of many kinetic equations. There is an obvious link with diffusion or hydrodynamic limits. A key feature of our approach is that it clearly distinguishes the mechanisms of relaxation at *microscopic level* (convergence towards a local equilibrium, in velocity space) and *macroscopic level* (convergence of the spatial density to a steady state), where the rate is given by a spectral gap which has to do with the underlying diffusion equation for the spatial density. See [3] for more details.

First non-constructive results were obtained by Ukai et al., see for instance [9]. Constructive methods inspired from *hypoelliptic theory* (see e.g. [7]) were then brought into the field of kinetic theory by F. Hérau and F. Nier, see for instance [6] in case of the Vlasov–Fokker–Planck equation. In a recent paper, [5], F. Hérau studied with such tools the case of an operator of zeroth order in the derivatives, which is known in the kinetic literature as the linear Boltzmann relaxation operator. Our approach is done in the spirit of [5] but in a simplified framework for which the order of the operator plays no role. Moreover, explicit estimates on the relaxation rate easily follow, weaker assumptions on the external potential than in [5] are needed and the method applies to more general relaxation operators of which we shall give an example. This example is based on kinetic equations which have been studied in [2] and give equations of fast diffusion in the diffusion limit.

The hypoelliptic theory is mainly focused on the regularization properties of the evolution operator, but in some cases the hypoelliptic estimates also imply a result of relaxation to equilibrium. However both questions are independent and have to be distinguished. In the *hypocoercivity* approach, the purpose is centered on the asymptotic behavior and the quantification of the relaxation rates. More precisely our goal is to construct a Lyapunov functional, or generalized entropy, and establish an inequality relating the entropy and its time derivative along the flow defined by the evolution equation. To establish the inequality is then equivalent to proving an exponential rate. Such an approach has systematically been tackled by C. Villani, see [10,11], and has been successfully applied to various models, see for instance [8]. It is also related to recent works on non-linear Boltzmann and Landau equations, see e.g. [4]. In this Note, we develop a new approach based on operators with less algebraic properties than the ones of the hypoelliptic theory, but which are better adapted to the micro–macro decomposition of the distribution function and give a much simpler insight of the mechanisms responsible for the relaxation at both levels. We illustrate our approach on two examples: the linear BGK and the linearized fast diffusion models. We refer the interested reader to a forthcoming paper, [3], in which the theory will be developed at a more general and abstract level.

2. Main results

Let V be a given *external potential* on $\mathbb{R}^d$, $d \geqslant 1$, and consider the kinetic equation

$$\partial_t f + \mathsf{T} f = \mathsf{L} f, \quad f = f(t,x,v), \quad x \in \mathbb{R}^d, \quad v \in \mathbb{R}^d, \tag{1}$$

where $\mathsf{T} := v \cdot \nabla_x - \nabla_x V \cdot \nabla_v$ is a transport operator, and the linear relaxation operator L is defined by

$$\mathsf{L} f = \Pi f - f, \quad \Pi f := \frac{\rho}{\rho_F} F(x,v), \quad \rho = \rho(f) := \int_{\mathbb{R}^d} f \,\mathrm{d}v \quad \text{and} \quad \rho_F = \rho(F) \tag{2}$$

for some function $F(x,v) > 0$ which only depends on $|v|^2/2 + V(x)$. On $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v)$, we consider the measure $\mathrm{d}\mu(x,v) = F(x,v)^{-1}\,\mathrm{d}x\,\mathrm{d}v$ where F is a positive probability measure. Unless it is explicitly specified, the scalar product and the norm are the ones of $L^2(\mathrm{d}\mu)$: $\langle f, g\rangle = \int_{\mathbb{R}^d \times \mathbb{R}^d} f g \,\mathrm{d}\mu$ and $\|f\|^2 = \langle f, f\rangle$. Throughout this paper, Eq. (1) is supplemented with a nonnegative initial datum $f_0 \in L^2(\mathrm{d}\mu)$ such that $\int_{\mathbb{R}^d \times \mathbb{R}^d} f_0 \,\mathrm{d}x\,\mathrm{d}v = 1$. We shall assume that a unique solution globally exists. This is granted under additional technical assumptions, see for instance [2]. The goal of this Note is to state hypocoercivity results in the two following cases:

