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Number Theory

Hypergeometric functions for function fields and transcendence

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Abstract

We study hypergeometric functions for $\mathbb{F}_q[T]$, and show in the entire (non-polynomial) case the transcendence of their special values at nonzero algebraic arguments which generate extension of the rational function field with less than q places at infinity. We also characterize in the balanced case the algebraicity of hypergeometric functions. ***To cite this article: D.S. Thakur et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Fonctions hypergéométriques pour les corps de fonctions et transcendance. Nous étudions les fonctions hypergéométriques pour $\mathbb{F}_q[T]$, et démontrons dans le cas entier (non polynomial) la transcendance de leurs valeurs spéciales aux arguments algébriques non nuls qui engendrent des extensions du corps de fonctions rationnelles avec au plus $q-1$ places à l'infini. Nous caractérisons aussi dans le cas équilibré l'algébricité des fonctions hypergéométriques. ***Pour citer cet article : D.S. Thakur et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit $p \geqslant 2$ un nombre premier et $q = p^w$ avec $w \geqslant 1$ un entier. Nous désignons par $\mathbb{F}_q$ le corps fini à q éléments, par $\mathbb{F}_q[T]$ l'anneau intègre des polynômes en T à coefficients dans $\mathbb{F}_q$, et par $\mathbb{F}_q(T)$ le corps de fractions de $\mathbb{F}_q[T]$. Pour tous les $P, Q \in \mathbb{F}_q[T]$ avec $Q \neq 0$, nous définissons

$$|P/Q|_\infty := q^{\deg P - \deg Q},$$

et appelons $|\cdot|_\infty$ la valeur absolue ∞-adic sur $\mathbb{F}_q(T)$. Nous désignons par $\mathbb{F}_q((T^{-1}))$ le complété topologique de $\mathbb{F}_q(T)$ pour $|\cdot|_\infty$, et par $\mathbf{C}_\infty$ le complété topologique d'une clôture algébrique fixée de $\mathbb{F}_q((T^{-1}))$.

Rappelons maintenant la définition de la fonction hypergéométrique ${}_rF_s$ pour les corps de fonctions introduite dans [14]. Pour la motivation et les propriétés diverses, voir [14], [17], et [19, §6.5].

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Fixons $a \in \mathbb{Z}$. Pour tous les entiers $n \geqslant 0$, définissons

$$(a)_n = \begin{cases} D_{n+a-1}^{q^{-(a-1)}} & \text{si } a \geqslant 1, \\ (-1)^{-a-n} L_{-a-n}^{-q^n} & \text{si } n \leqslant -a \text{ et } a \leqslant 0, \\ 0 & \text{si } n > -a \geqslant 0, \end{cases}$$

où pour tous les entiers $j \geqslant 0$, D_j est le produit de tous les polynômes unitaires de degré j dans $\mathbb{F}_q[T]$, et L_j est le plus petit multiple commun (unitaire) de tous les polynômes de degré j dans $\mathbb{F}_q[T]$.

Pour tous les entiers $r, s \geqslant 0$ et pour tous les $a_i, b_j \in \mathbb{Z}$ ($1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$) avec $b_j > 0$ (de sorte qu'il n'existe pas de dénominateurs nuls ci-dessous), considérons la série formelle

$${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_r)_n}{D_n (b_1)_n \cdots (b_s)_n} z^{q^n}.$$

Nous la désignons souvent par ${}_rF_s(z)$, quand les paramètres en question sont bien compris.

