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Numerical Analysis

An asymptotic preserving relaxation scheme for a moment model of radiative transfer

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Abstract

A new relaxation scheme is exhibited to approximate the weak solutions of the M_1 model to simulate radiative transfer. This numerical method uses better wavespeed approximations than the current schemes. In addition, it is proved to satisfy all the required stability properties and the asymptotic preserving property. ***To cite this article: C. Berthon et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Schéma de relaxation asymptotic-preserving pour le transfert radiatif. Les solutions faibles du modèle M_1 pour le transfert radiatif sont approchées par un nouveau schéma de relaxation. Cette méthode propose une meilleure approximations des vitesses d'onde que celle habituellement considérées par les schémas actuels. De plus, nous établissons les propriétés de stabilité nécessaires ainsi que la satisfaction de la limite diffusive. ***Pour citer cet article : C. Berthon et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Dans le présent travail, nous nous intéressons à l'approximation numérique des solutions d'un système hyperbolique avec terme source. Ce système, définit par (1), gouverne le modèle M_1 introduit dans [7,10]. Une des particularités de ce système réside dans son comportement asymptotique lorsque le nombre de Knudsen tend vers zéro. En effet, pour ce régime asymptotique, (1) devient une équation de diffusion de coefficient $1/(3\sigma)$. Ce comportement asymptotique soulève de nombreuses difficultés numériques et plusieurs travaux [6,8] ont été réalisés pour les lever. Notons que les schémas numériques doivent satisfaire simultanément cette propriété dite *asymptotic preserving* et preserver le domaine invariant associé au système (positivité de l'énergie et limitation du flux).

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Dans un premier temps et après [3], le terme source présent dans (1) est pris en compte dans l'opérateur d'ordre un. A cet effet, nous introduisons $\Sigma(x)$ définit par $\sigma(x) = \partial_x \Sigma(x)$. Notons que la réécriture du système, sous la forme (3), trouve une forme conservative. Nous introduisons alors un modèle de relaxation (5) pour approcher les solutions de (3) et donc de (1). Sous la condition $\mu = 0$, nous montrons que (5) est hyperbolique et que tous ces champs sont linéairement dégénérés. La solution du problème de Riemann associée à (5) est donnée.

Une particularité de ce modèle, et à la différence des travaux menés dans [6], réside dans l'introduction de vitesse d'onde b_L et b_R indépentantes. Un tel choix permet de réduire la diffusion numérique. Des conditions portant sur b_L et b_R sont apportées afin d'assurer à la solution du problème de Riemann de satisfaire en tout point la positivité de l'énergie ansi que la limitation du flux.

Ces résultats sont alors utilisés pour proposer une méthode numérique robuste. En effet, nous établissons la preservation du domaine invariant. De plus, après avoir introduit le nombre de Knudsen, nous montrons que le schéma numérique possède le comportement asymptotique attendu. En particuliers, la forme discrète de l'équation de diffusion est obtenue avec le bon coefficient de diffusion.

Il est important de noter dans la méthode proposée, que les estimations numériques des vitesses des ondes (b_L et b_R) sont réalisées indépendament. Cette approche permet de réduire la diffusion numérique. Des résultats numériques illustrent l'intérêt de la méthode. Le lecteur est également renvoyé à [2] où une extension 2D de la méthode est proposée et où des cas tests numériques 2D attestent le rôle crucial joué par la diffusion numérique.

1. Introduction

The present Note is devoted to the numerical approximation of solutions of the M_1 model as introduced in [7] (see also [10] in the framework of the limited flux diffusion). This model arises when simulating the radiative field and its interaction with the matter [6,5]. In the present work, we just consider the scattering part and we omit the role played by the temperature. The reader is referred to [2] where the full model is considered. The system governing such a model reads:

$$\partial_t E + \partial_x F = 0, \qquad \partial_t F + c^2 \partial_x P(E, F) = -c\sigma F, \tag{1}$$

where $E \geqslant 0$ is the energy, F is the flux and P is the radiative pressure defined as follows:

$$P(E, F) = E\chi(f), \quad f = F/cE, \quad \chi(f) = \left(3 + 4f^2\right)/\left(5 + 2\sqrt{4 - 3f^2}\right).$$

The parameter c denotes the speed of light while $\sigma \geqslant 0$ is the absorption coefficient. This system is equipped with the following convex state space: $\mathcal{A} = \{(E, F) \in \mathbb{R}^2 \mid E \geqslant 0, |f| \leqslant 1\}$.

