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Partial Differential Equations

Asymptotic behavior of solutions for linear parabolic equations with general measure data

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Abstract

In this Note we deal with the asymptotic behavior as t tends to infinity of solutions for linear parabolic equations whose model is

$$\begin{cases} u_t - \Delta u = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0 & \text{in } \Omega, \end{cases}$$

where μ is a general, possibly singular, Radon measure which does not depend on time, and $u_0 \in L^1(\Omega)$. We prove that the duality solution, which exists and is unique, converges to the duality solution (as introduced by Stampacchia (1965)) of the associated elliptic problem. ***To cite this article: F. Petitta, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Comportement asymptotique des solutions des équations paraboliques linéaires avec données de mesures générales. Dans cette Note nous traitons le comportement asymptotique, quand t tend vers l'infini, des solutions des équations paraboliques linéaires dont le modéle est :

$$\begin{cases} u_t - \Delta u = \mu & \text{dans } (0,T)\times\Omega, \\ u(0,x) = u_0 & \text{dans } \Omega, \end{cases}$$

oú μ est une mesure de Radon générale, éventuellement singulière, qui ne dépend pas de t, et où $u_0 \in L^1(\Omega)$. Nous montrons que la solution de dualité, qui existe et est unique, converge vers la solution de dualité (introduite par Stampacchia (1965)) du probléme elliptique associé. ***Pour citer cet article : F. Petitta, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Un grand nombre d'articles a déjà été consacré à l'étude du comportement asymptotique des solutions de problèmes paraboliques dans des contextes différents : pour une revue des résultats classiques voir [4,1,11], et les références incluses. Plus récemment, dans [7] et [6], nous avons considéré le cas des opérateurs monotones non-linéaires, et les problèmes quasi-linéaires avec des termes absorbants ayant une croissance naturelle par rapport au gradient ; en parti-

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culier, dans [7], nous avons considéré des mesures non négatives μ absolument continues par rapport à la p-capacité parabolique (appelées des *mesures diffuses*). Ici, nous analysons le cas des opérateurs linéaires avec des mesures générales, éventuellement singulières, et sans condition de signe sur les données. Soient $\Omega \subseteq \mathbb{R}^N$ un ensemble ouvert borné, $N \geqslant 2$ et $T > 0$; nous dénotons par Q le cylindre $(0,T) \times \Omega$. Nous nous intéressons à l'étude des propriétés principales et au comportement asymptotique par rapport au temps de la solution du problème parabolique linéaire (1) avec $\mu \in \mathcal{M}(Q)$, l'espace des mesures de Radon à variation totale bornée sur Q, $u_0 \in L^1(\Omega)$, et $L(u) = -\mathrm{div}(M(x)\nabla u)$, où M est une matrice avec éléments bornés, mesurables, et satisfaisant la condition d'ellipticité (2).

Notre résultat principal est le suivant :

Théorème 1. *Soit $\mu \in \mathcal{M}(Q)$ indépendant de t. Soit u la solution de dualité du probléme* (1) *avec $u_0 \in L^1(\Omega)$, et soit v la solution de dualité du problème elliptique associé* (3)*. Alors $u(T,x)$ converge vers $v(x)$ dans $L^1(\Omega)$ quand T tend vers à l'infini.*

1. Introduction

A large number of papers has been devoted to the study of asymptotic behavior for solutions of parabolic problems under various assumptions and in different contexts: for a review on classical results see [4,1,11], and references therein. More recently in [7] and [6] the case of nonlinear monotone operators, and quasilinear problems with nonlinear absorbing terms having natural growth, have been considered; in particular, in [7], we dealt with nonnegative measures μ absolutely continuous with respect to the parabolic p-capacity (the so called *soft measures*). Here we analyze the case of linear operators with possibly singular general measures and no sign assumptions on the data.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geqslant 2$, $T > 0$; we denote by Q the cylinder $(0,T) \times \Omega$. We are interested in the study of main properties and in the asymptotic behavior with respect to the time variable t of the solution of the linear parabolic problem

$$\begin{cases} u_t + L(u) = \mu & \text{in } (0,T) \times \Omega, \\ u(0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1}$$

with $\mu \in \mathcal{M}(Q)$ the space of Radon measures with bounded total variation on Q, $u_0 \in L^1(\Omega)$, and

$$L(u) = -\mathrm{div}\big(M(x)\nabla u\big),$$

where M is a matrix with bounded, measurable entries, and satisfying the ellipticity assumption

$$M(x)\xi \cdot \xi \geqslant \alpha|\xi|^2, \tag{2}$$

for any $\xi \in \mathbb{R}^N$, with $\alpha > 0$.

