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Group Theory

Nilpotent subalgebras of semisimple Lie algebras

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Abstract

Let $\mathfrak{g}$ be the Lie algebra of a semisimple linear algebraic group. Under mild conditions on the characteristic of the underlying field, one can show that any subalgebra of $\mathfrak{g}$ consisting of nilpotent elements is contained in some Borel subalgebra. In this Note, we provide examples for each semisimple group G and for each of the torsion primes for G of nil subalgebras not lying in any Borel subalgebra of $\mathfrak{g}$. ***To cite this article: P. Levy et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sous-algèbres nilpotentes d'algèbres de Lie semi-simples. Soit $\mathfrak{g}$ l'algèbre de Lie d'un groupe algébrique linéaire semi-simple. Si l'on impose certaines conditions à la caractéristique du corps de définition, on peut montrer que toute sous-algèbre de $\mathfrak{g}$ ne contenant que des éléments nilpotents est contenue dans une sous-algèbre de Borel. Dans cette Note, nous donnons des exemples, pour chaque groupe semi-simple G et pour chaque nombre premier de torsion pour G, de sous-algèbres d'éléments nilpotents qui ne sont contenues dans aucune sous-algèbre de Borel de $\mathfrak{g}$. ***Pour citer cet article : P. Levy et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit k un corps algébriquement clos de caractéristique $p > 0$. Par « groupe algébrique sur k » nous entendons un schéma en groupes affine de type fini sur k. Soit G un groupe algébrique semi-simple défini sur k (G est lisse et connexe) et soit U un sous-groupe (algébrique) unipotent de G. Si U est réduit, on sait que U est contenu dans un sous-groupe de Borel de G (cf. [4, 30.4]). Nous nous intéressons au cas où U n'est pas réduit, plus précisément au cas des p-sous-algèbres de Lie de $\mathrm{Lie}(G)$.

Théorème 0.1. *Supposons que p ne soit pas un nombre premier de torsion de G. Alors tout sous-groupe unipotent* (*non nécessairement réduit*) *de G est contenu dans un sous-groupe de Borel de G.*

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La démonstration repose essentiellement sur [7, Theorem A].

Théorème 0.2. *Supposons que p soit un nombre premier de torsion pour G. Il existe un sous-groupe unipotent de G, de dimension* 0*, qui n'est contenu dans aucun sous-groupe de Borel de G.*

On démontre ce théorème en construisant des p-sous-algèbres de Lie de $\mathrm{Lie}(G)$, formées d'éléments nilpotents, et qui ne sont contenues dans aucune sous-algèbre de Borel. Il y a deux types de constructions :

a) Si $\tilde{G} \to G$ est le revêtement universel de G et si p divise l'ordre du noyau (schématique) de $\tilde{G} \to G$, on peut construire une p-sous-algèbre commutative de $\mathrm{Lie}(G)$, formée d'éléments nilpotents, dont l'image réciproque dans $\mathrm{Lie}(\tilde{G})$ n'est pas commutative ; une telle sous-algèbre n'est pas contenue dans une sous-algèbre de Borel de G. Lorsque G est simple, l'algèbre ainsi construite est de dimension 2, et elle est annulée par la puissance p-ième.
b) Si p est de torsion pour le système de racines de G (par exemple $p = 2$, 3, ou 5 si G est de type E_8), il existe une p-sous-algèbre commutative de $\mathrm{Lie}(G)$, de dimension 3, annulée par la puissance p-ième, et non contenue dans une sous-algèbre de Borel.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$ and let G be a semisimple linear algebraic group over k. Let $\mathfrak{g}$ be the Lie algebra of G. Under mild conditions on G and p it is straightforward to show that any nil subalgebra of $\mathfrak{g}$, that is, a subalgebra consisting of nilpotent elements, is contained in a Borel subalgebra (see Section 2 below). J.-P. Serre has asked the following question: is it true that if p is a torsion prime for G then there exists a nil subalgebra of $\mathfrak{g}$ which is contained in no Borel subalgebra? In this Note, we establish a positive answer to this question. Moreover, if p is not a torsion prime for G, every nil subalgebra of $\mathfrak{g}$ lies in a Borel subalgebra. Our argument in fact applies to the more general setting of unipotent subgroup schemes of a semisimple group scheme over k.