One of the main property satisfied by the model concerns the asymptotic behavior for a large opacity. In this case, the Knudsen number ϵ can be introduced, writing

$$\epsilon \partial_t E + \partial_x F = 0, \qquad \epsilon \partial_t F + c^2 \partial_x P(E, F) = -\frac{c\sigma}{\epsilon} F. \tag{2}$$

As ϵ goes to zero, we recover the following formal limit [2,6]: $F = 0$ and $\partial_t E - \partial_x (\frac{c}{3\sigma} \partial_x E) = 0$. This asymptotic preserving property turns out to be crucial in the numerical approximations. A useful asymptotic preserving scheme was proposed in [8] and next extend to (1) in [6]. Their approach is based on a splitting technique. In the first step, the transport part is considered and it is approximated using a relaxation scheme [9]. The second step is devoted to the source term and a well-balanced scheme is assumed.

In the present work, we show that the two-steps scheme can be understood as a usual relaxation scheme [3,9]. This interpretation is used to perform better wavespeed approximations and then to enforce less diffusive approximate transport solutions. Numerical experiments will be given to highlight this diffusiveless property of the method.

2. A full relaxation model

To approximate the solutions of (1), we propose to approximate the solution of a relevant system with singular perturbation: the relaxation model [1,3,9]. To access such an issue, we first plug the source term $-c\sigma F$ into the first-order system. After [3], we set $\sigma(x) = \partial_x \Sigma(x)$ with $\partial_t \Sigma(x) = 0$ to get

$$\partial_t E + \partial_x F = 0, \quad \partial_t F + c^2 \partial_x P(E,F) + cF\partial_x \Sigma = 0, \quad \partial_t \Sigma = 0. \tag{3}$$

This system is hyperbolic with the eigenvalues $\lambda^0 = 0$ and

$$\lambda^{\pm} = \big(cf\sqrt{3} \pm 2c\big(\sqrt{4-3f^2}-1\big)\big)/\big(\sqrt{3\big(4-3f^2\big)}\big). \tag{4}$$

The main difference with [3] stays in the conservation form of (3) which arises when writing $F\partial_x\Sigma = \partial_t(\Sigma E) + \partial_x(\Sigma F)$. Now, the following relaxation model is considered:

$$\begin{cases} \partial_t E + \partial_x \phi = 0, \\ \partial_t \phi - b_L b_R \partial_x E + (b_L + b_R)\partial_x \phi + c\phi \partial_x \Sigma = \mu(F - \phi), \\ \partial_t F + c\partial_x \Pi + cF\partial_x \Sigma = 0, \\ \partial_t \Pi + (b_L + b_R)\partial_x \Pi - \dfrac{b_L b_R}{c}\partial_x F + (b_L + b_R)F\partial_x \Sigma = \mu\big(P(E,F) - \Pi\big), \\ \partial_t \Sigma = 0. \end{cases} \tag{5}$$

The parameters $b_L < 0 < b_R$ denote suitable approximations of the wavespeeds. They will be chosen to enforce several stability properties. For the sake of simplicity in the notations, we set

$$\mathbf{w} = {}^t(E, \phi, F, \Pi, \Sigma), \quad \mathcal{U} = {}^t(E, F), \quad \mathcal{F}(\mathcal{U}) = {}^t\big(F, c^2 P(E,F)\big).$$

With $\mu = 0$, let us note from now on that (5) is hyperbolic. The eigenvalues of the system are b_L, 0, b_R, and they are associated with linearly degenerate fields. As a consequence, the Riemann problem can be easily solved.