2.1. Maxwellian case

We assume that $F(x,v) := M(v) e^{-V(x)}$ with $M(v) := (2\pi)^{-d/2} e^{-|v|^2/2}$, where $V(x) = C_k (1+|x|^2)^{k/2}$ for $k > 1$ and C_k is chosen appropriately or, more generally, satisfies the following assumptions:

(H1) *Regularity*: $V \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$.
(H2) *Normalization*: $\int_{\mathbb{R}^d} e^{-V}\,\mathrm{d}x = 1$.
(H3) *Spectral gap condition*: there exists a positive constant Λ such that $\int_{\mathbb{R}^d} |u|^2 e^{-V}\,\mathrm{d}x \leqslant \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2 e^{-V}\,\mathrm{d}x$ for any $u \in H^1(e^{-V}\,\mathrm{d}x)$ such that $\int_{\mathbb{R}^d} u e^{-V}\,\mathrm{d}x = 0$.
(H4) *Pointwise condition* 1: there exists $c_0 > 0$ and $\theta \in (0,1)$ such that $\Delta V \leqslant \frac{\theta}{2} |\nabla_x V(x)|^2 + c_0$ $\forall x \in \mathbb{R}^d$.
(H5) *Pointwise condition* 2: there exists $c_1 > 0$ such that $|\nabla_x^2 V(x)| \leqslant c_1 (1 + |\nabla_x V(x)|)$ $\forall x \in \mathbb{R}^d$.
(H6) *Growth condition*: $\int_{\mathbb{R}^d} |\nabla_x V|^2 e^{-V}\,\mathrm{d}x < \infty$.

Theorem 2.1. *For any $\eta > 0$, there exists an explicit, positive constant $\lambda = \lambda(\eta)$ such that, under the above assumptions, the solution of* (1) *satisfies*:

$$\forall t \geqslant 0, \quad \left\| f(t) - F \right\|^2 \leqslant (1+\eta) \|f_0 - F\|^2 e^{-\lambda t}.$$

2.2. Fast diffusion case

For some $\beta > 0$ to be specified later, we assume that $F(x,v) := \omega(\frac{1}{2}|v|^2 + V(x))^{-(k+1)}$, $V(x) = (1+|x|^2)^\beta$ where ω is a normalization constant chosen such that $\int_{\mathbb{R}^d\times\mathbb{R}^d} F\,\mathrm{d}x\,\mathrm{d}v = 1$ and $\rho_F = \omega_0 V^{d/2-k-1}$ for some $\omega_0 > 0$. More general choices for V and corresponding assumptions can be found in [3].

Theorem 2.2. *Let $d \geqslant 1$, $k > d/2 + 1$. There exists a constant $\beta_0 > 1$ such that, for any $\beta \in (\min\{1, (d-4)/(2k-d-2)\}, \beta_0)$, there are two positive, explicit constants η and λ for which the solution of* (1) *satisfies*:

$$\forall t \geqslant 0, \quad \|f(t) - F\|^2 \leqslant (1+\eta)\|f_0 - F\|^2 e^{-\lambda t}.$$

2.3. A Lyapunov functional

On $L^2(\mathrm{d}\mu)$, Π is the orthogonal projection onto the space of local equilibria. Let us define

$$\mathsf{b}f := \Pi(vf), \quad \mathsf{a}f := \mathsf{b}(\mathsf{T}f), \quad \hat{\mathsf{a}}f := -\Pi(\nabla_x f) \quad \text{and} \quad \mathsf{A} := (1 + \hat{\mathsf{a}}\cdot\mathsf{a}\Pi)^{-1}\hat{\mathsf{a}}\cdot\mathsf{b}.$$

