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Functional Analysis

Cyclicity of bicyclic operators

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Abstract

We study cyclicity of injective operators on separable Banach spaces which admit a bicyclic vector such that the norms of its images under the iterates of the operator satisfy certain growth conditions. Our results apply in particular to the shift operator acting on the weighted spaces of sequences $\ell^2_\omega(\mathbb{Z})$. We also prove completeness results of translates in certain Banach spaces of functions on $\mathbb{R}$. ***To cite this article: E. Abakumov et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Cyclicité de certains opérateurs bicycliques. Nous étudions la cyclicité de certains opérateurs bicycliques agissant sur des espaces de Banach séparables. Nos résultats s'appliquent en particulier aux opérateurs de décalage sur les espaces de suites pondérés $\ell^2_\omega(\mathbb{Z})$. Nous prouvons également des résultats concernant la complétude des translatés d'une fonction dans certains espaces de Banach de fonctions sur $\mathbb{R}$. ***Pour citer cet article : E. Abakumov et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit X un espace de Banach séparable complexe, et T un opérateur borné sur X injectif et bicyclique : il existe un vecteur $x_0 \in R^\infty(T) = \bigcap_{n\geqslant 1} \mathrm{Im}(T^n)$ tel que les vecteurs $T^n x_0$, $n \in \mathbb{Z}$, engendrent un sous-espace dense de X. Le but de cette Note est de présenter une méthode permettant de déduire que T est en fait cyclique lorsque les normes $\|T^n x_0\|$, $n \in \mathbb{Z}$, vérifient certaines conditions de croissance. Voici notre résultat principal :

Théorème 0.1. *Soit $T \in \mathcal{B}(X)$ un opérateur injectif admettant un vecteur bicyclique x_0. Supposons qu'il existe un entier $k \geqslant 0$ et une suite sous-multiplicative $(\rho(n))_{n\geqslant 0}$ de nombres strictement positifs telle que*

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$$\lim_{n\to+\infty}\frac{\log^+\rho(n)}{\sqrt{n}}=0$$

et

(1) $\|T^n x_0\| = \mathrm{O}(n^k)$ *lorsque n tend vers l'infini* ;
(2) $\|T^{-n} x_0\| = \mathrm{O}(\rho(n))$ *lorsque n tend vers l'infini.*

Si le spectre ponctuel $\sigma_p(T^*)$ de T^* ne contient pas le cercle unité, alors T est cyclique.

Ce théorème permet par exemple de retrouver le fait que le décalage à droite $S:(a_n)_{n\in\mathbb{Z}}\mapsto(a_{n-1})_{n\in\mathbb{Z}}$ est cyclique sur $\ell^p(\mathbb{Z})$ pour tout $p>1$. Plus généralement :

Théorème 0.2. *Soit X un espace de Banach dont le dual est séparable, et U un opérateur unitaire sur X (c'est-à-dire une isométrie surjective). Si U est bicyclique, alors U est cyclique.*

Une deuxième conséquence du Théorème 0.1 concerne le décalage à droite agissant sur l'espace de suites

$$\ell^2_\omega(\mathbb{Z})=\left\{(a_n)_{n\in\mathbb{Z}}\in\mathbb{C}^{\mathbb{Z}};\ \left\|(a_n)\right\|_{\ell^2_\omega(\mathbb{Z})}=\left(\sum_{n\in\mathbb{Z}}|a_n|^2\omega(n)^2\right)^{1/2}<+\infty\right\},$$

où ω est une suite sous-multiplicative de nombres strictement positifs. C'est une question ouverte que de caractériser les poids ω ayant la propriété que S est cyclique sur $\ell^2_\omega(\mathbb{Z})$. La première partie du théorème suivant s'ensuit directement du Théorème 0.1 :

Théorème 0.3. *Supposons que $\omega(n)=1$ pour tout $n\geqslant 0$:*

(1) *si $\log^+\omega(n)/\sqrt{n}$ tend vers 0 lorsque n tend vers l'infini, alors S est cyclique sur $\ell^2_\omega(\mathbb{Z})$;*
(2) *si la série $\sum_{n>0}\log\omega(-n)/n^2$ diverge et si ω est suffisamment régulier pour que le Théorème de Volberg sur l'intégrale logarithmique s'applique, alors S n'est pas cyclique sur $\ell^2_\omega(\mathbb{Z})$.*

