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ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 344 (2007) 559–563
COMPTES RENDUS MATHEMATIQUE
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Partial Differential Equations

# Fully nonlinear degenerate operators associated with the Heisenberg group: barrier functions and qualitative properties

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Received 13 September 2006; accepted 6 March 2007

Available online 26 April 2007

Presented by Pierre-Louis Lions

**Abstract**

The aim of this Note is to state some continuity property, up to the boundary, for viscosity solutions to fully nonlinear Dirichlet problems on the Heisenberg group and to obtain qualitative properties of the Hadamard, Liouville and Harnack type. For this purpose, a key ingredient is the construction of some barrier functions for the Pucci–Heisenberg operators. ***To cite this article: A. Cutrì, N. Tchou, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***


**Résumé**

**Opérateurs completement non-linéaires associés au groupe de Heisenberg : fonctions barrière et propriétés qualitatives.** Nous déterminons des propriétés de continuité jusqu'au bord des solutions de viscosité pour le problème de Dirichlet pour des opérateurs completement non-linéaires associés au groupe de Heisenberg. Pour ces opérateurs nous provons aussi des propriétés de Hadamard, Liouville et Harnack. L'outil essentiel est la constructions de fonctions barrière pour les opérateurs de Pucci–Heisenberg. ***Pour citer cet article : A. Cutrì, N. Tchou, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***


## Version française abrégée

Dans cette Note nous étudions les problèmes complètement non linéaires :

$$F\big(\xi, D^2_{H^n}u\big) + H(\xi, \nabla_{H^n}u) = 0, \tag{1}$$

où $F$ est un opérateur non-linéaire uniformément elliptique avec constantes d'ellipticité $0 < \lambda \leqslant \Lambda$ et $H$ vérifie les hypothèses (4). Soient $i, j = 1, \dots, n$, nous avons appellé

$$\nabla_{H^n}\phi \doteq (X_1\phi, \dots, X_{2n}\phi), \qquad \big\{D^2_{H^n}\phi\big\}_{ij} \doteq (X_iX_j\phi)_{\mathrm{sym}}, \tag{2}$$

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1631-073X/$ – see front matter 
doi:10.1016/j.crma.2007.03.003

et les $2n$ champs des vecteurs de Heisenberg sont définis par ($\xi \in \mathbb{R}^{2n+1}$) :

$$X_i \doteq \frac{\partial}{\partial \xi_i} + 2\xi_{i+n}\frac{\partial}{\partial \xi_{2n+1}}, \qquad X_{i+n} \doteq \frac{\partial}{\partial \xi_{i+n}} - 2\xi_i \frac{\partial}{\partial \xi_{2n+1}}. \tag{3}$$

Nous soulignons que l'Éq. (1) est dégénérée à cause de la dégénéressance de $\nabla_{H^n}$ et de $D^2_{H^n}$.

D'abord nous montrons que si $H=0$ et $F(\xi,0)=0$ le problème de Dirichlet associé à l'Éq. (1) dans un domaine $\Omega$ qui vérifie une condition uniforme de sphère extérieure (10) est bien posé si la donnée au bord est continue, c'est-à-dire qu'il existe une et une unique solution de viscosité [5], continue jusqu'au bord. L'outil essentiel est la constructions de fonctions barrière pour les opérateurs de Pucci-Heisenberg, plus précisément nous explicitons l'action de l'opérateur de Pucci–Heisenberg sur les fonctions radiales par rapport à $\rho$ (voir (9)). La construction des barrières pour les opérateurs de Pucci–Hesenberg nous permet de démontrer des résultats de type « Hadamard » et d'en déduire un théorème de Liouville en petite dimension et une inégalité de Harnack faible pour les sursolutions de viscosité de $F \geqslant 0$.

Ensuite nous généralisons les résultats d'existence et unicité au cas $H \neq 0$ pour des domaines de type « couronnes instrinsèques » et pour des $H$ qui vérifient l'hypothèse (4). Les preuves des résultats annoncés dans cette Note sont contenues dans [7].

## 1. Introduction

We will consider the viscosity solution $u$ of the fully nonlinear problem (1), obtained by the composition of a fully nonlinear uniformly elliptic operator $F$ with ellipticity constants $0<\lambda \leqslant \Lambda$, such that $F(\xi,0)=0$ with the degenerate Heisenberg Hessian $D^2_{H^n}$, and whose the first order term satisfies:

$$\left|H(\xi,\nabla_{H^n}u)\right| \leqslant M\left|\nabla_{H^n}\rho(\xi)\right|^2 + K\left|\nabla_{H^n}\rho(\xi)\right|\left|\nabla_{H^n}u\right|. \tag{4}$$

