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Homological Algebra

On the p-modular cohomology algebra of a finite p-group and a theorem of Serre

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Abstract

We solve a problem posed by E. Yalçin on the cohomology length of a p-group P, by providing bounds for the group theoretical invariant $\mathbf{s}(P)$ when $p > 2$. These bounds improve the known bounds on the cohomology length of p-groups for odd p. ***To cite this article: K. Thas, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

L' algèbre de cohomologie p-modulaire d'un p-groupe fini. On obtient une borne pour la longueur cohomologique d'un p-groupe fini, $p > 2$, résolvant ainsi un problème posé par E. Yalçin. ***Pour citer cet article : K. Thas, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soient p un nombre premier et P un p-groupe fini. Notons

$$H^*(P) = H^*(P, \mathbb{F}_p) = \bigoplus_{i=0}^{\infty} H^i(P, \mathbb{F}_p)$$

l'algèbre de cohomologie de P à coefficients dans $\mathbb{F}_p$, et $\mathbf{chl}(P)$ la longueur cohomologique de P (voir Section 1).

Je me propose de démontrer le théorème suivant, qui résout un problème posé par E. Yalçin :

Théorème 0.1. *Supposons que $|P| = p^{2n+1}$ pour $p > 2$ et $n \geqslant 3$. Si P est un p-groupe extra-spécial de type* (d)*, on a*

$$p^{n-2}\big(p^2 + (\sqrt{2} - 1) - 5/2\big) + 1 \leqslant \mathbf{chl}(P) \leqslant p^{n-2}\big(p^2 + p - 1\big),$$

ou P_2 est un p-groupe extra-spécial de type (d) *avec $|P_2| = p^5$.*

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1. Introduction and notation

Throughout this Note, for a finite p-group P,

$$H^*(P) = H^*(P, \mathbb{F}_p) = \bigoplus_{i=0}^{\infty} H^i(P, \mathbb{F}_p)$$

will denote the p-modular cohomology algebra of P.

A theorem of J.-P. Serre [7] states that if P is a p-group which is not elementary Abelian, then there exist non-zero elements $u_1, u_2, \ldots, u_m \in H^1(P, \mathbb{F}_p)$ such that

$$\prod_{i=1}^{m} u_i = 0 \quad \text{if } p = 2 \quad \text{and} \quad \prod_{i=1}^{m} \beta(u_i) = 0 \quad \text{if } p > 2, \tag{$*$}$$

where β is the Bockstein homomorphism. The smallest integer m such that relation $(*)$ is satisfied is referred to as the *cohomology length* of P, and is denoted by $\mathbf{chl}(P)$ throughout. Several papers on the calculation of the cohomology length have appeared; see, for instance, O. Kroll [3], J.-P. Serre [8], T. Okuyama and H. Sasaki [6], P.A. Minh [5] and E. Yalçin [10].

Suppose P is a p-group which is not p-central (not all elements of order p belong to the center). Define a *representing set* S of P as a subset that includes at least one non-central element from each maximal elementary Abelian subgroup of P. Then define $\mathbf{s}(P)$ as the minimum cardinality of a representing set in P.

Theorem 1.1. *(E. Yalçin [10]) If P is an extra-special p-group which is not p-central, then* $\mathbf{chl}(P) \leqslant \mathbf{s}(P)$. *Moreover, if P has a self-centralizing maximal elementary Abelian subgroup, then equality holds.*

Theorem 1.1 was applied in [10] to prove the following theorem, which yields the best known bound for $\mathbf{chl}(P)$:

Theorem 1.2. *(E. Yalçin [10]) If P is a p-group and $k = \dim_{\mathbb{F}_p} H^1(P, \mathbb{F}_p)$, then*

$$\mathbf{chl}(P) \leqslant p + 1$$

if $k \leqslant 3$, and for $k > 3$ we have

$$\mathbf{chl}(P) \leqslant (p^2 + p - 1)p^{\lfloor k/2 \rfloor - 2}.$$

In this Note, we give an inductive bound when p is an odd prime which yields a new lower bound. As such, we solve Problem 7.2 of E. Yalçin [10].

The precise statement of the main result will be made in the next section.

2. Extra-special p-groups and statement of the main result

Let P be an extra-special p-group, which in this Note we define by the following group extension:

$$1 \mapsto \mathbb{Z}/p \mapsto P \mapsto V \mapsto 1,$$

V being a vector space over $\mathbb{F}_p$. Put $k = \dim_{\mathbb{F}_p} V$.

