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Algebraic Geometry

# Holomorphic connections on some complex manifolds

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**Abstract**

Let $M$ be a compact connected complex manifold equipped with a holomorphic submersion to a complex torus such that the fibers are all rationally connected. Then any holomorphic vector bundle over $M$ admitting a holomorphic connection actually admits a flat holomorphic connection. A similar statement is valid for any finite quotient of $M$. ***To cite this article: I. Biswas, J.N. Iyer, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***


**Résumé**

**Connexions holomorphes sur quelques variétés complexes.** Soit $M$ une variété complexe compacte connexe, munie d'une submersion holomorphe $M \to T$, où $T$ est un tore complexe, telle que les fibres soient rationnellement connexes. Soit $E$ un fibré vectoriel holomorphe sur $M$ admettant une connexion. Alors $E$ admet une connexion holomorphe plate. Un énoncé similaire vaut pour tout quotient fini de $M$. ***Pour citer cet article : I. Biswas, J.N. Iyer, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***


## 1. Introduction

The notion of a holomorphic connection on holomorphic fiber bundles was introduced by Atiyah [2]. It is a long-standing question whether any vector bundle on a compact complex manifold admitting a holomorphic connection admits a flat holomorphic connection.

Here we consider holomorphic connections on the following type of compact complex manifolds.

Let $M$ be a connected compact complex manifold such that there is a finite étale cover

$$\gamma : \widetilde{M} \longrightarrow M \tag{1}$$

of the following type: there is a holomorphic submersion

$$f : \widetilde{M} \longrightarrow T \tag{2}$$

to a complex torus $T$ with the property that for any point $x \in T$, the fiber $f^{-1}(x)$ is a rationally connected complex projective manifold.

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**Theorem 1.1.** *Let $E$ be a holomorphic vector bundle over $M$ admitting a holomorphic connection. Then $E$ admits a flat holomorphic connection.*

Examples of $M$ are: (1) holomorphic fiber bundles over a complex torus $T$ with Fano manifolds as fiber, in particular, any flag bundle associated to any holomorphic vector bundle over $T$; (2) any compact Kähler manifold with numerically effective tangent bundle in the sense of [5] (this includes hyperelliptic varieties); and (3) moduli space of stable vector bundles of rank $r$ and degree $\delta$ over a compact Riemann surface $Y$, with $r$ and $\delta$ mutually coprime, and also the symmetric product $\mathrm{Sym}^d(Y)$, where $d > 2(\mathrm{genus}(Y) - 1)$.

## 2. Fiber bundles over a torus

A smooth complex projective variety $Z$ is called rationally connected if for any two points of $Z$, there is an irreducible rational curve on $Z$ that contains them; see [8, p. 433, Theorem 2.1] and [8, p. 434, Definition–Remark 2.2] for other equivalent conditions.

Let $E$ be a holomorphic vector bundle over a complex manifold $Y$. A *holomorphic connection* on $E$ is a first order holomorphic differential operator

$$D : E \longrightarrow E \otimes \Omega_Y^1$$

satisfying the Leibniz identity $D(fs) = fD(s) + s \otimes \mathrm{d}f$, where $f$ is any locally defined holomorphic function and $s$ is any locally defined holomorphic section of $E$; here $\Omega_Y^1$ is the holomorphic cotangent bundle. The *curvature* $D \circ D$ of $D$ is a holomorphic section of $\mathrm{End}(E) \otimes \Omega_Y^2$ over $Y$. A *flat* (or *integrable*) holomorphic connection is one whose curvature vanishes identically.

Let $T$ be a complex torus. Let $X$ be a compact connected complex manifold and

$$\phi : X \longrightarrow T \tag{3}$$

a holomorphic submersion. We also assume that for each point $t \in T$, the fiber $\phi^{-1}(t)$ is a rationally connected complex projective manifold.

**Proposition 2.1.** *Let $E$ be a holomorphic vector bundle over $X$ admitting a holomorphic connection. Then there is a unique, up to an isomorphism, holomorphic vector bundle $V$ over $T$ such that $\phi^* V$ is holomorphically isomorphic to $E$, where $\phi$ is the projection in* (3)*. Furthermore, the vector bundle $V$ admits a holomorphic connection.*

**Proof.** Let $D$ be any holomorphic connection on $E$. Fix a point $t_0 \in T$. Let $Z_0$ denote the fiber $\phi^{-1}(t_0)$, where $\phi$ is the projection in (3). The restriction of $E$ to $Z_0$ will be denoted by $E_0$. Let $D_0$ denote the holomorphic connection on $E_0$ induced by the holomorphic connection $D$.

By our assumption on $\phi$, the compact complex manifold $Z_0$ is rationally connected. From [8] we know that given any point $x \in Z_0$, there is a holomorphic map

$$f : \mathbb{CP}^1 \longrightarrow Z_0 \tag{4}$$

such that $x \in \mathrm{image}(f)$ and the vector bundle $f^* T Z_0$ over $\mathbb{CP}^1$ is ample; this follows from [8, p. 430, Corollary 1.3] and [8, p. 434, Claim 2.3.1], and this assertion can actually be found in [8, p. 436, Proof (2.6)] (see also [1]).