Lemma 2.1. *Let $\mathbf{w}_{L,R}$ be equilibrium constant states: $\phi_{L,R} = F_{L,R}$ and $\Pi_{L,R} = P_{L,R}$. Define $\mathbf{w}_0(x) = \mathbf{w}_L$ if $x < 0$ and $\mathbf{w}_R$ otherwise as initial data of* (5) *but for $\mu = 0$. Assume $b_L < 0 < b_R$. The weak solution of* (5) *reads*:

$$\mathbf{w}(x,t) = \mathbf{w}_L \quad \text{if } \frac{x}{t} < b_L, \qquad \mathbf{w}_L^\star \quad \text{if } b_L < \frac{x}{t} < 0, \qquad \mathbf{w}_R^\star \quad \text{if } 0 < \frac{x}{t} < b_R, \qquad \mathbf{w}_R \quad \text{if } \frac{x}{t} > b_R, \tag{6}$$

where $\Sigma_{L,R}^\star = \Sigma_{L,R}$ and

$$E_{L,R}^\star = \frac{1}{b_R - b_L + c(\Sigma_R - \Sigma_L)}\left(b_R\left(E_R - \frac{F_R}{b_R}\right) - b_L\left(E_L - \frac{F_L}{b_L}\right) + c(\Sigma_R - \Sigma_L)\left(E_{L,R} - \frac{F_{L,R}}{b_{L,R}}\right)\right),$$

$$F_{L,R}^\star = \frac{b_R F_R - b_L F_L - c^2(P_R - P_L)}{b_R - b_L + c(\Sigma_R - \Sigma_L)},$$

$$\phi_{L,R}^\star = \frac{b_R F_R - b_L F_L + b_L b_R (E_R - E_L)}{b_R - b_L + c(\Sigma_R - \Sigma_L)}, \qquad \Pi_{L,R}^\star = cP_{L,R} + \frac{b_{L,R}}{c}(F_{L,R}^\star - F_{L,R}).$$

Let $\mathcal{U}_{L,R}$ be in $\mathcal{A}$. Assume

$$b_L \leqslant c\min\left(0, f_L, \frac{f_L - \chi(f_L)}{1 - f_L}, \frac{f_L + \chi(f_L)}{1 + f_L}\right), \qquad b_R \geqslant c\max\left(0, f_R, \frac{f_R - \chi(f_R)}{1 - f_R}, \frac{f_R + \chi(f_R)}{1 + f_R}\right). \tag{7}$$

Then $\mathcal{U}_{L,R}^\star = {}^t(E_{L,R}^\star, F_{L,R}^\star)$ belongs to $\mathcal{A}$.

The establishment of the conditions (7) can be found in [2] while the computation of (6) is standard [1,3].

3. A relaxation scheme

The above relaxation model (5) is used to propose a relevant numerical method [1,3,9] to approximate the weak solutions of (1). At time t^n, we assume known the approximate solution $\mathcal{U}^n(x) = \mathcal{U}_i^n$ for $x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. This solution is evolved in time involving two steps.

First, the solution is evolved in time when considering the Cauchy problem for (5), with $\mu = 0$, where the initial data is prescribed by $\mathbf{w}^n(x) = \mathbf{w}_i^n$ if $x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. The equilibrium states $\mathbf{w}_i^n$ satisfy $\phi_i^n = F_i^n$ and $\Pi_i^n = P_i^n$. Assume the CFL restriction

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left(\left|b_L^{i-\frac{1}{2}}\right|,\ b_R^{i+\frac{1}{2}}\right)\leqslant\frac{1}{2} \tag{8}$$

so that the approximate solution $\mathbf{w}^h(x,t)$ is made of the juxtaposition of the non-interacting Riemann problem solutions set at the cell interfaces $x_{i+\frac{1}{2}}$ for all $i\in\mathbb{Z}$. At time t^{n+1}, we define

$$\mathbf{w}_i^{n+1,-}=\frac{1}{\Delta x}\int\limits_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\mathbf{w}^h(x,t)\,\mathrm{d}x.$$