We outline two separate cases. First, assume that G is simply connected. The scheme-theoretic center Z of G is a finite group scheme. Now by a *Heisenberg-type subalgebra* of $\mathfrak{g}$, we mean a p-subalgebra which is a central extension of an abelian nil algebra by a 1-dimensional algebra. If p divides the order of Z, we exhibit a Heisenberg-type restricted subalgebra of $\mathfrak{g}$ whose center is central in $\mathfrak{g}$. This gives a construction of a suitable nil algebra in $\mathrm{Lie}(G_{ad})$, where G_{ad} is the corresponding adjoint group. Secondly, assume p is a torsion prime for the root system of G. Then we will exhibit a commutative 3-dimensional restricted nil subalgebra of $\mathfrak{g}$ which is not contained in any Borel subalgebra.

In [3], Draisma, Kraft and Kuttler study subspaces of $\mathfrak{g}$, rather than subalgebras, consisting of nilpotent elements; they exhibit examples in Lie algebras defined over fields of certain small characteristics of subspaces of maximal possible dimension which do not lie in a Borel subalgebra. We refer the reader as well to the article of Vasiu [12] in which he studies normal unipotent subgroup schemes of reductive groups.

2. Good characteristics

Throughout this Note, k is an algebraically closed field of characteristic $p > 0$. By 'linear algebraic group defined over k' we mean an affine group scheme of finite type over k. Let G be a semisimple linear algebraic group over k; in particular, G is a smooth group scheme with restricted Lie algebra $\mathfrak{g}$, the p-operation being denoted by $X \mapsto X^p$. Let T be a fixed maximal torus of G, $W = W(G,T)$ the Weyl group of G, $\Phi = \Phi(G,T)$ the root system, Φ^+ a positive system in Φ, $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ the corresponding basis and $B \subset G$ the associated Borel subgroup containing T. For $\alpha \in \Phi$, let $\alpha^\vee$ denote the corresponding coroot. If Φ is an irreducible root system then there is a unique root of maximal height with respect to Δ, noted here by β. Write $\beta = \sum_{i=1}^{\ell} m_i\alpha_i$ and $\beta^\vee = \sum_{i=1}^{\ell} m'_i\alpha_i^\vee$. Recall that p is **bad** for Φ if $m_i = p$ for some i, $1 \leqslant i \leqslant \ell$, and p is **torsion** for Φ if $m'_i = p$ for some i, $1 \leqslant i \leqslant \ell$. (If the Dynkin diagram is simply-laced then $m_i = m'_i$ for all i.) We say that p is **good** for Φ if p is not bad for Φ and that p is **very good** for Φ if p is good for Φ and $p \nmid (\ell + 1)$ when Φ is of type A_ℓ. Finally, we will say p is good (respectively, very good) for G if p is good (resp. very good) for every irreducible component of $\Phi = \Phi(G,T)$. We will say that p is bad for G if p is bad for some irreducible component of Φ and that p is **torsion for** G if p is torsion for some irreducible component of Φ or p divides the order of the fundamental group of G. (See [11] for a discussion of torsion primes.)

Before considering the case of non-torsion primes, we introduce one further definition:

Definition 2.1 *([8, Exposé XVII, 1.1]).* An algebraic group U over k is said to be *unipotent* if U admits a composition series whose successive quotients are isomorphic to some subgroup scheme of the algebraic group $\mathbf{G}_a$.

Theorem 2.2. *Let G be a semisimple group and p a non-torsion prime for G. Let U be a unipotent subgroup scheme of G. Then U is contained in a Borel subgroup of G.*

Proof. Consider first the case where G is of type A_ℓ. The result follows from [8, 3.2, Exposé XVII] and induction if $G = \mathrm{SL}_{\ell+1}$. For the other cases, as p does not divide the order of the fundamental group of G, we have a separable isogeny $\pi : \mathrm{SL}_{\ell+1} \to G$ which induces a bijection on the set of Borel subgroups, whence the result follows.

In case $G = \mathrm{Sp}_{2\ell}$, we argue similarly: a unipotent subgroup of G fixes a non-zero, isotropic vector in the natural representation of G and again by induction lies in a Borel subgroup of G. Indeed, this argument works as well for the orthogonal groups when $p \neq 2$.