If $P \cong P^* \times \mathbb{Z}/p$ for some subgroup $P^* \subset P$, then $\mathbf{chl}(P) = \mathbf{chl}(P^*)$ and $\mathbf{s}(P) = \mathbf{s}(P^*)$. Without loss of generality we suppose that P is not of this form, that is, P has no proper direct factors. Then, if P is represented by the extension class $[\alpha] \in H^1(V, \mathbb{F}_p)$, there exists a basis such that $[\alpha]$ is of one of the following forms (cf. P.A. Minh [5]):

$$\begin{cases} \text{for } p = 2 \text{ and } k = 2n, & \text{(a) } X_1Y_1 + X_2Y_2 + \cdots + X_nY_n \text{ or} \\ \text{for } p = 2 \text{ and } k = 2n, & \text{(b) } X_1^2 + Y_1^2 + X_1Y_1 + X_2Y_2 + \cdots + X_nY_n; \\ \text{for } p = 2 \text{ and } k = 2n+1, & \text{(c) } X_0^2 + X_1Y_1 + X_2Y_2 + \cdots + X_nY_n; \\ \text{for } p > 2 \text{ and } k = 2n, & \text{(d) } X_1Y_1 + X_2Y_2 + \cdots + X_nY_n \text{ or} \\ \text{for } p > 2 \text{ and } k = 2n, & \text{(e) } \beta(X_1) + X_1Y_1 + X_2Y_2 + \cdots + X_nY_n; \\ \text{for } p > 2 \text{ and } k = 2n+1, & \text{(f) } \beta(X_0) + X_1Y_1 + X_2Y_2 + \cdots + X_nY_n. \end{cases}$$

When P is an extra-special group of type (e) or (f), it is well known that $\mathbf{chl}(P) \leqslant p$. In cases (a) and (d), $C_P(E) = E$ for any maximal elementary Abelian subgroup $E \leqslant P$, so that equality holds in Theorem 1.1. For case (a), E. Yalçin obtained the best possible bound in [10]. Theorem 1.2 represents a general bound which is valid for all cases. In that same paper, Problem 7.2 asks for a calculation of $\mathbf{s}(P) = \mathbf{chl}(P)$ in terms of n and p for groups of type (d).

This calculation is the objective of the present note, so as to obtain at the same time a bound for $\mathbf{chl}(P)$ of such a group P, and more generally, of any p-group for odd p.

Theorem 2.1. *Let P be an extra-special p-group of order p^{2n+1} where p is odd. If P is of type* (d)*, we have $p^{n-2}(p^2 + (\sqrt{2} - 1) - 5/2) + 1 \leqslant \mathbf{chl}(P) \leqslant p^{n-2}(p^2 + p - 1)$. Moreover,*

$$p^{n-2}\big(\mathbf{chl}(P_2) - 1\big) + 1 \leqslant \mathbf{chl}(P) \leqslant p^{n-2} \cdot \mathbf{chl}(P_2),$$

where P_2 is an extra-special p-group of type (d) *with order p^5.*

In the rest of this Note, we will only consider extra-special groups of type (d); if P is a p-group, p odd, which is not elementary Abelian and $k = \dim_{\mathbb{F}_p}(H^1(P, \mathbb{F}_p)) \in \{2n, 2n+1\}$, then P has a factor group P_n isomorphic to some group of type (d), (e) or (f). So $\mathbf{chl}(P) \leqslant \mathbf{chl}(P_n)$.

3. Proof of the main result

Suppose $P = P_n$ is a group of type (d), and note that $|P_n| = p^{2n+1}$. Suppose $\mathcal{W} = \mathcal{W}(2n-1, p)$ is the variety in the $(2n-1)$-dimensional projective space $\mathbf{PG}(2n-1, p)$ over $\mathbb{F}_p$ which is determined by the bilinear alternating form induced by the quadratic form displayed in (d) of the previous section. So $\mathcal{W}$ is a 'non-singular symplectic polar space'. Define $\mathbf{s}(\mathcal{W})$ as the minimal cardinality of a set of points of $\mathbf{PG}(2n-1, p)$ which meets every maximal totally isotropic subspace ('generator') of $\mathcal{W}$. Then it holds that $\mathbf{s}(P_n) = \mathbf{s}(\mathcal{W})$ [11]. Note that the Witt index of $\mathcal{W}$ is n, so that $(n-1)$ is the dimension of a generator of $\mathcal{W}$. An easy counting argument[1] shows that the number of points of such a set is at least $p^n + 1$ [9], and in case of equality, one speaks of an 'ovoid' of $\mathcal{W}(2n-1, p)$. More generally, if B is a point set of $\mathcal{W}(2n-1, p)$ meeting each generator, call it a *blocking set*.

Theorem 3.1. *(See, e.g., the survey paper [9] for (i) and [2] for (ii).)*

(i) $\mathcal{W}(2m+1, p)$ *has no ovoids for* $m \geqslant 1$.
(ii) $\mathbf{s}(\mathcal{W}(3, p)) \geqslant p^2 + (\sqrt{2} - 1)p - 3/2$.