Nous formulons la conjecture suivante :

Conjecture 0.1. *Si $\omega(-n)=\mathrm{e}^{\sqrt{n}}$ pour tout $n>0$ et $\omega(n)=1$ pour tout $n\geqslant 0$, alors S n'est pas cyclique sur $\ell^2_\omega(\mathbb{Z})$.*

Herrero pose dans [7] la question suivante : si ω est un poids quelconque, l'un des deux opérateurs S ou S^* doit-il nécessairement être cyclique sur $\ell^2_\omega(\mathbb{Z})$? Nous prouvons :

Théorème 0.4. *La réponse à la question ci-dessus est positive lorsque*

$$\sum_{n\in\mathbb{Z}}\frac{\log\|S^n\|}{1+n^2}<+\infty.$$

Soit $w:\mathbb{R}\to[1,+\infty)$ un poids sous-multiplicatif mesurable sur $\mathbb{R}$. Alors $L^1_w(\mathbb{R})$ est une algèbre de Banach pour la convolution. La méthode employée pour prouver le Théorème 0.1 permet de retrouver simplement un résultat de [2], qui affirme que si Λ est un sous-ensemble de $\mathbb{R}$ dont le rayon de Beurling–Malliavin $R(\Lambda)$ est infini, et si w est un poids non quasianalytique, il existe une fonction f appartenant à l'algèbre de Beurling $L^1_w(\mathbb{R})$ telle que l'ensemble des translatés $f(\cdot-\lambda)$, $\lambda\in\Lambda$, est dense dans $L^1_w(\mathbb{R})$. Voici un autre résultat qui peut être obtenu par la même méthode :

Théorème 0.5. *Soit X un espace de Banach séparable, π une représentation fortement continue de $\mathbb{R}$ sur X, et Λ un sous-ensemble de $\mathbb{R}$ tel que $R(\Lambda)=+\infty$. Si π est non quasianalytique et cyclique, il existe un vecteur $x\in X$ tel que la famille des $\pi(\lambda)x$, $\lambda\in\Lambda$, engendre un sous-espace dense de X.*

1. Cyclicity of bicyclic operators on Banach spaces

Our aim in this Note is to present conditions on a bounded operator T acting on a separable Banach space X which imply that if T is *bicyclic*, then T must in fact be *cyclic*. Proofs will be presented elsewhere.

Definition 1.1. Let T be an injective operator on X, and define $R^\infty(T) = \bigcap_{n\geqslant 1} \mathrm{Ran}(T^n)$ to be the intersection of the ranges of all the operators T^n. We say that T is *bicyclic* if there exists a vector $x_0 \in R^\infty(T)$ (a bicyclic vector for T) such that the linear span of the vectors $T^n x_0$, $n \in \mathbb{Z}$, is dense in X. Recall that T is said to be *cyclic* if there exists a vector $x \in X$ such that the linear span of the vectors $T^n x$, $n \geqslant 0$, is dense in X.

Recall that a sequence $(\rho(n))_{n\in A}$, $A = \mathbb{Z}^+$ or $\mathbb{Z}$, is submultiplicative if $\rho(n+m) \leqslant \rho(n)\rho(m)$ for $n, m \in A$.

Theorem 1.2. *Let T be a bounded injective operator on X which admits a bicyclic vector $x_0 \in R^\infty(T)$. Suppose that the following conditions are satisfied: there exist a non-negative integer k and a submultiplicative sequence $(\rho(n))_{n\geqslant 0}$ of positive numbers with*

$$\lim_{n\to+\infty} \frac{\log^+ \rho(n)}{\sqrt{n}} = 0$$

such that

(1) $\|T^n x_0\| = \mathrm{O}(n^k)$ *as n goes to infinity*;
(2) $\|T^{-n} x_0\| = \mathrm{O}(\rho(n))$ *as n goes to infinity.*

If the point spectrum $\sigma_p(T^*)$ of T^* does not include the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$, then T is cyclic.

In particular if T is an invertible operator this yields:

Theorem 1.3. *Let T be an invertible bicyclic operator on X such that*

(1) *there exists a non-negative integer k such that $\|T^n\| = \mathrm{O}(n^k)$ as n goes to infinity*;
(2) $\log \|T^{-n}\|/\sqrt{n}$ *tends to* 0 *as n goes to infinity.*

If $\sigma_p(T^*)$ does not include the unit circle, then T is cyclic.