Consider now the case where $G = G_2$ and $p = 3$. By the result for SO_7, we know that U fixes a nontrivial singular vector in the action of G on its 7-dimensional orthogonal representation. One checks that the stabilizer of such a vector is a parabolic subgroup of G_2. Indeed this is clear for the group of k-points as the long root parabolic lies in the stabilizer and is a maximal subgroup. One checks directly that the stabilizer in $\mathfrak{g}$ of a maximal vector with respect to the fixed Borel subgroup is indeed a parabolic subalgebra with Levi factor a long root $\mathfrak{sl}_2$.

Now consider the case where p is a very good prime for G. As G is separably isogenous to a simply connected group, we may take G to be simply connected. Then G satisfies the following so-called *standard hypotheses* for a reductive group G (cf. [5, 5.8]):

- p is good for each irreducible component of the root system of G,
- the derived subgroup (G, G) is simply connected, and
- there exists a non-degenerate G-equivariant symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$.

We proceed by induction on $\dim G$, the case where $\dim G = 3$ and $G = \mathrm{SL}_2$ having been handled above. By [8, 3.5], U has a nontrivial center $Z(U)$ and either there exists $X \in \mathrm{Lie}(Z(U))$ with $X^p = 0$ and so $U \subset C_G(X)$ or there exists $u \in Z(U)$ with $u^p = 1$ and $U \subset C_G(u)$. By [10, 3.12] there exists a G-equivariant bijective morphism between the variety of nilpotent elements and the variety of unipotent elements; so applying Theorem A of [7] we have that U lies in a proper parabolic subgroup P of G. Let L be a Levi subgroup of P; then L satisfies the standard hypotheses as well. Taking the image of U in $P/R_u(P)$, we obtain a unipotent subgroup scheme of (L, L) which is, by induction on the dimension of G, contained in a Borel subgroup B_L of L. We then have that $B_L \cdot R_u(P)$ is a Borel subgroup of G containing U.

It remains to consider the case where the root system of G is not irreducible and p is not a very good prime for G. In this case, G is separably isogenous to a direct product of simply connected almost simple groups, and the result follows as in the case of type A_ℓ above. □

Remarks.

a) Given an arbitrary nil subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, that is not necessarily a restricted subalgebra, one can check via a faithful representation $\mathfrak{g} \to \mathfrak{gl}(V)$ that the p-closure $\bar{\mathfrak{n}}$ of $\mathfrak{n}$ in $\mathfrak{g}$ is again nil. Assume now that p is a non-torsion prime for G. Then by the preceding theorem, the infinitesimal unipotent subgroup scheme $\bar{\mathfrak{n}}$ lies in a Borel subalgebra of G and hence $\mathfrak{n}$ does as well.

b) We note that the conclusion of Theorem 2.2 holds for reduced unipotent subgroup schemes even if the characteristic is a torsion prime for G. (See [4, 30.4].)

Before presenting our examples, we fix some additional notation. If G is separably isogenous to a simply connected group then we can and will choose a Chevalley basis $\{h_i, e_\alpha, f_\alpha\colon 1 \leqslant i \leqslant \ell,\ \alpha \in \Phi^+\}$ for $\mathfrak{g}$, satisfying the usual relations. If G is not separably isogenous to a simply connected group, then we can choose $\{h_i, e_\alpha, f_\alpha\colon 1 \leqslant i \leqslant \ell, \alpha \in \Phi^+\}$ satisfying the usual Chevalley relations; however, the h_i will not be linearly independent and a basis of $\mathfrak{g}$ can be obtained by extending $\{h_i\colon 1 \leqslant i \leqslant \ell\}$ to a basis of $\mathrm{Lie}(T)$. We use the structure constants given in [9] for $\mathfrak{g}$ of type F_4; for $\mathfrak{g}$ of type E_ℓ, we use those given in [6]. Our labeling of Dynkin diagrams is taken as in [2]. It will sometimes be convenient to represent roots as the ℓ-tuple of integers giving the coefficients of the simple roots, arranged as in a Dynkin diagram.