S'il existe un entier i ($1 \leqslant i \leqslant r$) tel que $a_i \leqslant 0$, alors $(a_i)_n = 0$ pour tout $n \geqslant -a_i$. Dans ce cas ${}_rF_s$ est un polynôme en z. Pour éviter cette trivialité, nous supposerons dans la suite

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \quad \text{et} \quad 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s.$$

Par un calcul direct, nous obtenons tout de suite que le rayon de convergence R de ${}_rF_s$ satisfait à

$$R = \begin{cases} 0 & \text{si } r > s+1, \\ q^{-\sum_{i=1}^{r}(a_i-1) + \sum_{j=1}^{s}(b_j-1)} & \text{si } r = s+1, \\ +\infty & \text{si } r < s+1. \end{cases}$$

Si $R = +\infty$, alors ${}_rF_s$ est une fonction entière mais pas un polynôme selon notre hypothèse, elle est donc une fonction transcendante (voir par exemple Théorème 5 de [24]).

Voici les résultats principaux :

Théorème 1. *Soit $r, s \geqslant 0$ deux entiers tels que $r < s + 1$. Soit*

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \quad \textit{et} \quad 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

des entiers. Pour tout $\gamma \in \mathbf{C}_\infty \setminus \{0\}$ algébrique sur $\mathbb{F}_q(T)$ tel que $\mathbb{F}_q(T)(\gamma)$ ait au plus $q - 1$ places sur la place à l'infini de $\mathbb{F}_q[T]$ (en particulier γ peut donc être rationnel ou algébrique non nul de degré $< q$), alors ${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \gamma)$ est transcendant sur $\mathbb{F}_q(T)$.

Pour le cas équilibré $r = s + 1$, nous avons la caractérisation suivante :

Théorème 3. *Soit $r, s \geqslant 0$ deux entiers tels que $r = s + 1$. Soit*

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \quad \textit{et} \quad 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

des entiers. Alors les propriétés suivantes sont équivalentes :

(i) *$a_j \geqslant b_{j-1}$, pour tous les entiers j ($1 \leqslant j \leqslant r$)* ;
(ii) *$({}_rF_s(z))^{q^\ell} \in \mathbb{F}_q[T][[z]]$, avec $\ell = \max(a_r, b_s)$* ;
(iii) *${}_rF_s(z)$ est une fonction algébrique.*

1. Statements of main results

Let $\mathbb{F}_q$ be the finite field of characteristic p with q elements, let $\mathbb{F}_q[T]$ be the ring of polynomials in T with coefficients in $\mathbb{F}_q$, and let $\mathbb{F}_q(T)$ be the fraction field of $\mathbb{F}_q[T]$. For all $P, Q \in \mathbb{F}_q[T]$ and $Q \neq 0$, set

$$|P/Q|_\infty := q^{\deg P - \deg Q}.$$

We denote by $\mathbb{F}_q((T^{-1}))$ the topological completion of $\mathbb{F}_q(T)$ with respect to $|\cdot|_\infty$, and by $\mathbf{C}_\infty$ the topological completion of a fixed algebraic closure of $\mathbb{F}_q((T^{-1}))$. The latter is topologically complete and algebraically closed, and plays the role of $\mathbb{C}$ in our study. However, unlike $\mathbb{C}$, $\mathbf{C}_\infty$ is not locally compact.

For all integers $j \geqslant 0$, let D_j be the product of all monic polynomials of degree j in $\mathbb{F}_q[T]$, and let L_j be the (monic) least common multiple of all polynomials of degree j in $\mathbb{F}_q[T]$. Then

$$D_j = \prod_{k=0}^{j-1} [j-k]^{q^k} \quad \text{and} \quad L_j = \prod_{k=1}^{j} [k],$$

where $[k] = T^{q^k} - T$, for all integers $k \geqslant 0$. These polynomials $[j]$, D_j and L_j are fundamental for the arithmetic on $\mathbb{F}_q[T]$ and also occur in the formulas for Carlitz module structure, and in the expansions of its exponential, logarithm, etc. For more details on this subject, see [8] and [19].

Now we recall the definition of ${}_rF_s$, the hypergeometric function for function fields introduced in [14]. For motivation, various properties such as solutions of differential-difference equations, specializations, etc., see [14,17], and [19, §6.5].