Here is another consequence of Theorem 1.2 concerning surjective isometries of Banach spaces:

Theorem 1.4. *Let U be a unitary operator on a separable Banach space X.*

(1) *if U is bicyclic and $\sigma_p(U^*)$ does not include $\mathbb{T}$, then U is cyclic*;
(2) *if X has separable dual and U is bicyclic, then U is cyclic.*

For unitary operators acting on a Hilbert space this result is known and follows from a more general result [3] on cyclicity of normal operators. Theorem 1.4 applies in particular to all the unweighted bilateral shifts $S_p : (a_n)_{n\in\mathbb{Z}} \mapsto (a_{n-1})_{n\in\mathbb{Z}}$ on the spaces $\ell^p(\mathbb{Z})$ for $p > 1$. This result for S_p was proved independently by A. Olevskiĭ in 1998 by a different method (unpublished manuscript).

2. Cyclicity of S on $A_{\rho,0}(\mathbb{T})$

Let ρ be a submultiplicative weight with $\rho(n) \geqslant 1$ for every $n \in \mathbb{Z}$. The main argument of the proof of Theorem 1.2 involves a study of the multiplication operator by $\mathrm{e}^{\mathrm{i}\theta}$, denoted by S, on the Banach space

$$A_\rho(\mathbb{T}) = \left\{ f \in C(\mathbb{T});\ \|f\|_\rho = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| \rho(n) < +\infty \right\}.$$

Definition 2.1. A submultiplicative sequence $\rho : \mathbb{Z} \to [1, +\infty)$ such that

$$\sum_{n\in\mathbb{Z}} \frac{\log \rho(n)}{1+n^2} < +\infty$$

will be called a *Beurling sequence*.

If ρ is a Beurling sequence, it is known that $A_\rho(\mathbb{T})$ is a regular Banach algebra, so it contains functions with arbitrarily small support. The closed ideal $A_{\rho,0}(\mathbb{T})$ defined as the closure in $A_\rho(\mathbb{T})$ of the set of the functions which vanish identically in a neighborhood of the point 1 (which is clearly invariant under the action of S) appears as a natural space on which to study the cyclicity properties of S:

Theorem 2.2. *If ρ is a Beurling sequence, the operator S acting on $A_{\rho,0}(\mathbb{T})$ is cyclic.*

The proof of Theorem 2.2 relies on a Baire Category argument: we prove, using the fact that $A_\rho(\mathbb{T})$ is a regular Banach algebra, that given two non-empty open subsets of $A_{\rho,0}(\mathbb{T})$ there exists an analytic trigonometric polynomial p such that $p(S)(U) \cap V$ is non-empty. For more on this Baire Category method and its applications to universality questions, we refer the reader to the survey [5].

To deduce Theorem 1.2 from Theorem 2.2, the key argument is the following result of [1]: if $\rho(n) = (n+1)^k$ for $n \geqslant 0$ and $\lim_{n\to+\infty} \log^+ \rho(-n)/\sqrt{n} = 0$, then $A_{\rho,0}(\mathbb{T})$ coincides with the closed ideal generated by the function $(\mathrm{e}^{\mathrm{i}\theta} - 1)^{k+1}$, i.e. the closure of the set

$$Y^{(k)}_{\rho,0}(\mathbb{T}) = \left\{ \left(\mathrm{e}^{\mathrm{i}\theta} - 1\right)^{k+1} f;\ f \in A_\rho(\mathbb{T}) \right\}.$$

3. Cyclicity of shifts on weighted Hilbert spaces of sequences on $\mathbb{Z}$

The main motivation for Theorem 1.2 is the study of cyclicity properties of a particularly interesting class of operators: the shift operator on weighted Hilbert or Banach spaces of sequences on $\mathbb{Z}$. Let ω be a positive weight on $\mathbb{Z}$ such that $\sup_{n\in\mathbb{Z}} \omega(n+1)/\omega(n) < +\infty$. Then the shift operator $S : (a_n)_{n\in\mathbb{Z}} \mapsto (a_{n-1})_{n\in\mathbb{Z}}$ acting on the weighted ℓ^2-space