The second step is devoted to the relaxation procedure. The vector $\mathbf{w}_i^{n+1,-}$ is projected on the manifold $\{\phi=F,\Pi=P\}$. Put in other words, we exactly solve $\partial_t\mathbf{w}=\mu^t(0,F-\phi,0,P-\Pi,0)$, where the initial data is prescribed by $\mathbf{w}_i^{n+1,-}$, and we set μ goes to ∞. Hence, we obtain $\mathcal{U}_i^{n+1}=(E_i^{n+1,-},F_i^{n+1,-})$, $\Sigma_i^{n+1}=\Sigma_i^n$, while $\phi_i^{n+1}=F_i^{n+1}$ and $\Pi_i^{n+1}=P_i^{n+1}$. After usual computations, the final scheme reads as follows:

$$\begin{aligned}
&\mathcal{U}_i^{n+1}=\mathcal{U}_i^n-\frac{\Delta t}{\Delta x}(\alpha_{i+\frac{1}{2}}\tilde{\mathcal{F}}_{i+\frac{1}{2}}-\alpha_{i-\frac{1}{2}}\tilde{\mathcal{F}}_{i-\frac{1}{2}})+\frac{\Delta t}{\Delta x}\mathcal{R}_i^n,\\
&\tilde{\mathcal{F}}_{i+\frac{1}{2}}=\frac{1}{b_R^{i+\frac{1}{2}}-b_L^{i+\frac{1}{2}}}\big(b_R^{i+\frac{1}{2}}\mathcal{F}\big(\mathcal{U}_i^n\big)-b_L^{i+\frac{1}{2}}\mathcal{F}\big(\mathcal{U}_{i+1}^n\big)+b_L^{i+\frac{1}{2}}b_R^{i+\frac{1}{2}}\big(\mathcal{U}_{i+1}^n-\mathcal{U}_i^n\big)\big),\\
&\mathcal{R}_i^n={}^t\big(0,(\alpha_{i+\frac{1}{2}}-\alpha_{i-\frac{1}{2}})P_i^n+\big((1-\alpha_{i+\frac{1}{2}})b_L^{i+\frac{1}{2}}-(1-\alpha_{i-\frac{1}{2}})b_R^{i-\frac{1}{2}}\big)F_i^n\big),\\
&\alpha_{i+\frac{1}{2}}=\big(b_R^{i+\frac{1}{2}}-b_L^{i+\frac{1}{2}}\big)\big/\big(b_R^{i+\frac{1}{2}}-b_L^{i+\frac{1}{2}}+c(\Sigma_{i+1}-\Sigma_i)\big).
\end{aligned} \tag{9}$$

Involving Lemma 2.1, we immediately obtain the following stability result:

Theorem 3.1. *Let $\mathcal{U}_i^n$ be in $\mathcal{A}$ and assume the CFL condition* (8)*. Let $b_{L,R}^{i+\frac{1}{2}}$ satisfy*

$$\begin{aligned}
b_L^{i+\frac{1}{2}}&=\min\left(-\delta,f_i^n,\frac{f_i^n-\chi(f_i^n)}{1-f_i^n},\frac{f_i^n+\chi(f_i^n)}{1+f_i^n},\lambda^-\big(\mathcal{U}_i^n\big)\right),\\
b_R^{i-\frac{1}{2}}&=\max\left(\delta,f_i^n,\frac{f_i^n-\chi(f_i^n)}{1-f_i^n},\frac{f_i^n+\chi(f_i^n)}{1+f_i^n},\lambda^+\big(\mathcal{U}_i^n\big)\right),
\end{aligned} \tag{10}$$

where $\delta>0$ is a small parameter. Then, the updated solution $\mathcal{U}_i^{n+1}$ deduced from the above relaxation scheme (9) *satisfies the positiveness of the energy, $E_i^{n+1}\geqslant 0$, and the flux limitation $(f_i^{n+1})^2\leqslant 1$.*

