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Missing boundary data reconstruction by the factorization method

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Abstract

We consider the data completion problem for the Laplace equation in a cylindrical domain. The Neumann and Dirichlet boundary conditions are given on one face of the cylinder while there is no condition on the other face. This Cauchy problem has been known since Hadamard (1953) to be ill-posed. Here it is set as an optimal control problem with a regularized cost function. We use the factorization method for elliptic boundary value problems. For each set of Cauchy data, to obtain the estimate of the missing data one has to solve a parabolic Cauchy problem in the cylinder and a linear equation. The operator appearing in these problems satisfy a Riccati equation that does not depend on the data. ***To cite this article: A. Ben Abda et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Reconstruction de données frontières manquantes par la méthode de factorisation. On considère le problème de complétion de données pour l'équation de Laplace dans un domaine cylindrique. Les conditions de Dirichlet et Neumann sont données sur une face du cylindre alors qu'il n'y a pas de condition sur l'autre face. Depuis Hadamard (1953) on sait que ce problème de Cauchy est mal posé. On le formule ici comme un problème de contrôle avec une fonction coût régularisée. On utilise la méthode de factorisation des problèmes aux limites elliptiques. Pour chaque jeu de données de Cauchy, on obtient l'estimé de la donnée manquante en résolvant un problème de Cauchy parabolique dans le cylindre et une équation linéaire. L'opérateur qui apparait dans ces problèmes vérifie une équation de Riccati indépendante des données. ***Pour citer cet article : A. Ben Abda et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

In this Note, we consider the following problem:

$$(P_0)\begin{cases}\Delta u=\frac{\partial^2 u}{\partial x^2}+\Delta_y u=0 & \text{in } \Omega,\\ u=0 & \text{on } \Sigma,\\ u=T\in H_{00}^{\frac{1}{2}}(\mathcal{O}) \quad \text{on } \Gamma_0, \quad \frac{\partial u}{\partial n}=\Phi\in (H_{00}^{\frac{1}{2}}(\mathcal{O}))' & \text{on } \Gamma_a,\end{cases}$$

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where Ω is the cylinder $]0,a[\times\mathcal{O}$ with axis along the coordinate x. The section $\mathcal{O}\subset\mathbb{R}^{n-1}$ is a bounded domain with coordinates y. The lateral boundary is $\Sigma=]0,a[\times\partial\mathcal{O}$, and the faces are $\Gamma_0=\{0\}\times\mathcal{O}$ and $\Gamma_a=\{a\}\times\mathcal{O}$. Using notations of [6] we define $H_{00}^{\frac{1}{2}}(\mathcal{O})$ as the interpolate $[H_0^1(\mathcal{O}),L^2(\mathcal{O})]_{1/2}$.

On Γ_0 one has the Cauchy data

$$(\Phi,T)\in\big(H_{00}^{\frac{1}{2}}(\mathcal{O})\big)'\times H_{00}^{\frac{1}{2}}(\mathcal{O})$$

whereas no boundary condition is available on Γ_a. The aim of this Note is the recovering of these missing boundary data exploiting the over specified one on Γ_0. This problem is known to be ill-posed in Hadamard's sense [4]. In particular the existence of solution is not insured for arbitrary Cauchy data (Φ,T). This problem is treated in [1,2] in a general domain Ω having a boundary: $\partial\Omega=\Gamma_0\cup\Gamma_1$. In [1] the missing boundary data recovery problem is formulated as a control one with a double state, each component satisfying one condition on Γ_0, the control being the overdetermined boundary condition on Γ_a. This method is an iterative one and therefore requires a resolution of the full problem for each new data (Φ,T). In this work we use the same formulation as in [1] in terms of optimal control but we add a regularization term to insure the existence and uniqueness of the optimality system. We make use of the factorization method which transforms the elliptic boundary value problem into two parabolic ones. It allows the direct evaluation of the missing boundary data for all Cauchy data (Φ,T).

2. The boundary data recovering as an optimal control problem

As in [1] we consider the optimal control problem with control $(\eta,\tau)\in(H_{00}^{\frac{1}{2}}(\mathcal{O}))'\times H_{00}^{\frac{1}{2}}(\mathcal{O})$:

$$\begin{cases}\Delta u_1=0 & \text{in }\Omega,\\ u_1=0 & \text{on }\Sigma,\\ u_1=T & \text{on }\Gamma_0,\quad \frac{\partial u_1}{\partial n}=\eta\quad\text{on }\Gamma_a,\end{cases}\tag{1}$$

$$\begin{cases}\Delta u_2=0 & \text{in }\Omega,\\ u_2=0 & \text{on }\Sigma,\\ \frac{\partial u_2}{\partial n}=\Phi & \text{on }\Gamma_0,\quad u_2=\tau\quad\text{on }\Gamma_a.\end{cases}\tag{2}$$

If (ϕ,T) are compatible then $u_1=u_2$ when $(\eta,\tau)=(\varphi,t)$. We define the cost function E and its regularization E_ε by:

$$E(\eta,\tau)=\int_\Omega(\nabla u_1-\nabla u_2)^2\,\mathrm{d}x\,\mathrm{d}y,\qquad E_\varepsilon(\eta,\tau)=E(\eta,\tau)+\varepsilon\Big(\|\eta\|^2_{(H_{00}^{\frac{1}{2}}(\mathcal{O}))'}+\|\tau\|^2_{H_{00}^{\frac{1}{2}}(\mathcal{O})}\Big).$$

The strict convexity of E_ε yields the existence and uniqueness of a minimizer $(\varphi_\varepsilon,t_\varepsilon)$ (optimal control).