In the definition of A, we take the product coordinate by coordinate. These operators can be rewritten as

$$\mathsf{b}f = \frac{F}{\rho_F}\int_{\mathbb{R}^d} vf\,\mathrm{d}v, \quad \mathsf{a}f = \frac{F}{\rho_F}\left(\nabla_x\cdot\int_{\mathbb{R}^d} v\otimes vf\,\mathrm{d}v + \rho(f)\nabla_x V\right), \quad \text{and} \quad \hat{\mathsf{a}}f = -\frac{F}{\rho_F}\nabla_x\rho(f).$$

Let us also note that $\mathsf{AT} = (1 + \hat{\mathsf{a}}\cdot\mathsf{a}\Pi)^{-1}\hat{\mathsf{a}}\cdot\mathsf{a}$ and $\mathsf{a}\Pi f = \frac{F}{\rho_F}\frac{m_F}{d}\nabla_x(\frac{\rho(f)}{\rho_F})$ where $m_F := \int_{\mathbb{R}^d}|v|^2F(\cdot,v)\,\mathrm{d}v = d\int_{\mathbb{R}^d}|v_i|^2F(\cdot,v)\,\mathrm{d}v$ for any $i = 1,2,\ldots,d$. Define the functional

$$H(f) := \frac{1}{2}\|f\|^2 + \varepsilon\langle\mathsf{A}f, f\rangle.$$

The operator T is skew-symmetric on $L^2(\mathrm{d}\mu)$. If f is a solution of Eq. (1), then

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f-F) = D(f-F)$$
$$\text{with} \quad D(f) := \langle f, \mathsf{L}f\rangle - \varepsilon\langle\mathsf{AT}\Pi f, f\rangle - \varepsilon\langle\mathsf{AT}(1-\Pi)f, f\rangle + \varepsilon\langle\mathsf{TA}f, f\rangle + \varepsilon\langle\mathsf{L}f, (\mathsf{A}+\mathsf{A}^*)f\rangle. \tag{3}$$

The proof of Theorems 2.1 and 2.2 entirely relies on the following estimate with $\eta = 2\varepsilon/(1-\varepsilon)$:

Proposition 2.3. *Under the assumptions of Theorem* 2.1 *or* 2.2*, for any $\varepsilon > 0$ small enough, there exists an explicit constant $\lambda = \lambda(\varepsilon) > 0$ such that $D(f-F) + \lambda H(f-F) \leqslant 0$ and $\liminf_{\varepsilon\to 0}\lambda(\varepsilon)/\varepsilon > 0$.*

3. Proofs of Proposition 2.3

To simplify the computations, we replace f by $f - F$. Therefore, from now on we assume that $0 = \int_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathrm{d}x\,\mathrm{d}v = \langle f, F\rangle$. By definition of Π, $\int_{\mathbb{R}^d}(\Pi f - f)\,\mathrm{d}v = 0$. We have obviously $\langle Lf, f\rangle \leqslant -\|(1-\Pi)f\|^2$, and using the identity $\varepsilon xy \leqslant \frac{c}{2}x^2 + \frac{\varepsilon^2}{2c}y^2$, we get the estimates

$$-\varepsilon\langle\mathsf{AT}(1-\Pi)f, f\rangle = -\varepsilon\langle\mathsf{AT}(1-\Pi)f, \Pi f\rangle \leqslant \frac{c_2}{2}\|\mathsf{AT}(1-\Pi)f\|^2 + \frac{\varepsilon^2}{2c_2}\|\Pi f\|^2,$$
$$\varepsilon\langle\mathsf{TA}f, f\rangle = \varepsilon\langle\mathsf{TA}f, (1-\Pi)f\rangle \leqslant \frac{\varepsilon}{2}\|\mathsf{T}Af\|^2 + \frac{\varepsilon}{2}\|(1-\Pi)f\|^2,$$
$$\varepsilon\langle(\mathsf{A}+\mathsf{A}^*)\mathsf{L}f, f\rangle \leqslant \varepsilon\|(1-\Pi)f\|^2 + \varepsilon\|\mathsf{A}f\|^2,$$