In order to obtain uniqueness, in the elliptic case, the notion of duality solution of Dirichlet problem

$$\begin{cases} -\mathrm{div}\big(M(x)\nabla v\big) = \mu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

was introduced in [12].

Following the idea of [12] we can define a solution of problem (1) in a duality sense as follows:

Definition 1.1. A function $u \in L^1(Q)$ is a *duality solution* of problem (1) if

$$-\int_\Omega u_0 w(0)\,\mathrm{d}x + \int_Q ug\,\mathrm{d}x\,\mathrm{d}t = \int_Q w\,\mathrm{d}\mu, \tag{4}$$

for every $g \in L^\infty(Q)$, where w is the solution of the *backward problem*

$$\begin{cases} -w_t - \mathrm{div}\big(M^*(t,x)\nabla w\big) = g & \text{in } (0,T) \times \Omega, \\ w(T,x) = 0 & \text{in } \Omega, \\ w(t,x) = 0 & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{5}$$

where $M^*(t,x)$ is the transposed matrix of $M(t,x)$.

Notice that all terms in (4) are well defined thanks to standard parabolic regularity results (see [5]). Moreover, it is quite easy to check that any duality solution of problem (1) actually turns out to be a distributional solution of the same problem. Finally recall that any duality solution turns out to coincide with the renormalized solution of the same problem (see [8]); this notion was introduced in [3] for the elliptic case, and then adapted to the parabolic case in [8].

A unique duality solution for problem (1) exists, in fact we have the following:

Theorem 1.2. *Let $\mu \in \mathcal{M}(Q)$ and $u_0 \in L^1(\Omega)$, then there exists a unique duality solution of problem* (1).

The main result of this Note concerns the asymptotic behavior of the duality solution of problem (1), in the case where the measure μ do not depend on time.

First observe that by Theorem 1.2 a unique solution is well defined for all $t > 0$. We recall that by a duality solution of problem (3) we mean a function $v \in L^1(\Omega)$ such that

$$\int_{\Omega} v\, g \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega} z \,\mathrm{d}\mu, \tag{6}$$

for every $g \in L^\infty(\Omega)$, where z is the variational solution of the dual problem

$$\begin{cases} -\mathrm{div}\big(M^*(x)\nabla z\big) = g & \text{in } \Omega, \\ z(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

As we will see later, a duality solution of problem (1) turns out to be continuous with values in $L^1(\Omega)$. Let us state our main result:

Theorem 1.3. *Let $\mu \in \mathcal{M}(Q)$ be independent on the variable t. Let $u(t,x)$ be the duality solution of problem* (1) *with $u_0 \in L^1(\Omega)$, and let $v(x)$ be the duality solution of the corresponding elliptic problem* (3). *Then*

$$\lim_{T\to+\infty} u(T,x) = v(x),$$

in $L^1(\Omega)$.

2. Existence and uniqueness of the duality solution

Sketch of the proof of Theorem 1.2. We first check the result in the case $\mu \in L^1(Q)$ and u_0 smooth; let us fix $r, q \in \mathbb{R}$ such that $r,\ q > 1$, $\frac{N}{q} + \frac{2}{r} < 2$, and let us consider $g \in L^r(0,T;L^q(\Omega)) \cap L^\infty(Q)$. Let w be the solution of problem (5); standard parabolic regularity results (see again [5]) imply that w is continuous on Q and $\|w\|_{L^\infty(Q)} \leqslant C\|g\|_{L^r(0,T;L^q(\Omega))}$; therefore, the linear functional $\Lambda : L^r(0,T;L^q(\Omega)) \mapsto \mathbb{R}$, defined by $\Lambda(g) = \int_Q w\,\mathrm{d}\mu + \int_\Omega u_0 w(0)$, is well defined and continuous, since

$$\big|\Lambda(g)\big| \leqslant \big(\|\mu\|_{\mathcal{M}(Q)} + \|u_0\|_{L^\infty(\Omega)}\big)\|w\|_{L^\infty(Q)} \leqslant C\|g\|_{L^r(0,T;L^q(\Omega))}.$$

So, by *Riesz's representation theorem* there exists a unique $u \in L^{r'}(0,T;L^{q'}(\Omega))$ such that

$$\Lambda(g) = \int_Q u\, g \,\mathrm{d}x\,\mathrm{d}t,$$

for any $g \in L^r(0,T;L^q(\Omega))$. So we have that, if $\mu \in L^1(Q)$ and u_0 is smooth, then there exists a (unique by construction) duality solution of problem (1).