3. Heisenberg-type subalgebras

Here we take G to be simply connected. For $G = \mathrm{SL}_{mp}$, let E_{ij} denote the elementary $mp \times mp$ matrix with (r, s) entry $\delta_{ir}\delta_{js}$. Set $X = \sum_{j=0}^{m-1}\sum_{i=1}^{p-1} E_{jp+i,jp+i+1}$ and $Y = \sum_{j=0}^{m-1}\sum_{i=1}^{p-1} i E_{jp+i+1,jp+i}$. Then $X^p = 0 = Y^p$, $[X, Y] = I$ and hence the Lie algebra generated by X and Y is nilpotent.

Similar examples exist for other types with a nontrivial center:

- if $p = 2$ and $G = \mathrm{Spin}(2\ell + 1, k)$ then let $X = e_{\alpha_\ell}$ and $Y = f_{\alpha_\ell}$;
- if $p = 2$ and $G = \mathrm{Sp}(2\ell, k)$ then let $X = \sum_{i=1}^{\lceil \ell/2\rceil} e_{\alpha_{2i-1}}$ and $Y = \sum_1^\ell i f_{\alpha_i}$;
- if $p = 2$ and $G = \mathrm{Spin}(2\ell, k)$ then let $X = e_{\alpha_{\ell-1}} + e_{\alpha_\ell}$ and $Y = f_{\alpha_{\ell-1}} + f_{\alpha_\ell}$;
- if $p = 3$ and G is of type E_6 then let $X = e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_5} + e_{\alpha_6}$ and $Y = f_{\alpha_1} - f_{\alpha_3} + f_{\alpha_5} - f_{\alpha_6}$;
- if $p = 2$ and G is of type E_7 then let $X = e_{\alpha_2} + e_{\alpha_5} + e_{\alpha_7}$ and $Y = f_{\alpha_2} + f_{\alpha_5} + f_{\alpha_7}$.

In each of the above cases $X^p = 0 = Y^p$ and $[X, Y]$ is a nontrivial element of $\mathfrak{z}(\mathfrak{g})$, the center of $\mathfrak{g}$; in particular $[X, Y]$ is a nontrivial semisimple element. Hence there does not exist a Borel subalgebra of $\mathfrak{g}$ which contains both X and Y.

Now let G_{ad} denote an adjoint type group with root system Φ and $\pi : G \to G_{ad}$ the corresponding central isogeny (cf. §22 of [1]); then $\ker(\mathrm{d}\pi)$ is central in $\mathfrak{g}$. Applying 22.6 of [1], we see that π induces a bijection between Borel subgroups of G and Borel subgroups of G_{ad}. Moreover, by [1, 22.4], $\mathrm{d}\pi$ is bijective on nilpotent elements in the unipotent radical of a Borel subgroup. We deduce that there is no Borel subalgebra of $\mathrm{Lie}(G_{ad})$ which contains both $\mathrm{d}\pi(X)$ and $\mathrm{d}\pi(Y)$. Setting $\mathfrak{h} = k\,\mathrm{d}\pi(X) + k\,\mathrm{d}\pi(Y)$, we have our desired example.

Suppose now that the root system of G is not irreducible. Set $X = \sum_{i=1}^{\ell} e_{\alpha_i} \in \mathfrak{g}$, so $X \in \mathrm{Lie}(B)$. Then there exists a cocharacter $\tau : \mathbf{G}_m \to T$ with X in $\mathfrak{g}(\tau; 2)$, the 2-weight space with respect to τ and $\mathrm{Lie}(B) = \bigoplus_{i \geqslant 0} \mathfrak{g}(\tau; i)$. In particular, $\mathrm{ad}(X) : \mathfrak{g}(\tau; i) \to \mathfrak{g}(\tau; i + 2)$ for all $i \in \mathbb{Z}$. It is clear that $\mathrm{ad}(X) : \mathfrak{g}(\tau; -2) \to \mathfrak{g}(\tau; 0) = \mathrm{Lie}(T)$ is surjective.

Suppose now that G_0 is isogenous to G and p divides the order of the fundamental group of G_0. Let $\pi : G \to G_0$ be a central isogeny; our assumption on p implies that there exists $0 \neq W \in \ker(\mathrm{d}\pi)$. Then $W \in \mathrm{Lie}(T)$; hence there exists a unique $Y \in \mathfrak{g}(\tau; -2)$ for which $[X, Y] = W$. Set $\mathfrak{h} \subset \mathrm{Lie}(G_0)$ to be the restricted subalgebra generated by $\mathrm{d}\pi(X)$ and $\mathrm{d}\pi(Y)$. The proof that $\mathfrak{h}$ does not lie in any Borel subalgebra of $\mathrm{Lie}(G_0)$ goes through as above. Note that in most cases, $X^p \neq 0$.

