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Mathematical Analysis

Riesz products on the ring of p-adic integers

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Abstract

We introduce a class of probability measures, called Riesz products, on the ring $\mathbb{Z}_p$ of p-adic integers. We prove a result on the almost everywhere convergence, with respect to a Riesz product, of some related series and then obtain the Hausdorff dimension of the Riesz product. Other properties of these measures are also discussed, like the mutual absolute continuity, the quasi-invariance with respect to the shift transformation and the quasi-Bernoulli property. ***To cite this article: A. Fan, X.Y. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Produits de Riesz sur l'anneau des entiers p-adiques. Nous introduisons une classe de mesures de probabilités, appelées produits de Riesz, sur l'anneau $\mathbb{Z}_p$ des entiers p-adiques. Nous prouvons un résultat concernant la convergence presque partout, par rapport à un produit de Riesz, de certaines séries liées et puis nous obtenons la dimension de Hausdorff du produit de Riesz. D'autres propriétés de ces mesures sont aussi discutées, comme la continuité absolue mutuelle, la quasi-invariance par rapport au décalage et la propriété de quasi Bernoulli. ***Pour citer cet article : A. Fan, X.Y. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***
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Soit $p \geqslant 3$ un nombre premier et soit $\mathbb{Z}_p$ l'anneau des entiers p-adiques dans le corps $\mathbb{Q}_p$ des nombres p-adiques. Désignons par dx la mesure de Haar normalisée sur $\mathbb{Z}_p$ qui est considéré comme un groupe additif et dont groupe dual est noté $\widehat{\mathbb{Z}}_p$. Les caractères non-triviaux dans $\widehat{\mathbb{Z}}_p$ sont de la forme

$$\gamma_{n,k}(x) := \exp\bigl(2\pi i\bigl\{p^{-n}kx\bigr\}\bigr) \quad \bigl(n \geqslant 1,\ 1 \leqslant k < p^n,\ p \nmid k\bigr)$$

où le symbole $\{y\}$ désigne $\sum_{j=-n}^{-1} y_j p^j$, la partie fractionnelle d'un nombre p-adique $y = \sum_{j=-n}^{\infty} y_j p^j$ (voir [6,7]). Nous noterons simplement $\gamma_{n,1}$ par γ_n. Il est facile à vérifier que l'ensemble $\Gamma := \{\gamma_n\colon n \geqslant 1\} \subset \widehat{\mathbb{Z}}_p$ est dissocié au

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sens de [3]. Alors, pour toute suite de nombres complexes $a = (a_n)_{n \geqslant 1}$ satisfaisant $|a_n| \leqslant 1$ pour tous $n \geqslant 1$, nous pouvons définir une mesure de probabilité sur $\mathbb{Z}_p$, appelée produit de Riesz et notée

$$\mu_a = \prod_{n=1}^{\infty} \big(1 + \operatorname{Re} a_n \gamma_n(x)\big) \tag{1}$$

qui est la limite faible des produits partiels.

Soit $\{f_k\}_{k \geqslant 1}$ une suite de fonctions analytiques définies dans un domaine contenant le disque unité $\{z \in \mathbb{C}\colon |z| \leqslant 1\}$. Soit $\{\alpha_k\}_{k \geqslant 1}$ une suite de nombres complexes. Nous considérons la série suivante

$$\sum_{k=1}^{\infty} \alpha_k \big(f_k \circ \gamma_k(x) - \mathbb{E}_{\mu_a} f_k \circ \gamma_k\big). \tag{2}$$

Théorème 0.1. *Soit $\{c_j^{(k)}\}$ la suite des coefficients de Taylor de f_k au point zero. Supposons que*

$$\sum_{j=1}^{\infty} \sqrt{1 + \log j}\, \sup_{k \geqslant 1} \big|c_j^{(k)}\big| < \infty. \tag{3}$$

Alors, pour toute suite $\{\alpha_k\}_{k \geqslant 1} \in \ell^2$, la série (2) *converge pour μ_a-presque tout x.*

Rappelons que $\mathbb{Z}_p$ est muni de la norme p-adique standard et que la dimension de Hausdorff d'une mesure est définie au sens de [2]. Comme application du Théorème 0.1, nous avons :

Théorème 0.2. *La dimension de Hausdorff du produit de Riesz μ_a est égale à*

$$\dim \mu_a = 1 - \frac{1}{\log p} \limsup_{n \to \infty} \frac{\mathbb{E}_{\mu_a} \log P_{a,n}}{n}$$

où $P_{a,n}(x) = \prod_{k=1}^{n} (1 + \operatorname{Re} a_k \gamma_k(x))$.