The hypotheses on $F$ and $H$ imply that the following inequality holds true in the viscosity sense, see e.g. [3]:

$$\begin{aligned}&\widetilde{\mathcal{M}}^-_{\lambda,\Lambda}\big(D^2u\big) - K\left|\nabla_{H^n}\rho(\xi)\right|\left|\nabla_{H^n}u\right| - M\left|\nabla_{H^n}\rho(\xi)\right|^2\\ &\quad\leqslant F\big(\xi, D^2_{H^n}u\big) + H(\xi,\nabla_{H^n}u)\\ &\quad\leqslant \widetilde{\mathcal{M}}^+_{\lambda,\Lambda}\big(D^2u\big) + K\left|\nabla_{H^n}\rho(\xi)\right||\nabla_{H^n}u| + M\left|\nabla_{H^n}\rho(\xi)\right|^2,\end{aligned} \tag{5}$$

where the *Pucci–Heisenberg operators* $\widetilde{\mathcal{M}}^{\mp}$ are defined by the composition of the Pucci operators $\mathcal{P}^{\mp}$ (see [10]) with the Heisenberg Hessian $D^2_{H^n}$, that is:

$$\widetilde{\mathcal{M}}^{\mp}_{\lambda,\Lambda}\big(D^2u\big) \doteq \mathcal{P}^{\mp}(D^2_{H^n}u) = -\Lambda \sum_{e_i>0\ (e_i<0)} e_i - \lambda \sum_{e_i<0\ (e_i>0)} e_i, \quad \text{where } e_i \text{ denote the eigenvalues of } D^2_{H^n}u.$$

For $\lambda=\Lambda$ all the operators $F$ coincide with the Heisenberg Laplacian $-\Delta_{H^n}\cdot \doteq -\operatorname{tr}(D^2_{H^n}\cdot)$. In the linear case, the operator $F=L_A=-\operatorname{tr}(AD^2_{H^n}\cdot)$, where $A=P^{\mathrm T}P$ is a symmetric $2n\times 2n$ matrix with eigenvalues in between $[\lambda,\Lambda]$, arises in the stochastic theory as the infinitesimal generator of the degenerate diffusion process governed by the stochastic differential equation

$$\begin{cases} \mathrm{d}X = \sigma^{\mathrm T}P^{\mathrm T}\,\mathrm{d}W,\\ X(0)=\xi \end{cases} \tag{6}$$

where $\sigma$ is the matrix whose rows are the vector fields $X_i$ and $W$ denotes the standard $2n$-dimensional Brownian motion. The operator $\widetilde{\mathcal{M}}^+$ being the supremum of $L_A$, is involved in controlled diffusion processes of the same type as (6).

We shall prove some continuity property, up to the boundary, for viscosity solutions to fully nonlinear Dirichlet problems on the Heisenberg group (see Section 4) and we shall obtain qualitative properties of Hadamard, Liouville type and Harnack inequality (see Section 5). At this purpose, a key ingredient is the construction of some barrier functions for $\widetilde{\mathcal{M}}^{\mp}$. The qualitative properties generalize the ones obtained in [6,4] for the uniformly elliptic case.

The existence results generalize the ones obtained by Bardi and Mannucci in [2] which are only given for constant boundary data and for $F=\widetilde{\mathcal{M}}^+_{\lambda,\Lambda}$ and where $H$ does not degenerate at the characteristic points of the boundary, which is assumed to be either the Heisenberg ball (see (9)) or the Euclidean one. On the other hand, their existence results are more general than ours since they are related to more general, fully nonlinear, degenerate elliptic equations.

The proofs of the results below are contained in [7].

## 2. Settings

The vector fields (3) generate the whole Lie algebra of left-invariant vectorfields on the homogeneous Heisenberg group $H^n = (\mathbb{R}^{2n+1}, \circ)$, defined by:

$$\eta \circ \xi = \left(\xi_1 + \eta_1, \ldots, \xi_{2n} + \eta_{2n}, \xi_{2n+1} + \eta_{2n+1} + 2\sum_{i=1}^{n}(\xi_i \eta_{i+n} - \xi_{i+n}\eta_i)\right), \tag{7}$$

$$\delta_\lambda(\xi) = \left(\lambda\xi_1, \ldots, \lambda\xi_{2n}, \lambda^2\xi_{2n+1}\right) \quad (\lambda > 0). \tag{8}$$

We call

$$\rho(\xi) = \left(\left(\sum_{i=1}^{2n}\xi_i^2\right)^2 + \xi_{2n+1}^2\right)^{1/4} \quad \text{and} \quad B_R^H(\xi) = \left\{\eta \in \mathbb{R}^{2n+1}\colon\ \rho\left(\eta^{-1}\circ\xi\right) < R\right\} \tag{9}$$

respectively the *homogeneous norm* introduced by Korànyi and Vági (see [9]) and the Heisenberg ball, whose volume scales as $R^Q$ ($Q = 2n + 2$ being the homogeneous dimension of $H^n$).