$$\ell^2_\omega(\mathbb{Z}) = \left\{ (a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}};\ \left\|(a_n)\right\|_{\ell^2_\omega(\mathbb{Z})} = \left(\sum_{n\in\mathbb{Z}} |a_n|^2 \omega(n)^2 \right)^{1/2} < +\infty \right\}$$

is bounded on $\ell^2_\omega(\mathbb{Z})$. This operator S has been the subject of various investigations since the 60's (see [9,8]), concerning in particular non-trivial invariant and bi-invariant subspaces. The description of the invariant subspaces of the operator S acting on $\ell^2_\omega(\mathbb{Z})$ is closely connected to the description of the cyclic vectors of S. One of the main applications of Theorem 1.2 is to obtain a partial answer to the following open problem:

Problem 3.1. Characterize the weights ω such that S is cyclic on $\ell^2_\omega(\mathbb{Z})$.

This problem is mentioned in [6], and partial results were obtained by Herrero [7], who proved that S is cyclic if its point spectrum is non-empty, and not cyclic if the point spectrum of S^* is non-empty. Shifts on various Banach spaces of functions and hyperdistributions are studied extensively in [8]. Since S is always bicyclic in the sense of Definition 1.1 with bicyclic vector $(\delta_{0,n})_{n\in\mathbb{Z}}$, Theorem 1.2 can be applied here:

Theorem 3.2. *Let ω be a positive weight such that S is bounded on $\ell^2_\omega(\mathbb{Z})$. Suppose that there exist a non-negative integer k and a submultiplicative sequence $(\rho(n))_{n\geqslant 0}$ of positive numbers with*

$$\lim_{n\to+\infty} \frac{\log^+ \rho(n)}{\sqrt{n}} = 0$$

such that

(1) *$\omega(n) = \mathrm{O}(n^k)$ as n goes to infinity*;
(2) *$\omega(-n) = \mathrm{O}(\rho(n))$ as n goes to infinity.*

Then S is cyclic on $\ell^2_\omega(\mathbb{Z})$ if and only if

$$\sum_{n\in\mathbb{Z}} \frac{1}{\omega(n)^2} = +\infty.$$

For instance if for some $\alpha \in \mathbb{R}$, $\omega(n) = (|n|+1)^\alpha$ for every $n \in \mathbb{Z}$, then S is cyclic on $\ell^2_\omega(\mathbb{Z})$ if and only if $\alpha \leqslant 1/2$. The same conclusion holds true if $\omega(n) = (n+1)^\alpha$ for $n \geqslant 0$ and $\omega(-n) = \exp(\sqrt{n}/\log(n+1))$ for $n > 0$. Although Theorem 3.2 does not yield a complete characterization of the weights (with $\omega(n) = 1$ for every $n \geqslant 0$, for instance) such that S is cyclic on $\ell^2_\omega(\mathbb{Z})$, it leads us to a reasonable conjecture concerning this characterization. If we specialize to the case where $\omega(n) = 1$ for $n \geqslant 0$, then Theorem 3.2 tells us that S is cyclic on $\ell^2_\omega(\mathbb{Z})$ if $\omega(-n)$ grows "slower that $\mathrm{e}^{\sqrt{n}}$". The case where $\omega(-n) = \mathrm{e}^{\sqrt{n}}$ for $n > 0$ appears as a limit case, and our conjecture is that the shift is not cyclic for this weight:

Conjecture 3.3. *If ω_0 is the weight defined by $\omega_0(n) = 1$ for $n \geqslant 0$ and $\omega_0(-n) = \mathrm{e}^{\sqrt{n}}$ for $n > 0$, then S is not cyclic on $\ell^2_{\omega_0}(\mathbb{Z})$.*

On the other hand, it is not difficult to see that when $\omega(n) = 1$ for $n \geqslant 0$, $\omega(n) \geqslant 1$ for every $n \in \mathbb{Z}$,

$$\sum_{n<0} \frac{\log\omega(n)}{1+n^2} = +\infty$$

and ω is sufficiently regular, S is not cyclic on $\ell^2_\omega(\mathbb{Z})$: this follows directly from the known characterization of cyclic vectors for the shift on $\ell^2(\mathbb{Z})$ and Volberg's theorem on the logarithmic integral [10]. This holds true for instance if $\omega(n) = 1$ for every $n \geqslant 0$ and $\omega(-n) = \exp(n/\log(n+1))$ for every $n > 0$. But if $\omega(n) = \exp(n/\log(|n|+2))$ for every $n \in \mathbb{Z}$, we do not know whether S is cyclic or not on $\ell^2_\omega(\mathbb{Z})$.