The vector bundle $f^* E_0$ is equipped with the holomorphic connection $f^* D_0$. Any holomorphic connection on a Riemann surface is automatically flat, as a Riemann surface does not have nonzero holomorphic two-forms. Since $\mathbb{CP}^1$ is simply connected, any vector bundle over $\mathbb{CP}^1$ admitting a holomorphic connection must be holomorphically trivializable.

Let $\Theta_0$ denote the curvature of the holomorphic connection $D_0$ on $E_0$. So

$$\Theta_0 \in H^0\big(Z_0, \mathrm{End}(E_0) \otimes \Omega_{Z_0}^2\big). \tag{5}$$

Consider the pulled back section

$$f^* \Theta_0 \in H^0\big(\mathbb{CP}^1, f^*\big(\mathrm{End}(E_0) \otimes \Omega_{Z_0}^2\big)\big) = H^0\big(\mathbb{CP}^1, \mathrm{End}(f^* E_0) \otimes f^* \Omega_{Z_0}^2\big) \tag{6}$$

by the map $f$ in (4). We know that the vector bundle $f^*TZ_0$ is ample. Using this, together with the fact that any holomorphic vector bundle over $\mathbb{CP}^1$ splits holomorphically into a direct sum of line bundles [6], it follows immediately that $\bigwedge^2 f^*TZ_0$ is ample.

Since $\bigwedge^2 f^*TZ_0$ is ample, and $f^*E_0$ is holomorphically trivializable, the section $f^*\Theta_0$ in (6) vanishes identically. Since $x$ is an arbitrary fixed point, and $x \in \text{image}(f)$, we conclude that the section $\Theta_0$ in (5) vanishes identically.

In other words, the holomorphic connection $D_0$ is flat. As $Z_0$ is rationally connected, a theorem of [4] and [9] says that $Z_0$ is simply connected; see [7, p. 362, Proposition 2.3]. Since $E_0$ admits a flat holomorphic connection, and $Z_0$ is simply connected, we conclude that $E_0$ is holomorphically trivializable.

Since for each point $t \in T$, the restriction of $E$ to the fiber $\phi^{-1}(t)$ is holomorphically trivializable, where $\phi$ is the projection in (3), we conclude that the direct image

$$V := \phi_* E \tag{7}$$

on $T$, of the coherent analytic sheaf $E$, is a holomorphic vector bundle, and furthermore, the vector bundle $\phi^*V$ over $X$ is canonically isomorphic to $E$. In other words, the natural homomorphism $\phi^*\phi_*E \to E$ is an isomorphism.

We will now prove that the holomorphic connection $D$ on $E$ descends to a holomorphic connection on $V$.

Let $s$ be a holomorphic section of $V$ defined over an open subset $U \subset T$. Consider the holomorphic section $\phi^*s$ of the vector bundle $\phi^*V$ defined over the open subset $\phi^{-1}(U) \subset X$. Using the above mentioned canonical isomorphism of $\phi^*V$ with $E$, this section $\phi^*s$ gives a holomorphic section of $E$ over $\phi^{-1}(U)$. This holomorphic section of $E|_{\phi^{-1}(U)}$ will be denoted by $\hat{s}$.

Consider the holomorphic section

$$D(\hat{s}) \in H^0\big(\phi^{-1}(U), E \otimes \Omega_X^1\big), \tag{8}$$

where $D$ is the connection on $E$. Take any point $t \in U$. The inverse image $\phi^{-1}(t)$ will be denoted by $Z_t$. The restriction of $E$ to $Z_t$ will be denoted by $E_t$. Let

$$\tilde{s}_t \in H^0\big(Z_t, E_t \otimes \Omega_{Z_t}^1\big) \tag{9}$$

be the section obtained by restricting to $Z_t$ the section $D(\hat{s})$ in (8) and then using the natural projection of $\Omega_X^1|_{Z_t}$ to $\Omega_{Z_t}^1$.

We showed earlier that $E_t$ is holomorphically trivializable. We also have

$$H^0\big(Z_t, \Omega_{Z_t}^1\big) = 0,$$

as the complex projective manifold $Z_t$ is simply connected. Therefore, the section $\tilde{s}_t$ in (9) vanishes identically. Since $\tilde{s}_t$ vanishes identically for all $t \in U$, we claim that the section

$$D(\hat{s}) \in H^0\big(\phi^{-1}(U), (\phi^*V) \otimes \Omega_X^1\big)$$

in (8) is the pullback, to $\phi^{-1}(U)$, of a holomorphic section of $V \otimes \Omega_T^1$ over $U$. To prove this claim, first note that $D(\hat{s})$ is given by a holomorphic section of

$$E \otimes \big(\Omega_X^1/\Omega_\phi\big) = \phi^*\big(V \otimes \Omega_T^1\big)$$

over $\phi^{-1}(U)$, where $\Omega_\phi$ is the relative cotangent bundle for $\phi$. Next note that the restriction of the holomorphic vector bundle $\phi^*(V \otimes \Omega_T^1)$ to $Z_t$ is trivializable. Hence any holomorphic section of $\phi^*(V \otimes \Omega_T^1)$ over $Z_t$ must be a constant section, proving the claim.

