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Calculus of variations

Regularity of weak constant anisotropic mean curvature surfaces

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Abstract

In this Note a definition of weak constant anisotropic mean curvature surfaces and the expression in conformal coordinates of the anisotropic mean curvature of surfaces in $\mathbb{R}^3$ are obtained. Moreover, we prove that all weak constant anisotropic mean curvature surfaces in $\mathbb{R}^3$ are continuous. ***To cite this article: J. Zhai, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Régularité des solutions faibles du problème des surfaces à courbure moyenne anisotropique constante. Dans cette Note on donne une définition des solutions faibles du problème des surfaces à courbure moyenne anisotropique constante ; dans $\mathbb{R}^3$ on donne une représentation en coodonnées conformes des solutions. De plus, dans la cas de $\mathbb{R}^3$, nous démontrons la continuité de toutes les solutions. ***Pour citer cet article : J. Zhai, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

The mean curvature of a surface is the negative of the rate of change of surface area with respect to the volume swept out by deformation (see [10]). In Finsler geometry, for a surface Γ with anisotropic structure $\gamma(n)$ defined on the unit normal vectors n of the surface Γ, it is natural to consider the surface energy $\int_\Gamma \gamma(n)\,\mathrm{d}\sigma$ instead of the area (see [2,10]). Here we can extend γ to $\mathbb{R}^3$ by the assumption that $\gamma : \mathbb{R}^3 \to \mathbb{R}^+$ is positive homogeneous of degree one:

$$\gamma(\lambda p) = \lambda\gamma(p), \quad \text{for all } p \in S^2 \text{ and for all } \lambda \geqslant 0.$$

The change ratio of the surface energy $\int_\Gamma \gamma(n)\,\mathrm{d}\sigma$ per change of volume is called the anisotropic mean curvature of the surface Γ.

In the isotropic case, the regularity of weak constant mean curvature surfaces and related problems were studied extensively (see Heinz [5], Werner [13], Hildebrandt [6], Hildebrandt and Kaul [7], Wente [11,12], Grüter [4], Jost [8] and references therein). The paper [7] treated Plateau problem for surfaces of prescribed mean curvature in a three-dimensional Riemannian manifold. A solution was shown to exist for the boundary being a Jordan curve lying in

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a set, under a restriction on the sectional curvatures in the set. The classical existence theorem for Plateau problem for minimal surfaces was proved by Douglas [1] and Rado [9]. Wente [11,12] extended their theorem to the Plateau problem for surfaces of constant mean curvature. He proved the existence and continuity of a constant mean curvature surface of the type of the unit disk whose boundary is an oriented rectifiable Jordan curve in $\mathbb{R}^3$. Grüter [4] proved the regularity of surfaces with bounded prescribed mean curvature.

However, for the anisotropic case, very few results are known (see [2,10]). In this Note, a definition of weak constant anisotropic mean curvature surfaces, and the expression in conformal coordinates of the anisotropic mean curvature are given. Moreover, the regularity of the weak constant anisotropic mean curvature surfaces is proved.

Theorem 1.1. *Suppose $\gamma:\mathbb{R}^3 \to \mathbb{R}^+$ is positive homogeneous of degree one, and*

$$\inf_{|p|=1} \gamma(p) > 0, \qquad \sup_{|p|=1} \left|\nabla_p \gamma(p)\right| < \infty.$$

Then all weak constant anisotropic mean curvature surfaces are continuous.

Remark 1.2. The uniqueness and the existence of constant weakly anisotropic mean curvature spheres are proved in [3].

2. Conformal expression of anisotropic mean curvature

Suppose $G \subset \mathbb{R}^2$ is a domain and $\Gamma := \{x(u,v) \in \mathbb{R}^3\colon \forall (u,v) \in G\}$ is a surface in $\mathbb{R}^3$. Moreover, suppose x is conformal, that is

$$x_u \cdot x_v = 0, \quad |x_u|^2 = |x_v|^2, \quad \forall (u,v) \in G. \tag{1}$$

To obtain the conformal expression of anisotropic mean curvature, we calculate the first variation of the surface energy functional:

$$\mathcal{H}\big(\Phi_t(\Gamma)\big) = \int\limits_{\Phi_t(\Gamma)} \gamma\big(n_{\Phi_t(\Gamma)}(y)\big)\,\mathrm{d}\sigma(y) = \int\limits_G \gamma\big(n_{\Phi_t(M)}\big(y(u,v)\big)\big)|y_u \times y_v|\,\mathrm{d}u\,\mathrm{d}v$$

for smooth vector fields $X(x)$ of $\mathbb{R}^3$ with $X(x) = 0$ on ∂G, where

$$\Phi_t(\Gamma) := \big\{x + tX(x)\colon \forall x \in \Gamma\big\}$$

is an admissible surface and $n_{\Phi_t(\Gamma)}(y)$ denotes the unit normal vector of the surface $\Phi_t(\Gamma)$ at $y \in \Phi_t(\Gamma)$. The anisotropic mean curvature $H(x)$ of the surface Γ at x is defined by (see [2] or [10]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathcal{H}\big(\Phi_t(\Gamma)\big) = \int\limits_\Gamma H(x)\cdot X(x)\,\mathrm{d}\sigma(x).$$