4. On a question of Herrero

The following question was posed by Herrero in [7]:

Question 4.1. If ω is a positive weight on $\mathbb{Z}$, must either S or S^* be cyclic on $\ell^2_\omega(\mathbb{Z})$?

The following general theorem for operators on Banach spaces, as well as some easy considerations on supercyclic operators, yields a positive answer to Question 4.1 in the non-quasianalytic case:

Theorem 4.2. *Let X be a complex separable Banach space and T a bounded injective operator on X which admits a bicyclic vector $x_0 \in R^\infty(T)$. Suppose that the following conditions are satisfied: there exist a non-negative integer k and a Beurling sequence ρ such that*

(1) *$\|T^n x_0\| = \mathrm{O}(\rho(n))$ as $|n|$ goes to infinity*;
(2) *$\|T^n x_0\| \,.\, \|T^{-n} x_0\| = \mathrm{O}(n^k)$ as n goes to infinity.*

If $\sigma_p(T^*)$ does not include the unit circle, then T is cyclic.

Theorem 4.3. *Let S be a bounded invertible shift on $\ell^2_\omega(\mathbb{Z})$. Suppose that there exists a Beurling sequence ρ such that $\omega(n) = \mathrm{O}(\rho(n))$. Then either S or S^* is cyclic on $\ell^2_\omega(\mathbb{Z})$. This applies in particular if the operator S satisfies*

$$\sum_{n\in\mathbb{Z}} \frac{\log\|S^n\|}{1+n^2} < +\infty.$$

5. Completeness of translates

The Baire Category method combined with the regularity of the Banach algebra applies very well to the study of completeness of translates of functions in some Banach algebras of functions over $\mathbb{R}$ which are invariant by all translations. Here $w:\mathbb{R}\to[1,+\infty)$ will be a positive measurable function (weight) such that $w(t+s)\leqslant w(t)w(s)$ for every $t,s\in\mathbb{R}$. Then

$$L^1_w(\mathbb{R})=\left\{f;\ \|f\|_{L^1_w(\mathbb{R})}=\int_{\mathbb{R}}\big|f(t)\big|w(t)\,\mathrm{d}t<+\infty\right\}$$

is a Banach algebra with respect to convolution, and all the translation operators $f\mapsto f_\lambda=f(\cdot-\lambda)$, $\lambda\in\mathbb{R}$, are continuous on $L^1_w(\mathbb{R})$. If Λ is a subset of $\mathbb{R}$, Λ is called *generating* in $L^1_w(\mathbb{R})$ if there exists a function $f\in L^1_w(\mathbb{R})$ such that the translates f_λ, $\lambda\in\Lambda$, span a dense subspace of $L^1_w(\mathbb{R})$. The main result of [4] is that Λ is generating in $L^1(\mathbb{R})$ if and only if $R(\Lambda)=+\infty$, where $R(\Lambda)$ is the Beurling–Malliavin radius of the set Λ. The result of [4] is extended to non-quasianalytic Beurling algebras $L^1_w(\mathbb{R})$ by Blank in [2]. We give a straightforward proof of this result, using only the fact that if w is non-quasianalytic, $L^1_w(\mathbb{R})$ is a regular Banach algebra:

Theorem 5.1. *[2] If the weight w satisfies*

$$\int_{\mathbb{R}}\frac{\log w(t)}{1+t^2}<+\infty,$$

then Λ is generating for $L^1_w(\mathbb{R})$ if and only if $R(\Lambda)=+\infty$.

We deduce from Theorem 5.1 the following result concerning cyclic representations of $\mathbb{R}$ on Banach spaces:

Theorem 5.2. *Let π be a non-quasianalytic strongly continuous representation of $\mathbb{R}$ on a separable Banach space X, and Λ a subset of $\mathbb{R}$ such that $R(\Lambda)=+\infty$. If π is cyclic, then there exists a vector $x\in X$ such that the linear span of the vectors $\pi(\lambda)x$, $\lambda\in\Lambda$, is dense in X.*

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