1. Statement of results

Let $p \geqslant 3$ be a prime number and let $\mathbb{Z}_p$ be the ring of p-adic integers in the field $\mathbb{Q}_p$ of p-adic numbers. Denote by $\mathrm{d}x$ the normalized Haar measure on $\mathbb{Z}_p$ which is considered as an additive group and whose dual group is denoted by $\widehat{\mathbb{Z}}_p$. Nontrivial characters in $\widehat{\mathbb{Z}}_p$ are of the form

$$\gamma_{n,k}(x) := \exp\big(2\pi \mathrm{i} \big\{p^{-n} k x\big\}\big) \quad \big(n \geqslant 1,\ 1 \leqslant k < p^n,\ p \nmid k\big)$$

where the symbol $\{y\}$ denotes $\sum_{j=-n}^{-1} y_j p^j$ which is the fractional part of an arbitrary p-adic number $y = \sum_{j=-n}^{\infty} y_j p^j$ (see [6,7]). For simplicity, we write $\gamma_{n,1}$ by γ_n. It is easy to check that the set $\Gamma := \{\gamma_n\colon n \geqslant 1\} \subset \widehat{\mathbb{Z}}_p$ is a dissociate set in the sense of [3]. Thus, for any sequence of complex numbers $a = (a_n)_{n \geqslant 1}$ satisfying $|a_n| \leqslant 1$ for all $n \geqslant 1$, we can define a probability measure on $\mathbb{Z}_p$, called Riesz product and denoted by

$$\mu_a = \prod_{n=1}^{\infty} \big(1 + \operatorname{Re} a_n \gamma_n(x)\big) \tag{1}$$

which is the weak limit of the partial products.

Let $\{f_k\}_{k \geqslant 1}$ be a sequence of analytic functions defined in some domain containing the unit disk $\{z \in \mathbb{C}\colon |z| \leqslant 1\}$. Let $\{\alpha_k\}_{k \geqslant 1}$ be any sequence of complex numbers. We consider the following series

$$\sum_{k=1}^{\infty} \alpha_k \big(f_k \circ \gamma_k(x) - \mathbb{E}_{\mu_a} f_k \circ \gamma_k\big). \tag{2}$$

Theorem 1.1. *Let $\{c_j^{(k)}\}$ be the Taylor coefficients of f_k at the point zero. Suppose that*

$$\sum_{j=1}^{\infty} \sqrt{1+\log j}\, \sup_{k\geqslant 1} \big|c_j^{(k)}\big| < \infty. \tag{3}$$

Then for any sequence $\{\alpha_k\}_{k\geqslant 1} \in \ell^2$, the series (2) *converges for μ_a-almost every x.*

The p-adic norm will be denoted by $|\cdot|_p$. See [2] for the notions of dimension of a measure. As an application of Theorem 1.1, we have the following:

Theorem 1.2. *The Hausdorff dimension of the Riesz product μ_a is equal to*

$$\dim \mu_a = 1 - \frac{1}{\log p} \limsup_{n\to\infty} \frac{\mathbb{E}_{\mu_a} \log P_{a,n}}{n}$$

where $P_{a,n}(x) = \prod_{k=1}^{n}(1 + \operatorname{Re} a_k \gamma_k(x))$.