By a direct calculation, we can get the expression in conformal coordinates of the anisotropic mean curvature $H(x)$ of the surface Γ at x,

$$H\big(x(u,v)\big) = (-1)\frac{x_u(u,v) \times \partial_v \nabla_p \gamma(n(x(u,v))) - x_v(u,v) \times \partial_u \nabla_p \gamma(n(x(u,v)))}{|x_u(u,v) \times x_v(u,v)|}. \tag{2}$$

Definition 2.1. $x \in H^{1,2}(G,\mathbb{R}^3)$ is called a weak constant anisotropic mean curvature surface ($H \equiv 1$) if it is weakly conformal, that is

$$x_u \cdot x_v = 0, \quad |x_u|^2 = |x_v|^2, \quad \text{for almost all } (u,v) \in G,$$

and for all $\varphi \in H_0^{1,2}(G,\mathbb{R}^3)$

$$\int \big(x_v \times \nabla_p \gamma\big(n(x)\big)\big)\cdot \varphi_u - \big(x_u \times \nabla_p \gamma\big(n(x)\big)\big)\cdot \varphi_v + (x_u \times x_v)\cdot \varphi\,\mathrm{d}u\,\mathrm{d}v = 0. \tag{3}$$

3. Proof of Theorem 1.1

There are constants $c_1 > 0$ and $c_2 > 0$ such that

$$\inf_{|p|=1} \gamma(p) \geqslant c_1, \qquad \sup_{|p|=1} \left|\nabla_p \gamma(p)\right| \leqslant c_2.$$

Let $\rho > 0$, $\beta > 0$ and ψ be a cut-off function with

$$\psi(t) = \begin{cases} 1, & t \leqslant \dfrac{1}{2\rho^{\beta-1}}, \\ 0, & t \geqslant \dfrac{1}{\rho^{\beta-1}}, \end{cases} \qquad \psi'(t) \leqslant 0.$$

For any $z_0 \in G$ and $\sigma > 0$, fix $z_1 \in G$ with $|z_1 - z_0| < \sigma$, and let

$$\rho' = \inf_{z \in \partial B(z_0,\sigma)} \left|x(z) - x(z_1)\right|.$$

The density of x at z_1 is defined by:

$$I(\rho) := \int\limits_{B(z_0,\sigma)} \gamma\big(n(x)\big)|x_u \times x_v| \psi\left(\frac{|x - x(z_1)|}{\rho^\beta}\right) \mathrm{d}u\, \mathrm{d}v.$$

Note that

$$2I(\rho) - \rho I'(\rho) = \int\limits_{B(z_0,\sigma)} \gamma\big(n(x)\big)|x_u \times x_v| \left\{2\psi + \beta \frac{\psi'|x - x(z_1)|}{\rho^\beta}\right\} \mathrm{d}u\, \mathrm{d}v. \tag{4}$$

By a substitution of

$$\varphi(z) = \psi\left(\frac{|x(z) - x(z_1)|}{\rho^\beta}\right)\big(x(z) - x(z_1)\big)$$

in (3), we have:

$$-\int (x_u \times x_v) \cdot \big(x - x(z_1)\big)\psi \,\mathrm{d}u\, \mathrm{d}v \geqslant \int \gamma\big(n(x)\big)|x_u \times x_v| \left\{2\psi + \frac{2c_2}{c_1}\psi' \frac{|x - x(z_1)|}{\rho^\beta}\right\} \mathrm{d}u\, \mathrm{d}v. \tag{5}$$

Now choose

$$\beta = \frac{2c_2}{c_1}.$$

From (4), (5), we deduce the following monotonicity inequality,

$$\frac{I(\rho')}{\rho'^2} \geqslant \mathrm{e}^{(\rho-\rho')/c_1} \frac{I(\rho)}{\rho^2}, \quad \forall \rho \in (0, \rho'].$$

From [4] (Proposition 2.5), we get:

$$\liminf_{\rho \to 0} \frac{I(\rho)}{\rho^2} \geqslant \liminf_{\rho \to 0} \frac{c_1 \rho^{-2}}{2} \int\limits_{\{|x - x(z_1)| \leqslant \rho/2\}} |\nabla x|^2 \,\mathrm{d}u\, \mathrm{d}v \geqslant \frac{c_1 \pi}{8}.$$

Then

$$\inf_{z \in \partial B(z_0,\sigma)} \left|x(z) - x(z_1)\right| = \rho' \leqslant 2\sqrt{2}\left(\frac{I(\rho')}{c_1 \pi}\right)^{1/2} \mathrm{e}^{\rho'/(2c_1)} \to 0, \quad \text{as } \sigma \to 0.$$

For any $z_1 \in B(z_0, \sigma)$, $z_2 \in B(z_0, \sigma)$, there are $z_1' \in \partial B(z_0, \sigma)$ and $z_2' \in \partial B(z_0, \sigma)$, such that

$$\left|x(z_1) - x(z_2)\right| \leqslant \left|x(z_1) - x(z_1')\right| + \left|x(z_2) - x(z_2')\right| + \left|x(z_1') - x(z_2')\right|.$$

Note that the first and second terms in the right side go to zero as $\sigma \to 0$. As in [4] and [8] (Chapter 2), we use Courant–Lebesgue lemma to estimate $|x(z_1') - x(z_2')|$. Thus we get the continuity of x.

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