3. The factorization method as a tool for the boundary data recovering

We apply the method of spatial invariant embedding developed in [5]. One embeds problems (1) and (2), in a family of similar problems defined in $\Omega_s=]0,s[\times\mathcal{O}$, $0<s\leqslant a$. So we define u_1^s and u_2^s satisfying the same equation as u_1 and u_2 respectively restricted to Ω_s and we impose a Neumann boundary condition for u_1^s: $\frac{\mathrm{d}u_1^s}{\mathrm{d}x}|_{\Gamma_s}=\alpha$ and a Dirichlet condition for u_2^s: $u_2|_{\Gamma_s}^s=\beta$ on $\Gamma_s=\{s\}\times\mathcal{O}$. The Neumann–Dirichlet mapping: $\alpha\longmapsto u_1^s|\Gamma_s$ and Dirichlet–Neumann: $\beta\longmapsto\frac{\mathrm{d}u_2^s}{\mathrm{d}x}|_{\Gamma_s}$ are affine and one has:

$$u_1^s|_{\Gamma_s}=P(s)\alpha+w_1(s),\quad P(s)\in\mathcal{L}\big(H_{00}^{\frac{1}{2}}(\mathcal{O})',H_{00}^{\frac{1}{2}}(\mathcal{O})\big),\quad P(0)=0,\quad\text{and}\quad w_1(0)=T,\tag{3}$$

$$\left.\frac{\partial u_2^s}{\partial n}\right|_{\Gamma_s}=Q(s)\beta+w_2(s),\quad Q(s)\in\mathcal{L}\big(H_{00}^{\frac{1}{2}}(\mathcal{O}),H_{00}^{\frac{1}{2}}(\mathcal{O})'\big),\quad Q(0)=0,\quad\text{and}\quad w_2(0)=-\Phi.\tag{4}$$

Taking the derivative of (3), (4) and using the fact that α and β are arbitrary, one can show as in [5] that the self adjoint operators P and Q and the functions w_1 and w_2 satisfy the two following systems:

$$\begin{cases}\frac{\mathrm{d}P(x)}{\mathrm{d}x}-P(x)\Delta_yP(x)=I, & P(0)=0,\\ \frac{\mathrm{d}w_1(x)}{\mathrm{d}x}-P(x)\Delta_yw_1(x)=0, & w_1(0)=T,\end{cases}\tag{5}$$

$$\begin{cases}\frac{\mathrm{d}Q(x)}{\mathrm{d}x}+Q(x)^2=-\Delta_y, & Q(0)=0,\\ \frac{\mathrm{d}w_2(x)}{\mathrm{d}x}+Q(x)w_2=0, & w_2(0)=-\Phi.\end{cases}\tag{6}$$

Proposition 3.1. *Let A and its regularization A_ε be the operator matrices*

$$A=\begin{pmatrix} P(a) & -P(a)Q(a)\\ -Q(a)P(a) & Q(a)\end{pmatrix},\qquad A_\varepsilon=A+\varepsilon\begin{pmatrix}(-\Delta_y)^{-\frac{1}{2}} & 0\\ 0 & (-\Delta_y)^{\frac{1}{2}}\end{pmatrix}.$$

Up to a constant C the cost function $E_\varepsilon(\eta,\tau)$ can be expressed as

$$E_\varepsilon(\eta,\tau)=C+[\eta,\tau]A[\eta,\tau]'-2\int_{\Gamma_a}\big(P(a)w_2(a)\eta+Q(a)w_1(a)\tau\big)\,\mathrm{d}y+\varepsilon\Big(\|\eta\|^2_{(H^{\frac{1}{2}}_{00}(\mathcal{O}))'}+\|\tau\|^2_{H^{\frac{1}{2}}_{00}(\mathcal{O})}\Big).$$

Then the optimal control $(\varphi_\varepsilon,t_\varepsilon)$ is uniquely determined by

$$A_\varepsilon[\varphi_\varepsilon,t_\varepsilon]'=\big[P(a)w_2(a),\,Q(a)w_1(a)\big]'. \tag{7}$$

Proof. Using Green's formula and the fact that $w_1(a)$ (resp. $w_2(a)$) is the value of $u_1(a)$ (resp. $\frac{\partial u_2}{\partial n}(a)$) for $\eta=0$ (resp. $\tau=0$), one can show that

$$E(\eta,\tau)=C-\int_{\Gamma_a}\big(\eta w_1(a)+\tau w_2(a)\big)\,\mathrm{d}y-2\int_{\Gamma_a}u_1(a)\frac{\partial u_2(a)}{\partial n}\,\mathrm{d}y+\int_{\Gamma_a}\Big(\eta u_1(a)+\tau\frac{\partial u_2(a)}{\partial n}\Big)\,\mathrm{d}y.$$