for any $c_2 > 0$. Here we have used the identities $\mathsf{AT}(1-\Pi) = \Pi\mathsf{AT}(1-\Pi)$ and $\mathsf{TA} = (1-\Pi)\mathsf{TA}$, which are respectively consequences of the fact that the range of A is contained in $\Pi L^2(\mathrm{d}\mu)$ and that $\mathsf{TA}f = vF\nabla_x(\rho(\mathsf{A}f)/\rho_F)$. We moreover observe that $\langle\hat{\mathsf{a}}\cdot\mathsf{a}\Pi f, f\rangle = \frac{1}{d}\int_{\mathbb{R}^d}|\nabla_x(\frac{\rho(f)}{\rho_F})|^2 m_F\,\mathrm{d}x$. Let $g = \mathsf{A}f$, $u = \rho(g)/\rho_F$. An elementary computation shows that $-\nabla_x\int_{\mathbb{R}^d} vf\,\mathrm{d}v = \rho_F u - \frac{1}{d}\nabla_x(m_F\nabla_x u)$, from which one can deduce, after a few steps that we shall omit here, that $\|\mathsf{A}f\|^2 = \int_{\mathbb{R}^d} |u|^2 \rho_F \, dx$ and $\|\mathsf{TA}f\|^2 = \frac{1}{d}\int_{\mathbb{R}^d} |\nabla_x u|^2 m_F \, dx$ are such that $2\|\mathsf{A}f\|^2 + \|\mathsf{TA}f\|^2 \leqslant \|(1-\Pi)f\|^2$. We have proved that

$$D(f) \leqslant -\left(1 - \frac{5}{2}\varepsilon\right)\left\|(1-\Pi)f\right\|^2 - \varepsilon\langle \mathsf{AT}\Pi f, f\rangle + \frac{c_2}{2}\left\|\mathsf{AT}(1-\Pi)f\right\|^2 + \frac{\varepsilon^2}{2c_2}\|\Pi f\|^2.$$

To complete the proof of Proposition 2.3, it remains to estimate from above $-\langle \mathsf{AT}\Pi f, f\rangle$ and $\|\mathsf{AT}(1-\Pi)f\|^2$. As for the second of these two terms, we actually estimate $\|(\mathsf{AT}(1-\Pi))^* f\|^2$ as follows. Using $(\mathsf{AT}(1-\Pi))^* f = (\hat{\mathsf{a}} \cdot \mathsf{a}(1-\Pi))^* g$ with $g = (1 + \hat{\mathsf{a}} \cdot \mathsf{a}\Pi)^{-1} f$, we first observe that

$$\rho(f) = \rho_F u - \frac{1}{d}\nabla_x(m_F \nabla_x u) \tag{4}$$

where $u = \rho(g)/\rho_F$. Let $q_F := \int_{\mathbb{R}^d} |v_1|^4 F \, dv$, $u_{ij} := \partial^2 u/\partial x_i \partial x_j$. After some elementary but tedious computations, we also get

$$\left\|\left(\mathsf{AT}(1-\Pi)\right)^* f\right\|^2 = \frac{1}{3}\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \left[\left((2\delta_{ij}+1)q_F - \frac{3m_F^2}{d^2\rho_F}\delta_{ij}\right)u_{ii}u_{jj} + 2(1-\delta_{ij})q_F u_{ij}^2\right] dx. \tag{5}$$

Case of Theorem 2.1. In the *Maxwellian case,* various simplifications occur. With $\rho_F = e^{-V} = \frac{1}{d}m_F = q_F$, (4) becomes

$$\rho(f) = ue^{-V} - \nabla_x\left(e^{-V}\nabla_x u\right) \tag{6}$$

and it follows from (H3) that

$$\langle \mathsf{AT}\Pi f, f\rangle \geqslant \frac{\Lambda}{1+\Lambda}\|\Pi f\|^2.$$