The above statement can be completed when establishing the asymptotic preserving property of the numerical method. After introducing the Knudsen number ϵ, the scheme reads:

$$\begin{cases}
\epsilon\dfrac{\mathcal{U}_i^{n+1}-\mathcal{U}_i^n}{\Delta t}=\dfrac{1}{\Delta x}(\alpha_{i+\frac{1}{2}}\tilde{\mathcal{F}}_{i+\frac{1}{2}}-\alpha_{i-\frac{1}{2}}\tilde{\mathcal{F}}_{i-\frac{1}{2}})+\dfrac{\Delta t}{\Delta x}\mathcal{R}_i^n,\\
\alpha_{i+\frac{1}{2}}=\epsilon\big(b_R^{i+\frac{1}{2}}-b_L^{i+\frac{1}{2}}\big)\big/\big(\epsilon\big(b_R^{i+\frac{1}{2}}-b_L^{i+\frac{1}{2}}\big)+c(\Sigma_{i+1}-\Sigma_i)\big).
\end{cases} \tag{11}$$

Theorem 3.2. *Let the assumptions stated Theorem* 3.1 *be satisfied. With ϵ small, the diffusion limit of* (11) *is given by $F_i^n=0$ and*

$$\frac{E_i^{n+1}-E_i^n}{\Delta t}=\frac{c}{3\Delta x^2}\left(\frac{1}{\sigma_{i+\frac{1}{2}}}\big(E_{i+1}^n-E_i^n\big)+\frac{1}{\sigma_{i-\frac{1}{2}}}\big(E_{i-1}^n-E_i^n\big)\right),$$

where $\sigma_{i+\frac{1}{2}}=(\Sigma_{i+1}-\Sigma_i)/\Delta x=(\sigma_{i+1}+\sigma_i)/2$.