2. Basic facts

Before proving the theorems stated above, we present some basic useful facts. Let

$$\Gamma^* = \{1\} \cup \bigcup_{m=1}^{\infty} \Gamma_m^*, \quad \Gamma_m^* = \big\{\gamma_1^{\epsilon_1} \cdots \gamma_m^{\epsilon_m}\colon \epsilon_1, \ldots, \epsilon_{m-1} = -1, 0, 1; \epsilon_m = -1, 1\big\}.$$

The set Γ is said to be dissociate if all characters of the form $\gamma_1^{\epsilon_1}\gamma_2^{\epsilon_2}\cdots\gamma_n^{\epsilon_n}$ where $\epsilon_j = 0,\ 1$ or -1 for any $1 \leqslant j \leqslant n$ are all distinct. The proofs of the following Facts 1–3 are elementary. The Fact 4 is well known for Riesz products.

Fact 1. *The set Γ is dissociate.*
Fact 2. *For any $n \geqslant 1$, we have $\gamma_n^p = \gamma_{n-1}$ ($\gamma_0 = 1$ by convention).*
Fact 3. *If $k \in \mathbb{Z}$ and $\gamma_n^k \in \Gamma_m^*$, then $n = m$ and $k = \pm 1 \bmod p$.*
Fact 4. *If $\gamma \notin \Gamma^*$, $\hat{\mu}_a(\gamma) = 0$; if $\gamma = \gamma_1^{\epsilon_1}\cdots\gamma_n^{\epsilon_n} \in \Gamma^*$, $\hat{\mu}_a(\gamma) = a_1^{(\epsilon_1)}\cdots a_n^{(\epsilon_n)}$ where by $a^{(\epsilon)}$ we means 1, $\frac{a}{2}$ or $\frac{\bar{a}}{2}$ according to $\epsilon = 0, -1$ or 1. Recall that for a measure μ, we define its Fourier coefficients by $\hat{\mu}(\gamma) = \int \gamma \,\mathrm{d}\mu$ for all $\gamma \in \widehat{\mathbb{Z}}_p$.*

The following specific fact concerning $\mathbb{Z}_p$ will be very useful:

Fact 5. *Let $n \geqslant 1$. For any function $F(x)$ on $\mathbb{Z}_p$ depending only on the first $n-1$ coordinates and for any integer k such that $p \nmid k$, we have $\int_{\mathbb{Z}_p} F(x)\gamma_n^k(x)\,\mathrm{d}x = 0$.*

In fact, the Haar integral in question is equal to

$$p^{-n} \sum_{x_0, x_1, \ldots, x_{n-1} \in \mathbb{Z}/p\mathbb{Z}} F(x_0, x_1, \ldots, x_{n-2})\gamma_n^k(x_0, x_1, \ldots, x_{n-1}).$$

Write $\gamma_n^k(x) = G(x_0, x_1, \ldots, x_{n-2})\exp(2\pi k\mathrm{i}\frac{x_{n-1}}{p})$, where G is a function depending only on the first $n-1$ coordinates. Taking summation over x_{n-1} and using the hypothesis $p \nmid k$, we get

$$\sum_{x_{n-1}\in \mathbb{Z}/p\mathbb{Z}} \exp\left(2\pi k\mathrm{i}\frac{x_{n-1}}{p}\right) = 0.$$

Hence Fact 5 follows.

Let $B_t(x) = \{y \in \mathbb{Z}_p\colon |x-y|_p \leqslant p^{-t}\}$ denote a ball. For $n \geqslant 1$, we denote by $P_{a,n}$ the n-th partial product of (1), i.e.

$$P_{a,n}(x) = \prod_{j=1}^{n}\big(1 + \operatorname{Re} a_j\gamma_j(x)\big).$$

Fact 6. *We have* $\mu_a(B_n(x)) = p^{-n}P_{a,n}(x)$ *for any ball* $B_n(x)$.

In fact, observe that both $P_{a,n}(y)$ and the characteristic function $1_{B_n(x)}(y)$ of $B_n(x)$ depend only on the first n coordinates. Hence, by Fact 5, we have

$$\mu_a\big(B_n(x)\big) = \int 1_{B_n(x)}(y)\,\mathrm{d}\mu_a(y) = \int 1_{B_n(x)}(y)P_{a,n}(y)\,\mathrm{d}y = p^{-n}P_{a,n}(x).$$

Theorem 1.2 is a consequence of Theorem 1.1 , Fact 6 and a result in [2]. Let us emphasize Fact 6 which does not hold for Riesz products on $\mathbb{R}/\mathbb{Z}$.