On the other hand, from the above computation,

$$\left\|\left(\mathsf{AT}(1-\Pi)\right)^* f\right\|^2 \leqslant 2\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} |u_{ij}|^2 e^{-V} \, dx.$$

Let $\|u\|_0^2 := \int_{\mathbb{R}^d} |u|^2 e^{-V} \, dx$ and $W := |\nabla_x V|$. By multiplying (6) by u, we get after an integration by parts that $\|u\|_0^2 + \|\nabla_x u\|_0^2 \leqslant \|\Pi f\|^2$. By expanding the square in $|\nabla_x(ue^{-V/2})|^2$, one can prove using (H3) and (H4) that the following improved Poincaré inequality holds, with $\kappa = (1-\theta)/(2(2+\Lambda c_0))$:

$$\kappa\|Wu\|_0^2 \leqslant \|\nabla_x u\|_0^2 \tag{7}$$

for any $u \in H^1(e^{-V}\,dx)$ such that $\int_{\mathbb{R}^d} ue^{-V}\,dx = 0$.

Multiply (6) by $W^2 u$ and integrate by parts. By (H5), we get

$$\|Wu\|_0^2 + \|W\nabla_x u\|_0^2 - 2c_1\left(\|\nabla_x u\|_0 + \|W\nabla_x u\|_0\right) \cdot \|Wu\|_0 \leqslant \frac{\kappa}{8}\left\|W^2 u\right\|_0^2 + \frac{2}{\kappa}\|\Pi f\|^2. \tag{8}$$

Applying (7) to $Wu - \int_{\mathbb{R}^d} Wue^{-V}\,dx$, we get

$$\kappa\left\|W^2 u\right\|_0^2 \leqslant \int_{\mathbb{R}^d} \left|\nabla_x(Wu)\right|^2 e^{-V}\,dx + 2\kappa\int_{\mathbb{R}^d} Wue^{-V}\,dx \int_{\mathbb{R}^d} W^3 ue^{-V}\,dx.$$

On the one hand, by the Cauchy–Schwarz inequality, $\int_{\mathbb{R}^d} Wue^{-V}\,dx \leqslant \|W\|_0\|u\|_0 =: a$, and on the other hand, $\int_{\mathbb{R}^d} W^3 ue^{-V}\,dx \leqslant a\|W\|_0^2 + \frac{1}{4a}\|W^2 u\|_0^2$, so that $2\int_{\mathbb{R}^d} Wue^{-V}\,dx \int_{\mathbb{R}^d} W^3 ue^{-V}\,dx$ can be bounded by $\frac{1}{2}\|W^2 u\|_0^2 + 2\|W\|_0^4\|u\|_0^2$. Notice that $\|W\|_0$ is bounded by (H6). As for the other term of the r.h.s., we can simply write that $\int_{\mathbb{R}^d} |\nabla_x(Wu)|^2 e^{-V}\,dx$ is bounded by $2\|W\nabla_x u\|_0^2 + 4c_1^2(\|u\|_0^2 + \|Wu\|_0^2)$ using (H5). Hence we have

$$\kappa\left\|W^2 u\right\|_0^2 \leqslant 4\|W\nabla_x u\|_0^2 + 8c_1^2\left(\|u\|_0^2 + \|Wu\|_0^2\right) + 4\kappa\|W\|_0^4\|u\|_0^2.$$