Fix $a \in \mathbb{Z}$. For all integers $n \geqslant 0$, define

$$(a)_n = \begin{cases} D_{n+a-1}^{q^{-(a-1)}} & \text{if } a \geqslant 1, \\ (-1)^{-a-n} L_{-a-n}^{-q^n} & \text{if } n \leqslant -a \text{ and } a \leqslant 0, \\ 0 & \text{if } n > -a \geqslant 0. \end{cases}$$

For all integers $r, s \geqslant 0$ and for all $a_i, b_j \in \mathbb{Z}$ $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s)$ with $b_j > 0$ (so that there are no zero denominators below), consider the formal series

$${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_r)_n}{D_n (b_1)_n \cdots (b_s)_n} z^{q^n}.$$

We often denote it by ${}_rF_s(z)$, when the parameters are well understood.

Example. If $r = s = 0$, then we obtain the Carlitz exponential

$${}_0F_0(z) = e_C(z) := \sum_{n=0}^{+\infty} \frac{z^{q^n}}{D_n}.$$

Let $r = s = p = \operatorname{Char} \mathbb{F}_q$, $b_j = a_j + 1 = 2$ $(1 \leqslant j \leqslant p)$. Then

$$\left({}_rF_s(z)\right)^q = \sum_{n=0}^{+\infty} \frac{(1)_n^{pq}}{D_n^q (2)_n^{pq}} z^{q^{n+1}} = \sum_{n=0}^{+\infty} \frac{z^{q^{n+1}}}{[n+1]^{p-1} D_{n+1}} = \frac{e_C^{(p-1)}(z)}{(p-1)!},$$

where $e_C^{(p-1)}(z)$ is the $(p-1)$th derivative of $e_C(z)$ with respect to T. For $r = 0$, $s = 1$, and $b_1 = m + 1$, we get the Bessel–Carlitz function

$$J_m(z) := \sum_{n=0}^{+\infty} \frac{z^{q^{m+n}}}{D_{m+n} D_n^{q^m}} \quad \text{and} \quad {}_0F_1(-; m+1; z) = J_m^{q^{-m}}.$$

We note that the normalization here is slightly different from that of [7].

If there exists some integer i $(1 \leqslant i \leqslant r)$ such that $a_i \leqslant 0$, then for $n \geqslant -a_i$, we have $(a_i)_n = 0$. In this case ${}_rF_s$ is a polynomial in z. As we are interested in algebraicity and transcendence questions, we avoid this triviality, and in the following we shall always suppose

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \quad \text{and} \quad 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s.$$

We see that the radius of convergence R of ${}_rF_s$ is zero, positive or infinite according as whether r is more than, equal to or less than $s + 1$. If $R = +\infty$, then ${}_rF_s$ is entire, and not a polynomial by our hypothesis, so it is a transcendental function (see for example Theorem 5 of [24]).

The main results are the following:

Theorem 1. *Let $r, s \geqslant 0$ be integers such that $r < s + 1$, and let*

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \quad \text{and} \quad 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

be integers. Then for all $\gamma \in \mathbf{C}_\infty \setminus \{0\}$ algebraic over $\mathbb{F}_q(T)$ and such that $\mathbb{F}_q(T)(\gamma)$ has less than q places above the infinite place of $\mathbb{F}_q[T]$ (in particular, γ can be any nonzero rational or nonzero algebraic of degree less than q), then ${}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \gamma)$ is transcendental over $\mathbb{F}_q(T)$.

As a direct consequence of Theorem 1 (see examples of specializations above), we obtain

Theorem 2. *For all algebraic $\gamma \in \mathbf{C}_\infty \backslash \{0\}$ as in Theorem 1, all the $e_C(\gamma)$, $e_C^{(p-1)}(\gamma)$, and $J_m(\gamma)$ are transcendental over $\mathbb{F}_q(T)$, where $m \geqslant 0$ is an integer.*

Remarks. (1) For $e_C(\gamma)$ and $J_m(\gamma)$, Theorem 2 is a special case of the known result (where γ is allowed to be any nonzero algebraic). But for $e_C^{(p-1)}(\gamma)$, it is stronger than the one proved by L. Denis [6] where γ was assumed rational. Since the derivative of an algebraic element is algebraic, we have indeed established the transcendence of the lower derivatives also, while the higher derivatives are zero.