4. Commutative subalgebras

In this section we study the case where p is a torsion prime for an irreducible component of the root system of G. In each case we construct a 3-dimensional commutative restricted subalgebra of $\mathfrak{g}$ spanned by nilpotent elements e, X, Y, with $e^p = X^p = Y^p = 0$, which lies in no Borel subalgebra of G. It suffices to consider the case where G is simple. In what follows we will use the Bala–Carter–Pommerening notation for nilpotent orbits in $\mathfrak{g}$.

The case $p = 2$.

Here we take e to be an element of type A_1^3 if G is of type D_ℓ or E_ℓ, of type $A_1 \times \tilde{A}_1$ if G is of type B_ℓ or F_4, and of type $\tilde{A}_1$ if G is of type G_2.

If the Dynkin diagram of G is simply-laced then it has a (unique) subdiagram of type D_4. We will work within this subsystem subalgebra. Set

$$e = e_{10{}^{0}_{0}} + e_{00{}^{1}_{0}} + e_{00{}^{0}_{1}}, \qquad X = e_{11{}^{0}_{0}} + e_{01{}^{1}_{0}} + e_{01{}^{0}_{1}}, \qquad Y = f_{11{}^{1}_{0}} + f_{11{}^{0}_{1}} + f_{01{}^{1}_{1}}.$$

If G is of type B_ℓ or F_4 then the Dynkin diagram of G has a (unique) subdiagram of type B_3, which we label with roots $\beta_1, \beta_2, \beta_3$, where β_3 is short. Here we let $e = e_{\beta_1} + e_{\beta_3}$, $X = e_{110} + e_{011}$, $Y = f_{111} + f_{012}$.

Finally, if G is of type G_2 then let $e = e_{\alpha_1}$, $X = e_{11}$, $Y = f_{21}$.

The case $p = 3$.

Here either G is of type E_ℓ, $\ell = 6, 7, 8$ or G is of type F_4. We take e to be an element of type $A_2^2 \times A_1$ if G is of type E_ℓ and of type $A_1 \times \tilde{A}_2$ if G is of type F_4. If G is of type E_6, E_7 or E_8 then we can restrict to the (standard) subsystem of type E_6: let

$$e = e_{\substack{10000\\0}} + e_{\substack{01000\\0}} + e_{\substack{00010\\0}} + e_{\substack{00001\\0}} + e_{\substack{00000\\1}},$$
$$X = e_{\substack{11100\\0}} + e_{\substack{00110\\1}} + e_{\substack{00111\\0}} - e_{\substack{01100\\1}} + e_{\substack{01110\\0}},$$
$$Y = f_{\substack{11110\\0}} + f_{\substack{00111\\1}} + f_{\substack{11100\\1}} - f_{\substack{01111\\0}} + f_{\substack{01110\\1}}.$$

If G is of type F_4 then let $e = e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_4}$, $X = e_{0111} + e_{1110} - e_{0120}$ and $Y = 2f_{1111} - 2f_{1120} + f_{0121}$.

The case $p = 5$.

Here G is of type E_8. We choose e to be an element of type $A_4 \times A_3$. Let

$$e = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_6} + e_{\alpha_7} + e_{\alpha_8},$$
$$X = e_{\substack{1111000\\1}} + 2e_{\substack{0011110\\1}} + 2e_{\substack{1111100\\0}} + 2e_{\substack{0011111\\0}} + 2e_{\substack{0111110\\0}} - e_{\substack{0121000\\1}} - e_{\substack{0111100\\1}},$$
$$Y = f_{\substack{1111110\\0}} + f_{\substack{1121000\\1}} + f_{\substack{1111100\\1}} + 2f_{\substack{0011111\\1}} + 2f_{\substack{0111110\\1}} + f_{\substack{0121100\\1}} - 2f_{\substack{0111111\\0}}.$$

Note that in each of the above cases, there exists e_α (resp. e_β, f_γ) in the expression for e (resp. X, Y) such that $\alpha + \beta - \gamma = 0$.