Combined with (8), this proves that, for some $c_3 > 0$, $\|W\nabla_x u\|_0 \leqslant c_3\|\Pi f\|$. By multiplying (6) by Δu and integrating by parts, we get $\|\nabla_x^2 u\|_0^2 - (\|W\nabla_x u\|_0 + \|\Pi f\|)\|\nabla_x^2 u\|_0 \leqslant \|Wu\|_0\|\nabla_x u\|_0$. Altogether, this proves that $\|(\mathsf{AT}(1-\Pi))^* f\|^2 \leqslant c_4\|f\|^2$ for some $c_4 > 0$ and, as a consequence, $\|(\mathsf{AT}(1-\Pi))f\|^2 \leqslant c_4\|f\|^2$. Since $(1-\Pi)^2 = 1-\Pi$, we finally obtain $\|(\mathsf{AT}(1-\Pi))f\|^2 \leqslant c_4\|(1-\Pi)f\|^2$. Summarizing, with $\lambda_1 = 1 - \frac{1}{2}c_2c_4 - 5\varepsilon/2$ and $\lambda_2 = \frac{\Lambda\varepsilon}{1+\Lambda} - \frac{\varepsilon^2}{2c_2}$, we have proved that $D(f) \leqslant -\lambda_1\|(1-\Pi)f\|^2 - \lambda_2\|\Pi f\|^2$. With $c_2 = a\varepsilon$, $a > \frac{1}{2}(1+1/\Lambda)$ and $\varepsilon > 0$ small enough, λ_1 and λ_2 are positive and the result holds with $\lambda = \min\{\lambda_1, \lambda_2\}$. The explicit expression of c_4 can easily be retraced in the above computations.

Case of Theorem 2.2. In the *fast diffusion case,* we only sketch the main steps of the proof. For $p = 0, 1, 2$, let $w_p^2 := \omega_0 V^{p-q}$, where $q = k + 1 - d/2$, $w_0^2 := \rho_F$ and $V(x) = (1+|x|^2)^\beta$. Define $\|u\|_i^2 = \int_{\mathbb{R}^d} |u|^2 w_i^2\,\mathrm{d}x$. Notice that $\rho_F \in L^1(\mathbb{R}^d)$ means $\beta(d-2k-2)+d < 0$. This is the case if $\beta(d+2-2k)+d-4 < 0$ and $\beta \geqslant 1$. The proof in the Maxwellian case can be adapted as follows. Eq. (4) can be rewritten as

$$\rho(f) = w_0^2 u - \frac{2}{2k-d}\nabla_x\big(w_1^2\nabla_x u\big) \tag{9}$$

and (H3) is replaced by the Hardy–Poincaré inequality, see [1], $\|u\|_0^2 \leqslant \Lambda\|\nabla_x u\|_1^2$ for some $\Lambda > 0$, under the condition $\int_{\mathbb{R}^d} u w_0^2\,\mathrm{d}x = 0$. This holds true if $\beta \geqslant 1$. The fact that $\langle \mathsf{AT}\Pi f, f\rangle \geqslant \frac{\Lambda}{1+\Lambda}\|\Pi f\|^2$ then follows. We also need the following Hardy–Poincaré inequality

$$\int_{\mathbb{R}^d} V^{\alpha+1-q-\frac{1}{\beta}}|u|^2\,\mathrm{d}x - \frac{(\int_{\mathbb{R}^d} V^{\alpha+1-q-\frac{1}{\beta}} u\,\mathrm{d}x)^2}{\int_{\mathbb{R}^d} V^{\alpha+1-q-\frac{1}{\beta}}\,\mathrm{d}x} \leqslant \frac{1}{4(\beta_0-1)^2}\int_{\mathbb{R}^d} V^{\alpha+1-q}|\nabla_x u|^2\,\mathrm{d}x$$

which is responsible for the condition $\beta < \beta_0(\delta)$, $\delta > 0$. Observe that (9) multiplied by u gives, after an integration by parts, $\|u\|_0^2 + (q-1)^{-1}\|\nabla_x u\|_1^2 \leqslant \|\Pi f\|^2$. By multiplying (9) by $V^\alpha u$ with $\alpha := 1 - 1/\beta$ or by $V\Delta u$ and integrating by parts, we find directly that $\|\nabla_x^2 u\|_2^2$ is bounded by $\|\Pi f\|^2$. Computations which are quite similar to the ones of the Maxwellian case then allow to conclude. More details will be given in [3].

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