(2) The original proof of L.I. Wade [21] for e_C, and those of L. Denis [7] for J_m or [5] for $e_C^{(1)}(\gamma)$ used the ideas of functional equations, Drinfeld modules, and algebraic group theory. It seems difficult to apply these techniques in the case of the hypergeometric function.

For the balanced case $r = s + 1$, we have the following characterization:

Theorem 3. *Let $r, s \geqslant 0$ be integers such that $r = s + 1$, and let*

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \quad \text{and} \quad 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

be integers. Then the following properties are equivalent:

(i) *$a_j \geqslant b_{j-1}$, for all integers j $(1 \leqslant j \leqslant r)$;*
(ii) *$({}_rF_s(z))^{q^\ell} \in \mathbb{F}_q[T][[z]]$, with $\ell = \max(a_r, b_s)$;*
(iii) *${}_rF_s(z)$ is an algebraic function.*

In [14] and [19], there is another analog ${}_r\mathcal{F}_s$ (just as there are two kinds of Zeta, Gamma functions and cyclotomic theories in the function field context [19]) of hypergeometric functions, where the parameters are in $\mathbf{C}_\infty$. For these analogs, $(a)_n := e_n(a) := \prod(a - f)$, with f running over all polynomials of degree less than n. Since this is zero for large n, when a itself is a polynomial, the bad or trivial cases now are for $a \in \mathbb{F}_q[T]$. So the question now is for what a's in $\mathbb{F}_q(T) \setminus \mathbb{F}_q[T]$ do we get an algebraic function. We have the following partial results, compared to the complete results for the first analog.

Theorem 4. (1) *Any function ${}_{s+1}\mathcal{F}_s(a_i; b_j; z)$, with a_i being any proper fractions and b_j being fractions with denominators of degree one, is algebraic.*

(2) *If ${}_{s+1}\mathcal{F}_s(a_j; b_i; z)$ is algebraic, then ${}_{s+2}\mathcal{F}_{s+1}(a_j, a; b_i, b; z)$ is algebraic, where a is a proper fraction and b is a proper fraction with denominator of degree one.*

(3) *If ${}_{s+1}\mathcal{F}(a_1, \ldots, a_{s+1}; b_1, \ldots, b_s, z)$ and ${}_{r+1}\mathcal{F}_r(a'_1, \ldots, a'_{r+1}; b'_1, \ldots, b'_r; z)$ are both algebraic, then ${}_{s+r+2}\mathcal{F}_{s+r+1}(a_h, a'_i; b_j, b'_k, c; z)$ is algebraic if c is fraction with degree one denominator.*

(4) *If you stay in the balanced case, you can add or remove parameters having degree one denominators retaining algebraicity.*

(5) *The functions ${}_2\mathcal{F}_1(1/T, 1/T; 1/T^2; z)$ and ${}_{q+1}\mathcal{F}_q(1/T, \{(\theta T + 1)/T^3\}; 1/T^2, \ldots, 1/T^2; z)$, where θ varies in $\mathbb{F}_q$, are transcendental.*

(6) *When* $q = 2$, ${}_{q+1}\mathcal{F}_q(1/T^2, \{(\theta T + 1)/T^3\}; 1/T^2, \ldots, 1/T^2; z) = {}_2\mathcal{F}_1(1/T^2 + 1/T^3, 1/T^3; 1/T^2; z)$ *and* ${}_2\mathcal{F}_1(1/T^3, 1/T^3; 1/T^2; z)$ *are algebraic, whereas* ${}_2\mathcal{F}_1(1/T^2, 1/T; 1/(T+1)^2; z)$ *and* ${}_2\mathcal{F}_1(1/T^2, 1/T^2; 1/(T+1)^2; z)$ *are transcendental.*

Remarks. (1) In contrast to the situation for the first analog, the algebraicity is not equivalent to the integrality in this case.