3. Proof of Theorem 1.1

The proof consists of a series of lemmas. We observe that the series (2) can be decomposed into a sum of a finite number of martingales. The idea is inspired by J. Peyrière's work ([4]) on Riesz products on the unit circle in the complex plane. However the two concrete situations are different. Let $\mathfrak{B}_k = \sigma(\gamma_1,\ldots,\gamma_k)$ be the σ-algebra generated by $\gamma_1,\ldots,\gamma_k$.

Lemma 3.1. *For any* $-1 \leqslant r \leqslant p-2$, *we have* $\mathbb{E}_{\mu_a}(\gamma_k^r(x)|\mathfrak{B}_{k-1}) = \mathbb{E}_{\mu_a}\gamma_k^r(x)$.

Proof. What we have to prove is that for any ball $B = B(y_0,\ldots,y_{k-2})$,

$$\int_B \gamma_k^r(x)\,\mathrm{d}\mu_a(x) = \mu_a(B)\mathbb{E}_{\mu_a}\gamma_k^r(x).$$

The case $r = 0$ is trivial. For $r \neq 0$, by Fact 5, we have

$$\int_B \gamma_k^r(x)\,\mathrm{d}\mu_a(x) = \int_B \gamma_k^r(x)P_{a,k}(x)\,\mathrm{d}x = \frac{1}{2}\int_B P_{a,k-1}(x)\big[a_k\gamma_k^{r+1}(x) + \bar{a}_k\gamma_k^{r-1}(x)\big]\,\mathrm{d}x.$$

Recall that $\mathbb{E}_{\mu_a}\gamma_k^r(x) = 1, \frac{a_k}{2}, \frac{\bar{a}_k}{2}$ for $r = 0, -1, 1$ and $\mathbb{E}_{\mu_a}\gamma_k^r(x) = 0$ for $2 \leqslant r \leqslant p-2$ (see Facts 3–4). We conclude by distinguishing three cases: $r+1=0$, $r-1=0$ and $r \pm 1 \neq 0$ and by using Fact 5. □

Lemma 3.2.

$$\mathbb{E}_{\mu_a}\big(\gamma_k^n(x)|\mathfrak{B}_{k-1}\big) = \begin{cases} \gamma_k^n(x), & n \equiv 0 \bmod p,\\ \dfrac{\bar{a}_k}{2}\gamma_k^{n-1}(x), & n \equiv 1 \bmod p,\\ \dfrac{a_k}{2}\gamma_k^{n+1}(x), & n \equiv -1 \bmod p,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. This lemma follows from Lemma 3.1 and the following fact. Let $n = mp + r$ with $-1 \leqslant r \leqslant p-2$. Since $\gamma_k^p = \gamma_{k-1}$ (Fact 2), we have

$$\mathbb{E}_{\mu_a}\big(\gamma_k^n(x)|\mathfrak{B}_{k-1}\big) = \gamma_{k-1}^m(x)\mathbb{E}_{\mu_a}\big(\gamma_k^r(x)|\mathfrak{B}_{k-1}\big). \qquad \square$$

Lemma 3.3. *If* $d > \log_p \frac{p(|n|-1)}{p-2}$, *then for any* $k \geqslant 1$

$$\mathbb{E}_{\mu_a}\big(\gamma_{k+d}^n(x)|\mathfrak{B}_k\big) = \mathbb{E}_{\mu_a}\big(\gamma_{k+d}^n(x)\big).$$

In particular, for fixed n and fixed $0 \leqslant j \leqslant d-1$, $\{\gamma_{j+ld}^n(x) - \mathbb{E}_{\mu_a}\gamma_{j+ld}^n\}_{l\geqslant 0}$ *is a sequence of martingale differences.*

Proof. By Lemma 3.1, if $|n| < p-1$, we have

$$\mathbb{E}_{\mu_a}\big(\gamma_{k+1}^n(x)|\mathfrak{B}_k\big) = \mathbb{E}_{\mu_a}\big(\gamma_{k+1}^n(x)\big).$$