A standard approximation argument (see for instance Theorem 1.2 in [2]) shows that a unique solution also exists for problem (1) if $\mu \in \mathcal{M}(Q)$ and $u_0 \in L^1(\Omega)$. □

3. Asymptotic behavior

In this section we will prove Theorem 1.3. From now on we will denote by $T_k(s)$ the function $\max(-k, \{\min(k,s))$ and $\Theta_k(s)$ will indicate its primitive function, that is $\Theta_k(s) = \int_0^s T_k(\sigma)\,\mathrm{d}\sigma$.

Let us prove the following preliminary result:

Proposition 3.1. *Let $\mu \in \mathcal{M}(Q)$ be independent on time and let v be the duality solution of the elliptic problem* (3)*. Then v is the unique solution of the parabolic problem* (1)*, with $u_0 = v$, in the duality sense introduced in Definition* 1.1*, for any fixed $T > 0$.*

Proof. We have to check that v is a solution of problem (1); to do that let us choose $T_k(v)$ as test function in (5). We obtain

$$-\int_0^T \langle w_t, T_k(v)\rangle \,\mathrm{d}t + \int_Q M^*(x)\nabla w \cdot \nabla T_k(v)\,\mathrm{d}x\,\mathrm{d}t = \int_Q T_k(v)\; g\,\mathrm{d}x\,\mathrm{d}t.$$

Now, integrating by parts we have $-\int_0^T \langle w_t, T_k(v)\rangle\,\mathrm{d}t = \int_\Omega w(0)v(x) + \omega(k)$, where $\omega(k)$ denotes a nonnegative quantity which vanishes as k diverges, while

$$\int_Q T_k(v)\; g\,\mathrm{d}x\,\mathrm{d}t = \int_Q v\, g\,\mathrm{d}x\,\mathrm{d}t + \omega(k).$$

Finally, using Theorem 2.33 and Theorem 10.1 of [3], we have

$$\int_Q M^*(x)\nabla w \cdot \nabla T_k(v)\,\mathrm{d}x\,\mathrm{d}t = \int_Q M(x)\nabla T_k(v)\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega w\,\mathrm{d}\lambda_k(x)\,\mathrm{d}t,$$

where the λ_k are measures in $\mathcal{M}(\Omega)$ which converge to μ in the narrow topology of measures; thus, recalling that w is bounded continuous, and using the dominated convergence theorem, we have

$$\int_Q M^*(x)\nabla w \cdot \nabla T_k(v)\,\mathrm{d}x\,\mathrm{d}t = \int_Q w\,\mathrm{d}\mu + \omega(k).$$

Gathering together all these facts, we have that v is a duality solution of (1) having itself as initial datum.

Proposition 3.1 allows us to deduce that the duality solution of problem (1) u belongs to $C(0,T;L^1(\Omega))$ for any fixed $T > 0$; indeed, $z = u - v$ uniquely solves problem (1) with $u_0 - v$ as initial datum and $\mu = 0$ in the duality sense, and so $z \in C(0,T;L^1(\Omega))$. This is due to a result of [9], since z turns out to be an entropy solution in the sense of the definition given in [10].

Therefore, as we said before, for fixed μ and $g \in L^\infty(Q)$ one can uniquely determine u and w, solution of the above problems, defined for any time $T > 0$. □

Moreover, let us give the following definition:

Definition 3.2. A function $u \in L^1(Q)$ is a *duality supersolution* of problem (1) if

$$\int_Q u\, g\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_Q w\,\mathrm{d}\mu + \int_\Omega u_0 w(0)\,\mathrm{d}x,$$

for any bounded $g \geqslant 0$, and w solution of (5), while u is a *duality subsolution* if $-u$ is a duality supersolution.

By linearity we easily deduce

Lemma 3.3. *Let $\bar{u}$ and $\underline{u}$ be respectively a duality supersolution and a duality subsolution for problem* (1)*. Then $\underline{u} \leqslant \bar{u}$.*

Observe that, if the functions in Lemma 3.3 are continuous with values in $L^1(\Omega)$, then we actually have that $\underline{u}(t,x) \leqslant \bar{u}(t,x)$ for every fixed t, a.e. on Ω.

Proof of Theorem 1.3. We split the proof into a few steps.

Step 1. Let us first suppose $u_0 = 0$ and $\mu \geqslant 0$. If we consider a parameter $s > 0$ we have that both $u(t, x)$ and $u_s(t, x) \equiv u(t + s, x)$ are duality solutions of problem (1) with, respectively, 0 and $u(s, x) \geqslant 0$ as initial datum; so, from Lemma 3.3 we deduce that $u(t + s, x) \geqslant u(t, x)$ for $t, s > 0$. Therefore u is a monotone nondecreasing function in t and so it converges to a function $\tilde{v}(x)$ almost everywhere and in $L^1(\Omega)$ since, thanks to Proposition 3.1 and Lemma 3.3, $u(t, x) \leqslant v(x)$.