**Definition 1.** We say that $\Omega$ satisfies the uniform exterior Heisenberg ball condition if

$$\exists r_0 > 0 \text{ such that } \forall \xi_0 \in \partial\Omega,\ \exists \eta_0 \in \Omega^C \text{ such that } \overline{B_{r_0}^H(\eta_0)} \cap \overline{\Omega} = \xi_0. \tag{10}$$

## 3. Barrier functions

In this section we will construct some barrier functions for the operators $\widetilde{\mathcal{M}}^{\mp}_{\lambda,\Lambda}$. For this purpose, we will make use of the expression of $\widetilde{\mathcal{M}}^{\mp}_{\lambda,\Lambda}$ on functions $f$ depending on the homogeneous norm defined in (9). Let us observe that, for $\rho(\xi) > 0$,

$$D^2_{H^n} f(\rho) = f'(\rho) D^2_{H^n}\rho + f''(\rho)\nabla_{H^n}\rho \otimes \nabla_{H^n}\rho; \tag{11}$$

moreover, the following technical lemma, provides the expression of the intrinsic Hessian matrix $D^2_{H^n}\rho$:

**Lemma 1.** *Let $\rho(\xi)$ be defined in* (9)*, then*

$$D^2_{H^n}\rho = -\frac{3}{\rho}\nabla_{H^n}\rho \otimes \nabla_{H^n}\rho + \frac{1}{\rho}|\nabla_{H^n}\rho|^2 I_{2n} + \frac{2}{\rho^3}\begin{pmatrix} B & C \\ -C & B\end{pmatrix},$$

*where the entries of the $n \times n$ matrices $B$ and $C$ are respectively $b_{ij} = \xi_i\xi_j + \xi_{n+i}\xi_{n+j}$ and $c_{ij} = \xi_i\xi_{n+j} - \xi_j\xi_{n+i}$.*

**Lemma 2.** *The eigenvalues of $D^2_{H^n} f(\rho)$ are $\lambda_1$ (simple), $\lambda_2$ (simple) and $\lambda_3$ (with multiplicity $2n-2$), where:*

$$\lambda_1 = |\nabla_{H^n}\rho|^2 f''(\rho); \quad \lambda_2 = 3|\nabla_{H^n}\rho|^2\frac{f'(\rho)}{\rho}; \quad \lambda_3 = |\nabla_{H^n}\rho|^2\frac{f'(\rho)}{\rho} \tag{12}$$

*at the points where $\nabla_{H^n}\rho \neq 0$; otherwise all the eigenvalues of $D^2_{H^n} f(\rho)$ vanish identically.*

Using Lemma 2, we can construct radial functions $\Phi(\xi) = \varphi(\rho(\xi))$ which are classical solutions of the equation

$$\widetilde{\mathcal{M}}^{+}_{\lambda,\Lambda}\left(D^2\Phi\right) = 0 \quad \text{in } \mathbb{R}^{2n+1} \setminus \{0\}, \tag{13}$$

and are either convex and decreasing or concave and increasing. In the first case the solutions are given by

$$\varphi(\rho) = C_1\rho^{2-\beta} + C_2 \quad \text{where } \beta = \frac{\Lambda}{\lambda}(Q-1) + 1 \tag{14}$$

whereas in the second case the solutions are similar but depend on the exponent $\alpha = \frac{\lambda}{\Lambda}(Q-1)+1$ instead of $\beta$, and are of logarithmic type if $\alpha = 2$. Observe that the analogues solutions for $\widetilde{\mathcal{M}}^-_{\lambda,\Lambda}$ are simply obtained by changing the sign. Let us point out, moreover, that in the particular case in which $\lambda = \Lambda$, Eq. (13) reduces to the Heisenberg Laplace equation; in this case, it results $\alpha = \beta = Q$ and $\Phi$ coincides with the classical fundamental solution (see [8]).

## 4. Existence results

By using the just constructed barriers for $\widetilde{\mathcal{M}}^{\mp}$, we can prove the following theorem, which, combined with the Perron–Ishii method, produces the existence of a unique continuous viscosity solution of (15) in $\overline{\Omega}$:

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^{2n+1}$ be a bounded domain satisfying* (10) *and let $\psi$ be a bounded continuous function on $\partial\Omega$. Then there exist a lower barrier $\underline{u}$ and an upper barrier $\bar{u}$ for*

$$\begin{cases} F(\xi, D^2_{H^n}u) = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega \end{cases} \tag{15}$$