Write $n = mp + r$ with $-1 \leqslant r \leqslant p-2$. Since $\mathfrak{B}_k \subset \mathfrak{B}_{k+1}$ and $\gamma_{k+2}^p = \gamma_{k+1}$, we have

$$\mathbb{E}_{\mu_a}\big(\gamma_{k+2}^n(x)|\mathfrak{B}_k\big) = \mathbb{E}_{\mu_a}\big[\gamma_{k+1}^m(x)\mathbb{E}_{\mu_a}\big(\gamma_{k+2}^r(x)|\mathfrak{B}_{k+1}\big)|\mathfrak{B}_k\big] = \mathbb{E}_{\mu_a}\big(\gamma_{k+2}^r(x)\big)\mathbb{E}_{\mu_a}\big(\gamma_{k+1}^m(x)|\mathfrak{B}_k\big).$$

Hence if $|m| < p-1$, i.e. $|n-r| < p(p-1)$ and especially if $|n| < p(p-1)-(p-1)$ then

$$\mathbb{E}_{\mu_a}\big(\gamma_{k+2}^n(x)|\mathfrak{B}_k\big) = \mathbb{E}_{\mu_a}\big(\gamma_{k+2}^r(x)\big)\mathbb{E}_{\mu_a}\big(\gamma_{k+1}^m(x)\big) = \mathbb{E}_{\mu_a}\big(\gamma_{k+2}^n(x)\big).$$

By induction, we can prove the desired equality for $|n| < p^{d-1}(p-1) - p^{d-2}(p-1) - \cdots - (p-1)$, i.e. $d > \log_p \frac{p(|n|-1)}{p-2}$. □

Lemma 3.4. *If $d > \log_p \frac{p(|n|-1)}{p-2}$, then for each n fixed and for $0 \leqslant j \leqslant d-1$, $\{\gamma_{j+ld}^n(x) - \mathbb{E}_{\mu_a}\gamma_{j+ld}^n\}_{l\geqslant 0}$ is a L^2-bounded and orthogonal sequence in $L^2(\mu_a)$.*

Proof. The boundedness is trivial. The orthogonality is a consequence of Lemma 3.3. In fact, for $l_1 > l_2 \geqslant 0$,

$$\mathbb{E}_{\mu_a}\big(\gamma_{j+l_1d}^n(x)\bar{\gamma}_{j+l_2d}^n(x)\big) = \mathbb{E}_{\mu_a}\big[\bar{\gamma}_{j+l_2d}^n(x)\mathbb{E}_{\mu_a}\big(\gamma_{j+l_1d}^n(x)|\mathfrak{B}_{k+l_2d}\big)\big] = \mathbb{E}_{\mu_a}\gamma_{j+l_1d}^n(x)\mathbb{E}_{\mu_a}\bar{\gamma}_{j+l_2d}^n(x). \qquad \square$$

Lemma 3.5. *There exists a constant $C > 0$ such that for $n \neq 0$ and $\{\alpha_k\} \in \ell^2$, we have*

$$\int \sup_{N\geqslant 1}\left|\sum_{k=1}^N \alpha_k\big(\gamma_k^n - \mathbb{E}_{\mu_a}\gamma_k^n\big)\right|^2 \mathrm{d}\mu_a \leqslant C\big(1+\log|n|\big)\sum_{k=1}^\infty |\alpha_k|^2.$$

Proof. Take the smallest integer d such that $d > \log_p \frac{p(|n|-1)}{p-2}$. Let $S_N(x) = \sum_{k=1}^N \alpha_k(\gamma_k^n(x) - \mathbb{E}_{\mu_a}\gamma_k^n)$. We have $S_N(x) = \sum_{j=0}^{d-1} X_{j,m_j}(x)$ for some $m_j \leqslant N$, where

$$X_{j,m}(x) = \sum_{0\leqslant l\leqslant m} \alpha_{j+ld}\big(\gamma_{j+ld}^n(x) - \mathbb{E}_{\mu_a}\gamma_{j+ld}^n\big) \quad (m = 1, 2, \ldots).$$