We need a good bound for the size of a blocking set of symplectic polar spaces, which we will try to obtain now.

Let $\mathcal{W}(2r-1, p) \subset \mathcal{W}(2n-1, p)$, where we assume $r \geqslant 2$ and $n \geqslant 3$. Suppose η is the symplectic polarity defined by $\mathcal{W}(2n-1, p)$. Now let $\pi \subset \mathcal{W}(2n-1, p)$ be a projective subspace of $\mathbf{PG}(2n-1, p)$ of dimension $n-r-1$, such that $\mathcal{W}(2r-1, p) \subset \pi^\eta$ ($\mathcal{W}(2r-1, p)$ has no point in common with π). Suppose B is a blocking set of $\mathcal{W}(2r-1, p)$. Define B^* as the set of points of $\mathcal{W}(2n-1, p)$ which are on lines that contain a point of B and one of π, but not contained in π (B^* is a 'truncated cone' with base π and vertex B). Then one observes two facts:

(i) $|B^*| = \frac{p^{n-r}-1}{p-1}(p-1)|B| + |B| = |B|p^{n-r}$;
(ii) B^* contains at least one point of any generator of $\mathcal{W}(2n-1, p)$.

Now put $2r - 1 = 3$, and consider a point x of $\mathcal{W}(3, p)$. Let y be a point of $\mathcal{W}(3, p)$ which is not collinear with x on $\mathcal{W}(3, p)$. For any point z of $\mathcal{W}(3, p)$, denote by $z^\perp$ the set of points which are collinear with z on $\mathcal{W}(3, p)$ (including z). Also, if A is a point set of $\mathcal{W}(3, p)$, write $A^\perp$ for $\cap_{a \in A} a^\perp$, and $A^{\perp\perp}$ for $(A^\perp)^\perp$. Then clearly

$$B = \big(\big(x^\perp \setminus \{x, y\}^\perp\big) \cup \{x, y\}^{\perp\perp}\big) \setminus \{x\}$$

[1] Let S be a set of points of $\mathcal{W}(2n-1, p)$ meeting every generator. Count in two ways the number of pairs (p, π), where $p \in S$, π is a generator and $p \in \pi$. Then $|S|\cdot$(number of generators containing x) $\geqslant$ (total number of generators)$\cdot$1.

is a blocking set of $\mathcal{W}(3, p)$ of size $p^2 + p - 1$. So we get

$$|B^*| = p^{n-2}\big(p^2 + p - 1\big),$$

and hence $\mathbf{s}(\mathcal{W}(2n-1, p))$ is at most $p^{n-2} \cdot \mathbf{s}(\mathcal{W}(3, p))$. (Note that as thus we have obtained an alternative proof of a result of [10].)

Now suppose that for $k < n$, $k \in \mathbb{N} \setminus \{0, 1\}$, $\mathbf{s}(\mathcal{W}(2k-1, p)) \geqslant p^{k-2}(\mathbf{s}(\mathcal{W}(3, p)) - 1) + 1$.

We will use this induction hypothesis to show that the inequality also holds for $k = n$. The following argument was first made by K. Metsch in a slightly more particular setting (cf. [4, p. 284]), but was never published.

Let B^* be a blocking set of $\mathcal{W}(2n-1, p)$ which does not contain a blocking set of strictly smaller size. Then there is a generator of $\mathcal{W}(2n-1, p)$ that meets B^* in a unique point x. Each of the $\alpha := p^{n-1} + \cdots + p^2 + p$ other points of this generator sees in its quotient a blocking set of a $\mathcal{W}(2n-3, p)$, so besides x at least $\beta := p^{n-3}(\mathbf{s}(\mathcal{W}(3, p)) - 1)$ further points. As every point of $B^* \setminus \{x\}$ is counted at most $\gamma := p^{n-2} + \cdots + p + 1$ times, we have $|B^* \setminus x| \geqslant \alpha\beta/\gamma$.

This proves the main result.

Note that the geometrical results of this section are still valid if we replace the field $\mathbb{F}_p$ by $\mathbb{F}_q$ when q is any odd prime power.

The minimal size of a blocking set of $\mathcal{W}(3, 3)$, respectively $\mathcal{W}(3, 5)$, equals 11, respectively 29 | see [1, Remark 10]. So for these cases, we have $10 \times 3^{n-2} + 1 \leqslant \mathbf{s}(\mathcal{W}(2n-1, 3)) \leqslant 11 \times 3^{n-2}$ and $28 \times 5^{n-2} + 1 \leqslant \mathbf{s}(\mathcal{W}(2n-1, 5)) \leqslant 29 \times 5^{n-2}$.

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