*such that $\underline{u}(\xi) = \bar{u}(\xi) = \psi(\xi)$ on $\partial\Omega$.*

The same result holds true for operators depending on first order term $H$ satisfying (4). As a model case, let us consider an 'intrinsic annulus' with radii $0 \leqslant R_1 < R_2 < \infty$. By using the same technique, we can prove that there exists a unique continuous up to the boundary viscosity solution of

$$\begin{cases} F(\xi, D^2_{H^n}u) + H(\xi, \nabla_{H^n}u) = 0 & \text{in } B^H_{R_2}(0) \setminus B^H_{R_1}(0), \\ u = 0 & \text{on } \partial(B^H_{R_2}(0) \setminus B^H_{R_1}(0)), \end{cases} \tag{16}$$

as a consequence of the following result:

**Theorem 2.** *Let $F$ and $H$ be as before. Then, there exist a lower barrier $\underline{u}$ and an upper barrier $\bar{u}$ for the problem* (16) *which satisfy $\underline{u}(\xi) = \bar{u}(\xi) = 0$ on $\partial(B^H_{R_2}(0) \setminus B^H_{R_1}(0))$.*

The proof is achieved by using (5) and the just constructed barriers for $\widetilde{\mathcal{M}}^{\mp}$. More general domains could be treated, and this is extensively studied in [7], we point out that the regularity conditions on the boundary are connected with the behaviour of $H$ (as in (4)).

## 5. Hadamard and Liouville properties

**Theorem 3** *(Nonlinear Degenerate Hadamard Theorem). Let $\Omega$ be a domain of $\mathbb{R}^{2n+1}$ containing the closed intrinsic ball $\overline{B}^H_R(0)$ centered at the origin and with radius $R > 0$, and let $u \in LSC(\Omega)$ be a viscosity supersolution of*

$$F\big(\xi, D^2_{H^n}u\big) \geqslant 0 \quad \text{in } \Omega,$$

*then, for all $r < R$, $m(r) = \min_{\rho(\xi)\leqslant r} u(\xi)$ is a concave function of $r^{2-\beta}$, with $\beta$ defined in* (14). *More precisely, for every fixed $r_1 < R$ and for all $r_1 \leqslant r \leqslant R$, it results*

$$m(r) \geqslant \frac{m(r_1)(R^{2-\beta} - r^{2-\beta}) + m(R)(r^{2-\beta} - r_1^{2-\beta})}{(R^{2-\beta} - r_1^{2-\beta})}.$$

For supersolutions of $\widetilde{\mathcal{M}}^-_{\lambda,\Lambda}$, which is the lower operator in the class we consider, one can prove a better result depending on the exponent $\alpha$ which could be lower than two and which combined with the Strong Minimum Principle (see [1]) allows one to deduce the following *Liouville* type result:

**Theorem 4.** *Let $u \in LSC(\mathbb{R}^{2n+1})$ be a bounded from below viscosity solution of*

$$\widetilde{\mathcal{M}}^-_{\lambda,\Lambda}\big(D^2u\big) \geqslant 0 \quad \text{in } \mathbb{R}^{2n+1} \tag{17}$$

*if $Q \leqslant \frac{\Lambda}{\lambda} + 1$, then $u$ is constant.*

**Remark 1.** The previous result cannot be extended to every operator $F$. Indeed the function

$$u(\xi) = \min\{\rho^{2-\beta}(\xi), R^{2-\beta}\},$$

with $R > 0$ arbitrarily fixed, is a strictly positive nonconstant viscosity supersolution of

$$\widetilde{\mathcal{M}}^+_{\lambda,\Lambda}(D^2u) \geqslant 0 \quad \text{in } \mathbb{R}^{2n+1}. \tag{18}$$

Hence, Theorem 4 does not hold for $\widetilde{\mathcal{M}}^+_{\lambda,\Lambda}$.

Another consequence of the Hadamard theorem and of the Strong Minimum principle (see [1]) is the following version of the *weak Harnack inequality* for radial supersolutions:

**Theorem 5.** *Let $u \in LSC(B^H_{2R}(0))$ be a radial viscosity solution of*

$$u \geqslant 0, \quad F(\xi, D^2_{H^n}u) \geqslant 0 \quad \text{in } B^H_{2R}(0),$$

*then u satisfies the following weak Harnack inequality*:

$$\operatorname{meas}(B^H_{R/2}(0) \cap \{u(\rho) > t\}) \leqslant \frac{CR^Q}{t^{Q/(\beta-2)}}(u(R))^{Q/(\beta-2)} \quad \forall t > 0. \tag{19}$$

*with $\{u(\rho) > t\} = \{\xi \in B^H_{2R}(0)\colon u(\rho(\xi)) > t\}$ and* $\operatorname{meas}(E)$ *equals to the Lebesgue measure of the measurable set $E \subset \mathbb{R}^{2n+1}$.*

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