Now, recalling that u is obtained as limit of regular solutions with smooth data μ_ε, we can define $u_\varepsilon^n(t, x)$ as the solution of

$$\begin{cases} \left(u_\varepsilon^n\right)_t - \operatorname{div}\left(M(x)\nabla u_\varepsilon^n\right) = \mu_\varepsilon & \text{in } (0,1) \times \Omega, \\ u_\varepsilon^n(0, x) = u_\varepsilon(n, x) & \text{in } \Omega, \\ u_\varepsilon^n = 0 & \text{on } (0,1) \times \partial\Omega. \end{cases} \tag{8}$$

On the other hand, if $g \geqslant 0$, we define $w^n(t, x)$ as

$$\begin{cases} -w_t^n - \operatorname{div}\left(M^*(x)\nabla w^n\right) = g & \text{in } (0,1) \times \Omega, \\ w^n(1, x) = w(n+1, x) & \text{in } \Omega, \\ w^n = 0 & \text{on } (0,1) \times \partial\Omega. \end{cases} \tag{9}$$

Recall that, through the change of variable $s = T - t$, w solves a similar linear parabolic problem, so that if $g \geqslant 0$, by classical comparison results one has that $w(t, x)$ is decreasing in time. Moreover, by comparison principle, we have that w^n is increasing with respect to n and, again by comparison Lemma 3.3, we have that, for fixed $t \in (0, 1)$

$$w^n(1, x) \leqslant w^n(t, x) = w(n + t, x) \leqslant w(n, x) = w^{n-1}(1, x),$$

and so its limit $\tilde{w}$ does not depend on time and is the solution of elliptic dual problem (7). An analogous argument shows that also the limit of u^n does not depend on time. Thus, using u_ε^n in (9) and w^n in (8), integrating by parts, subtracting, and passing to the limit over ε, we obtain

$$\int_0^1\!\!\int_\Omega u^n g - \int_0^1\!\!\int_\Omega w^n \,\mathrm{d}\mu + \int_\Omega u^n(0) w^n(0)\,\mathrm{d}x - \int_\Omega u^n(1) w^n(1)\,\mathrm{d}x = 0.$$

Hence, we can pass to the limit on n using monotone convergence theorem obtaining

$$\int_\Omega \tilde{v} g - \int_\Omega \tilde{w}\,\mathrm{d}\mu\,\mathrm{d}x = 0, \tag{10}$$

and so $v = \tilde{v}$.

If g has no sign we can reason separately with g^+ and g^- obtaining (10) and then using the linearity of (4) to conclude.

If v is the duality solution of problem (3), we proved in Proposition 3.1 that v is also the duality solution of the initial boundary value problem (1) with v itself as initial datum. Therefore, by comparison Lemma 3.3, if $0 \leqslant u_0 \leqslant v$, we have that the solution $u(t, x)$ of (1) converges to v in $L^1(\Omega)$ as t tends to infinity; in fact, we proved it for the duality solution with homogeneous initial datum, while v is a nonnegative duality solution with itself as initial datum.

Step 2. Now, let us take $u_\lambda(t, x)$ the solution of problem (1) with $u_0 = \lambda v$ as initial datum for some $\lambda > 1$ and again $\mu \geqslant 0$. Hence, since λv does not depend on time, we have that it is a duality supersolution of the parabolic problem (1), and, observing that v is a subsolution of the same problem, we can apply again the comparison lemma finding that $v(x) \leqslant u_\lambda(t, x) \leqslant \lambda v(x)$ a.e. in Ω, for all positive t.

Moreover, thanks to the fact that the datum μ does not depend on time, we can apply the comparison result also between $u_\lambda(t + s, x)$ solution with $u_0 = u_\lambda(s, x)$, with s a positive parameter, and $u_\lambda(t, x)$, the solution with $u_0 = \lambda v$ as initial datum; so we obtain $u_\lambda(t + s, x) \leqslant u_\lambda(t, x)$ for all $t, s > 0$, a.e. in Ω. So, by virtue of this monotonicity result we have that there exists a function $\bar{v} \geqslant v$ such that $u_\lambda(t, x)$ converges to $\bar{v}$ a.e. in Ω as t tends to infinity. Clearly $\bar{v}$ does not depend on t and we can develop the same argument used before to prove that we can pass to the limit in the approximating duality formulation, and so, by uniqueness, we can obtain that $\bar{v} = v$. So, we have proved that the result holds for the solution starting from $u_0 = \lambda v$ as initial datum, with $\lambda > 1$ and $\mu \geqslant 0$. Since we proved before that the result holds true also for the solution starting from $u_0 = 0$, then, again applying a comparison argument, we can conclude in the same way that the convergence to v holds true for solutions starting from u_0 such that $0 \leqslant u_0 \leqslant \lambda v$ as initial datum, for fixed $\lambda > 1$.