The result is obtained by using (3) and (4). □

This result provides the optimal estimates of the boundary conditions φ_ε, t_ε on Γ_a without using an adjoint state, or in other words Eq. (7) is an uncoupled version of the classical optimality system for optimal control problems. If the problem has to be solved for several data (Φ,T) the Riccati equations for P and Q have to be solved only once and the equations for w_1, w_2 are uncoupled parabolic equations in opposite direction in x that must be solved for each data (Φ,T).

Remark 1. This method can also be applied to non-cylindrical domains with a smooth boundary Σ. The study of Riccati equations in such domains can be found in [3] using the speed vector field method. In the case of the cylinder the operators P, Q and $-\Delta_y$ commute and furthermore $Q=-P\Delta_y$. Then only one Riccati equation has to be solved.

Proposition 3.2. *The data (Φ,T) are compatible under any of the three equivalent following assertions*:

(i) $P(a)w_2(a)+w_1(a)\in\Im(I-P(a)Q(a))$,
(ii) $w_2(a)+Q(a)w_1(a)\in\Im(I-Q(a)P(a))$,
(iii) $[P(a)w_2(a),Q(a)w_1(a)]'\in\Im(A)$.

In this case the missing data are given through the unregularized problem by

$$\big(I-P(a)Q(a)\big)t=P(a)w_2+w_1,\quad\text{and}\quad\big(I-Q(a)P(a)\big)\varphi=Q(a)w_1+w_2.$$

4. Link with the Steklov–Poincaré operator

In this part we present the link between the previous approach and the calculation of [2] using the Steklov–Poincaré operator. For all $(\Phi,T)\in(H^{\frac{1}{2}}_{00}(\Gamma_a))'\times H^{\frac{1}{2}}_{00}(\Gamma_a)$ and for all $\tau\in H^{\frac{1}{2}}_{00}(\Gamma_a)$, we consider the modified problem (1), (2), where we replace u_1 by $\bar u_1$ which is controlled by the same Dirichlet condition as u_2: $\bar u_1=\tau$ on Γ_a. Following [2] we denote by $\bar u_1(\tau)$ (resp. $u_2(\tau)$) the value of $\bar u_1$ when $T=0$ (resp. u_2 when $\Phi=0$) and by $\bar u_1^0(T)$ (resp. $u_2^0(\Phi)$) the value of $\bar u_1$ (resp. u_2) when $\tau=0$. We define the bilinear forms $S(.,.)$, $S_D(.,.)$ and $S_N(.,.)$ on $H^{\frac{1}{2}}_{00}(\Gamma_a)\times H^{\frac{1}{2}}_{00}(\Gamma_a)$ and the linear form l on $H^{\frac{1}{2}}_{00}(\Gamma_a)$ by:

$$S(\tau,\delta)=S_D(\tau,\delta)-S_N(\tau,\delta)=\int_\Omega\nabla\bar u_1(\tau)\nabla\bar u_1(\delta)\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega\nabla u_2(\tau)\nabla u_2(\delta)\,\mathrm{d}x\,\mathrm{d}y,\quad\forall\tau,\delta\in H^{\frac{1}{2}}_{00}(\Gamma_a),$$

$$l(\delta) = -\int_\Omega \nabla \bar{u}_1^0(T)\nabla \bar{u}_1(\delta)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \nabla u_2^0(\Phi)\nabla u_2(\delta)\,\mathrm{d}x\,\mathrm{d}y - \big(\Phi, u_2(\delta)\big)_{L^2(\Gamma_0)}.$$

The boundary data recovery amounts to finding $t \in H_{00}^{\frac{1}{2}}(\Gamma_a)$ such that $\frac{\partial u_1}{\partial n}(t) = \frac{\partial u_2}{\partial n}(t)$ on the face Γ_a. It is equivalent to finding $t \in H_{00}^{\frac{1}{2}}(\Gamma_a)$ verifying $S(t,\delta) = l(\delta)$ for all $\delta \in H_{00}^{\frac{1}{2}}(\Gamma_a)$. These bilinear and linear forms can be expressed with the help of the operators $P(a)$ and $Q(a)$ satisfying the Riccati equations (5), (6) and with the residuals w_1 and w_2

$$S_N(\tau,\delta) = \big(Q(a)\tau,\delta\big), \qquad S_D(\tau,\delta) = \big(P(a)^{-1}\tau,\delta\big),$$
$$S(\tau,\delta) = \big(\big(P(a)^{-1} - Q(a)\big)\tau,\delta\big), \qquad l(\delta) = \big(P(a)^{-1}w_1 + w_2,\delta\big), \quad \forall \tau,\delta \in H_{00}^{\frac{1}{2}}(\Gamma_a).$$

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