Let $s' \in H^0(U, V \otimes \Omega_T^1)$ be the unique section such that $\phi^*s' = D(\hat{s})$.

Now it is easy to check that the map from the locally defined holomorphic sections of $V$ to the locally defined holomorphic sections of $V \otimes \Omega_T^1$, defined by the above prescription $s \mapsto s'$, satisfies the Leibniz identity; it follows from the fact that $D$ satisfies the Leibniz identity. Therefore, the vector bundle $V$ admits a holomorphic connection. This completes the proof of the proposition. □

**Proof of Theorem 1.1.** If $\psi : Y'' \to Y'$ is an étale covering map between complex projective varieties with $Y''$ smooth and rationally connected, then $Y'$ is also smooth and rationally connected. Hence $Y'$ is simply connected. Therefore, $\psi$ is an isomorphism. Since the fibers of $f$ (in (2)) are rationally connected, this implies that for any point $x \in T$, the inverse image $\gamma^{-1}(\gamma(f^{-1}(x)))$ (see (1)) is a finite union of copies of the rationally connected variety $\gamma(f^{-1}(x))$. We note that $\gamma(f^{-1}(x))$ does not admit any nonconstant holomorphic maps to $T$ (there are no nonconstant holomorphic maps from $\mathbb{CP}^1$ to $T$). Hence there is a finite subset $S_x \subset T$ such that $\gamma^{-1}(\gamma(f^{-1}(x))) = f^{-1}(S_x)$. Let $T'$ be the quotient of $T$ obtained by identifying the points of $S_x$ for each $x$. It is easy to see that the quotient map $\gamma_0 : T \to T'$ is an étale covering map (because $\gamma$ is étale), and also there is a natural holomorphic map $f_0 : M \to T'$, such that $\gamma$ is the pullback of $\gamma_0$ by $f_0$.

Let $E$ be a holomorphic vector bundle over $M$ equipped with a holomorphic connection $D$. Consider the holomorphic vector bundle $\gamma^*E$ on $\widetilde{M}$ equipped with the holomorphic connection $\gamma^*D$. From Proposition 2.1 we know that $\gamma^*E$ descends to $T$. Let $V$ be the holomorphic vector bundle over $T$ such that $f^*V = \gamma^*E$. We also know from Proposition 2.1 that $V$ admits a holomorphic connection $D^V$. Since $f^*V = \gamma^*E$ descends to $M$, and $\widetilde{M} = T \times_{T'} M$, the vector bundle $V$ descends to $T'$. Let $V'$ be the holomorphic vector bundle over $T'$ such that $\gamma_0^*V'$.

The holomorphic connection $D^V$ on $V$ induces a holomorphic connection on $V'$. To see this, first note that for any simply connected neighborhood $U_{x_0}$ of a point $x_0 \in T'$, the inverse image $\gamma_0^{-1}(U_{x_0})$ is a union of copies of $U_{x_0}$ parametrized by the finite set $\gamma_0^{-1}(x_0)$. Furthermore, the restriction of $V$ to each such copy of $U_{x_0}$ in $\gamma_0^{-1}(U_{x_0})$ is identified with $V'|_{U_{x_0}}$. Now for any holomorphic section $s \in \Gamma(U_{x_0}, V'|_{U_{x_0}})$, define

$$D^{V'}(s) := \frac{1}{\#\gamma_0^{-1}(x_0)} \sum_{\nu \in \gamma_0^{-1}(x_0)} D^V(s_\nu) \in \Gamma\big(U_{x_0}, (V'|_{U_{x_0}}) \otimes \Omega^1_{U_{x_0}}\big),$$

where $s_\nu$ is the section of $V$, over the copy of $U_{x_0}$ in $\gamma_0^{-1}(U_{x_0})$ indexed by $\nu$, given by $s$. It is easy to see that the map $V \to V' \otimes \Omega^1_{T'}$ defined by $s \mapsto D^{V'}(s)$ is a holomorphic connection on the holomorphic vector bundle $V'$ on $T'$.

Since $V'$ admits a holomorphic connection, from [3, Theorem 4.1] it follows that $V'$ admits a flat holomorphic connection. The proof of Theorem 1.1 is now completed using the fact that $f_0^*V' = E$.

We note that any Fano manifold is rationally connected [9, p. 766, Theorem 0.1]. This gives the first example in the list following Theorem 1.1. The second example follows from [5, p. 296, Main Theorem]. The last examples follow by considering the natural projection to $\mathrm{Pic}^d(Y)$.

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