Step 3. Now, let $u_0 \in L^1(\Omega)$ a nonnegative function and $\mu \geqslant 0$, and recall that, thanks to suitable Harnack inequality (see [13]), if $\mu \neq 0$, then $v > 0$ (which implies λv tends to $+\infty$ on Ω as λ diverges). Without loss of generality we can suppose $\mu \neq 0$ (the case $\mu \equiv 0$ is the easier one and it can be proved as in [7]); let us define the monotone nondecreasing (with respect to λ) family of functions $u_{0,\lambda} = \min(u_0, \lambda v)$.

As we have shown above, for every fixed $\lambda > 1$, $u_\lambda(t,x)$, the duality solution of problem (1) with $u_{0,\lambda}$ as initial datum, converges to v a.e. in Ω, as t tends to infinity. Moreover, using again standard compactness arguments, we also have that $T_k(u_\lambda(t,x))$ converges to $T_k(v)$ weakly in $H_0^1(\Omega)$ as t diverges, for every fixed $k > 0$.

So, thanks to *Lebesgue theorem*, we can easily check that $u_{0,\lambda}$ converges to u_0 in $L^1(\Omega)$ as λ tends to infinity. Therefore, using a stability result for renormalized solutions of the linear problem (1) (see [8]) we obtain that $T_k(u_\lambda(t,x))$ converges to $T_k(u(t,x))$ strongly in $L^2(0,T;H_0^1(\Omega))$ as λ tends to infinity.

On the other hand, since $z_\lambda = u - u_\lambda$ solves the problem (1) with $u_0 - u_{0,\lambda}$ as initial datum, then z_λ turns out to be an entropy solution of the same problem and so we have (see [10])

$$\int_\Omega \Theta_k(u-u_\lambda)(t)\,\mathrm{d}x \leqslant \int_\Omega \Theta_k(u_0-u_{0,\lambda})\,\mathrm{d}x,$$

for every $k,\ t > 0$. Dividing the above inequality by k, and passing to the limit as k tends to 0 we obtain

$$\big\|u(t,x)-u_\lambda(t,x)\big\|_{L^1(\Omega)} \leqslant \big\|u_0(x)-u_{0,\lambda}(x)\big\|_{L^1(\Omega)}, \tag{11}$$

for every $t > 0$. Hence, we have $\|u(t,x)-v(x)\|_{L^1(\Omega)} \leqslant \|u(t,x)-u_\lambda(t,x)\|_{L^1(\Omega)} + \|u_\lambda(t,x)-v(x)\|_{L^1(\Omega)}$; then, thanks to the fact that the estimate in (11) is uniform in t, for every fixed ϵ, we can choose $\bar\lambda$ large enough such that $\|u(t,x)-u_{\bar\lambda}(t,x)\|_{L^1(\Omega)} \leqslant \frac{\epsilon}{2}$, for every $t > 0$; on the other hand, thanks to the result proved above, there exists $\bar t$ such that $\|u_{\bar\lambda}(t,x)-v(x)\|_{L^1(\Omega)} \leqslant \frac{\epsilon}{2}$, for every $t > \bar t$, and this concludes the proof of the result in the case of nonnegative data μ and $u_0 \in L^1(\Omega)$.

Step 4. Let $\mu \in \mathcal{M}(Q)$ be independent on t and $u_0 \in L^1(\Omega)$ with no sign assumptions. We consider again the function $z(t,x) = u(t,x) - v(x)$; thanks to Proposition 3.1 it turns out to solve problem (1) with $u_0 - v$ as initial data and $\mu = 0$, and so, if either $u_0 \leqslant v$ or $u_0 \geqslant v$ then the result is true since $z(t,x)$ tends to zero in $L^1(\Omega)$ as t diverges thanks to what we proved above. Now, if $u^\oplus$ and $u^\ominus$ solve problem (1) with, respectively, $\max(u_0, v)$ and $\min(u_0, v)$ as initial data, then, by comparison, we have $u^\ominus(t,x) \leqslant u(t,x) \leqslant u^\oplus(t,x)$ for any t, a.e. in Ω, and this concludes the proof since the result holds true for both $u^\oplus$ and $u^\ominus$. $\square$

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