By Lemma 3.3, $\{X_{j,m}\}_{m\geqslant 1}$ is a martingale. Let $X_{j,N}^* = \sup_{0\leqslant m\leqslant N}|X_{j,m}|$ and $S_N^* = \sup_{0\leqslant m\leqslant N}|S_N(x)|$. Then by Doob's inequality, we have

$$\|S_N^*\|_{L^2(\mu_a)} \leqslant \sum_{j=0}^{d-1}\|X_{j,N}^*\|_{L^2(\mu_a)} \leqslant C_1\sum_{j=0}^{d-1}\|X_{j,N}\|_{L^2(\mu_a)}.$$

We finish the proof by using Lemma 3.4 which ensures that

$$\|X_{N,k}\|_{L^2(\mu_a)} \leqslant C_2\sqrt{\sum_{l=0}^N|\alpha_{k+ld}|^2} \leqslant C_2\sqrt{\sum_{k=1}^\infty|\alpha_k|^2}. \qquad \square$$

Let us finish the proof by showing a maximal inequality for the partial sums of the series (2). Let $F_N(x)$ be the Nth partial sum of (2). Let $F^*(x) = \sup_{N\geqslant 1}|F_N(x)|$ and $\varphi_{k,j}(x) = \gamma_k^j(x) - \mathbb{E}_{\mu_a}\gamma_k^j$. We have

$$\big|F_N(x)\big| = \left|\sum_{k=1}^N \alpha_k\sum_{j=1}^\infty c_j^{(k)}\varphi_{k,j}(x)\right| \leqslant \sum_{j=1}^\infty\left|\sum_{k=1}^N \alpha_k c_j^{(k)}\varphi_{k,j}(x)\right|.$$

Then, by Lemma 3.5, $\|F^*(x)\|_{L^2(\mu_a)}$ is bounded by

$$C\sum_{j=1}^\infty\sqrt{1+\log j}\left(\sum_{k=1}^\infty\big|\alpha_k c_j^{(k)}\big|^2\right)^{1/2} \leqslant C\left(\sum_{j=1}^\infty\sqrt{1+\log j}\sup_{k\geqslant 1}\big|c_j^{(k)}\big|\right)\left(\sum_{k=1}^\infty|\alpha_k|^2\right)^{1/2}.$$

4. Other properties

Let us state some other properties of the Riesz products whose proofs will be detailed in a forthcoming paper.

1. Two Riesz products μ_a and μ_b are mutually absolutely continuous if

$$\sum_{n=1}^{\infty} |a_n - b_n|^2 \left(1 + \frac{\cos^2(s_n - t_n)}{2 - |a_n + b_n|}\right) < +\infty$$

where s_n and t_n are respectively the arguments of $a_n + b_n$ and $a_n - b_n$. Formally this condition is better than that of Peyrière's for circle Riesz products.

2. The shift map on $\mathbb{Z}_p$ can be written as $Tx = \frac{x}{p} - \{\frac{x}{p}\}$. The Riesz product μ_a is never T-invariant unless $a_n \equiv 0$. However, if we assume $\sup_{n \geqslant 1} |a_n| < 1$, then μ_a is T-quasi-invariant iff $\sum_{n=1}^{\infty} |a_n - a_{n+1}|^2 < \infty$.

3. Assume that $|a_n| < 1$ for any $n \geqslant 1$. Then the Riesz products μ_a is quasi-Bernoulli in the sense that $\mu_a(B_{n+m}(x)) \approx \mu_a(B_n(x))\mu_a(B_m(T^n x))$ if and only if there exists a number a such that $|a| < 1$ and $\sum_{n=1}^{\infty} |a_n - a| < \infty$. Similar result is previously obtained for inhomogeneous Bernoulli infinite product measures by T. Langlet (personal communication).

4. It is possible to exactly compute the α-order energy of μ_a $(0 < \alpha < 1)$, defined by $\iint |x - y|_p^{-\alpha} \, \mathrm{d}\mu_a(x) \, \mathrm{d}\mu_a(y)$. Consequently, it is possible to calculate the energy dimension of μ_a (see [2] for the notion of energy dimension).

5. When $f_k(z) = z$, the condition $\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$ is also necessary for the series (2) to converge μ_a-almost everywhere ([1]. See also [5] for the circle case).

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