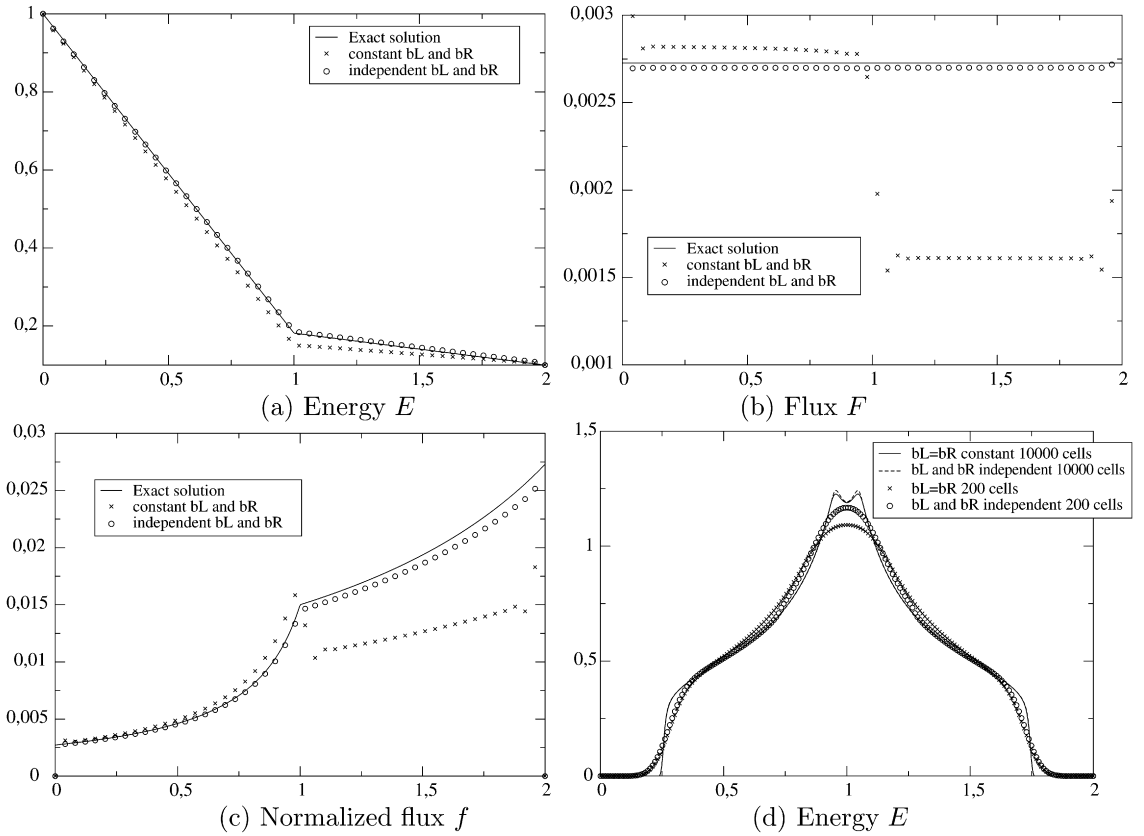

Fig. 1. (a)–(c): Steady problem. Comparison between exact solution (solid line), constant wavespeeds given by (12) (× symbol), variable wavespeeds given by (10) (∘ symbol). (d): Unsteady problem.

Fig. 1. (a)–(c) : Problème stationnaire. Comparaison entre la solution exacte (ligne pleine), vitesse d'onde constante donnée par (12) (×), vitesse d'onde variable donnée par (10) (∘). (d) : Problème instationnaire.

In [2], all the above stability properties are preserved when considering 2D radiative transfer simulations. Such results are obtained involving a relevant HLLC scheme to extend the present relaxation scheme.

To conclude the presentation of the method, let us note that the scheme proposed in [6] is deduced from (9) when considering

$$b_R^{i+\frac{1}{2}} = -b_L^{i+\frac{1}{2}} = c\chi\Big(\max_{i\in\mathbb{Z}}\big|f_i^n\big|\Big), \quad \forall i \in \mathbb{Z}. \tag{12}$$

Such a choice of the wavespeed satisfies the assumptions of Theorems 3.1 and 3.2 but for a larger numerical diffusion as emphasized by the numerical experiments (other choice of the wavespeed can be found in [4]).

In the first numerical experiment, a steady problem is considered. The test is performed on the domain $[0, 2]$ where we impose a discontinuous absorption coefficient. We set $\sigma(x) = 100$ if $x \in [0, 1)$ and $\sigma(x) = 10$ otherwise. Here, the speed of light c is fixed equal to 1. The boundary values of both energy and flux are given by $E(0, t) = 1$, $F(0, t) = 0$, $E(2, t) = 1/10$ and $F(2, t) = 0$ for all $t > 0$. One of the main interest of such a test lies on the knowledge of the exact solution. After easy computations, the exact steady flux solution remains constant: $F(x) = 2.72 \times 10^{-3}$ for all $x \in (0, 2)$, while $E(x)$ is solution of

$$E(x)\chi\big(F/E(x)\big) = 1/3 - 100xF, \quad \text{if } x \in (0, 1), \quad 1/30 + 10(2 - x)F, \quad \text{if } x \in (1, 2).$$

The exact solution $E(x)$ is next easily computed but its expression is too long to be given here. The numerical results obtained with the scheme (9) involving wavespeeds b_L and b_R given by (10) or (12) are respectively compared to the exact solution. The test is performed using 50 cells. For the sake of completeness, let us note that the initial data is

given by: $E(x,0) = 1$ and $F(x,0) = 0$ for all $x \in (0,2)$. The results are displayed in Fig. 1(a)–(c) and clearly they show the interest of wavespeeds given by (10) instead of (12)

The second experiment is directly derived from a test proposed in [6]. Once again, the domain $[0,2]$ is considered but for a continuous absorption coefficient $\sigma(x) = (1-x)^4$. The initial flux F is equal to 0 while $E(x,0) = 2$ if $x \in (1/2, 3/2)$ and $E(x,0) = 10^{-5}$ otherwise. We set $c = 1$. The numerical approximations are displayed Fig. 1(d) at time $t = 0.5$ with 200 and 10 000 cells were the wavespeeds are evaluated involving (10) or (12). This test illustrates the diffusiveless property of the scheme (9), (10) when compared to the scheme (9)–(12).

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