Proposition 4.1. *Let $\mathfrak{h} = ke + kX + kY$, with e, X, Y as above. Then $\mathfrak{h}$ is not contained in any Borel subalgebra of $\mathfrak{g}$.*

Proof. Suppose $\mathfrak{h}$ is contained in a Borel subalgebra. Then for some $g \in G$, $\operatorname{Ad} g(\mathfrak{h}) \subset \mathfrak{b}$, where $\mathfrak{b}$ is the Borel subalgebra corresponding to the positive Weyl chamber. By the Bruhat decomposition, we have $g = u'nu$, where $u, u' \in U^+$ and $n \in N_G(T)$. But now $\operatorname{Ad} g(\mathfrak{h}) \subset \mathfrak{b}$ if and only if $\operatorname{Ad}(nu)(\mathfrak{h}) \subset \mathfrak{b}$, thus we may assume that $u' = 1$. Let $w = nT \in W$. We will explain our argument for the case where G is of type D_4 and $p = 2$. Note that $\operatorname{Ad} u(e) = e + x$, where x is in the span of all positive root subspaces for roots of length greater than 1. Thus $\operatorname{Ad} nu(e) \in \mathfrak{b}$ implies, in particular, that $w(\alpha_1) \in \Phi^+$. Applying a similar argument to X and Y, we see that $w(\alpha_2 + \alpha_3) \in \Phi^+$ and $w(-(\alpha_1 + \alpha_2 + \alpha_3)) \in \Phi^+$. Taking the sum $w(\alpha_1) + w(\alpha_2 + \alpha_3) + w(-(\alpha_1 + \alpha_2 + \alpha_3)) = 0$, we have a contradiction. This argument works for all the examples given above, using the observation that if e_α and e_β have non-zero coefficients in the expression for e then α and β are not congruent modulo the subgroup $\mathbb{Z}\Phi$ (and similarly for X, Y). □

Finally, the examples of Section 3 and Proposition 4.1 give the following result:

Theorem 4.2. *Let G be a semisimple algebraic group over k and p a torsion prime for G. Then there exists a non-reduced unipotent subgroup scheme of G which does not lie in any Borel subgroup of G.*

We conclude with one further proposition which describes to some extent the nature of the 3-dimensional subalgebras defined above.

Proposition 4.3. *Let e, X and Y be as in Proposition* 4.1. *Any non-zero element of $\mathfrak{h} = ke \oplus kX \oplus kY$ is conjugate to e and $N_G(\mathfrak{h})/C_G(\mathfrak{h}) \cong \mathrm{SL}(3, k)$.*

Proof. In each case, e is a regular nilpotent element in $\operatorname{Lie}((L, L))$, for some Levi factor L of G normalized by T. Note that (L, L) is a commuting product of type A_m subgroups and hence p is good for (L, L). We choose τ to be a cocharacter of (L, L) (and hence a cocharacter of G), associated to e (see [5, 5.3]). In particular $e \in \mathfrak{g}(2; \tau)$. Then one checks that $\mathfrak{g}(\tau; -1) \cap C_{\mathfrak{g}}(e) = kX \oplus kY$. This then implies that the group $C = C_G(e) \cap C_G(\tau(k^\times))$ normalizes $\mathfrak{h}$. It can be checked that the adjoint representation induces a surjective morphism $C \to \mathrm{SL}(kX \oplus kY)$. But we can apply a similar argument to an analogous subgroup of $C_G(Y)$. Thus $N_G(\mathfrak{h})$ contains the subgroups $\mathrm{SL}(ke \oplus kX)$ and $\mathrm{SL}(kX \oplus kY)$, and hence contains $\mathrm{SL}(\mathfrak{h})$. In particular, all non-zero elements of $\mathfrak{h}$ are conjugate by an element of $N_G(\mathfrak{h})$. It follows from our remark on root elements in the expressions for e, X and Y that there can be no cocharacter in G for which e, X and Y are all in the sum of positive weight spaces. This then implies that $N_G(\mathfrak{h})/C_G(\mathfrak{h})$ is isomorphic to $\mathrm{SL}(\mathfrak{h})$. $\square$

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