(2) Our results so far are consistent with the statement (with naive analogy with Theorem 3) that if ${}_r\mathcal{F}_s$ (with $r = s + 1$) is an algebraic function, then the degree of the denominator of a_j is at least the degree of the denominator of b_{j-1} (or b_j), when arranged in order. But the last part of Theorem 4 shows that degree equalities can still lead to transcendental functions in contrast.

2. Proofs (sketches)

The proof of Theorem 1 is a rather technical application of the following criterion which improves and generalizes Theorem 1 in [25].

Theorem 5. *Let* $\mathbb{L}$ *be a valued field of characteristic* $p \geqslant 2$, *endowed with a nonarchimedean absolute value* $|\cdot|$. *Let* $\mathbb{A}$ *be a subring of* $\mathbb{L}$, *and* $\mathbb{K}$ *the fraction field of* $\mathbb{A}$ *in* $\mathbb{L}$. *Let* $\alpha \in \mathbb{L}$. *Then* α *is transcendental over* $\mathbb{K}$ *if and only if there exists a sequence* $(\alpha_n)_{n\geqslant 0}$ *in* $\mathbb{L}$ *satisfying the following two conditions*:

(i) *There exists a sequence* $(\delta_n)_{n\geqslant 0}$ *of positive real numbers such that for all integers* $n \geqslant 0$, *we have*

$$|\alpha - \alpha_n| \leqslant \delta_n;$$

(ii) *For all integers* $t \geqslant 1$, *there exist* $t+1$ *integers* $0 \leqslant \sigma_0 < \cdots < \sigma_t$ *such that for every* $(t+1)$*-tuple* $(A_0, \ldots, A_t)$ *of not all zero elements in* $\mathbf{Z}$, *there exist an infinite subset* $S \subseteq \mathbb{N}$ *and* $t+1$ *sequences* $\theta_j = (\theta_j(n))_{n\geqslant 0}$ $(0 \leqslant j \leqslant t)$ *of positive integers increasing to* $+\infty$ *such that for all integers* j $(0 \leqslant j \leqslant t)$ *with* $A_j \neq 0$, *we have*

$$\lim_{S \ni n \to +\infty} |\beta_n| / \delta_{\theta_j(n)}^{q^{\sigma_j}} = +\infty,$$

where β_n *is defined by*

$$\beta_n = \sum_{j=0}^{t} A_j \alpha_{\theta_j(n)}^{q^{\sigma_j}}.$$

Besides all the applications given in [25], the above criterion contains also all the five criteria obtained by V. Laohakosol et al. in [10] via Wade's method, which generalizes respectively the five classical results of L.I. Wade [21–23]. With the help of Theorem 5, we can also give another proof of the main result of [11] (originally proved by automata method) proving transcendence of the Carlitz–Goss gamma values at non-natural p-adic integers. Note that the special case giving the transcendence of the values at fractions treated in [15], and [2] by automata method has been vastly generalized to determination of full algebraic relations between the values at fractions in [4], but the period method of [3,12,4] does not apply to non-fractional p-adic integers.

For Theorem 3, we prove (i) $\leftrightarrow$ (ii) $\to$ (iii) directly and prove (iii) $\to$ (ii) in two ways: As in [20] using the Eisenstein theorem and characterizations of $(a)_n$'s in terms of products, or using the characterization of algebraic functions by H. Sharif and C. Woodcock [13] and by T. Harase [9] (see also [1]).

Theorem 4 is proved using some results of the first author and Anderson on the 'solitons' of Anderson (see [3,19,16], and especially [18] for the background on the name and references).

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