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Algebraic Geometry

On the D-affinity of quadrics in positive characteristic ☆

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Abstract

In this Note we deal with the rings of differential operators on quadrics of low dimension in positive characteristic. We prove a vanishing theorem for the first term of the p-filtration on the rings of differential operators on such quadrics. Such a vanishing is a necessary condition for the D-affinity of these varieties. We also discuss applications of this result to derived categories of coherent sheaves. ***To cite this article: A. Samokhin, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sur la D-affinité de quadriques en caractéristique positive. Dans cette Note nous étudions les anneaux des opérateurs différentiels sur les quadriques en petite dimension en caractéristique positive. Nous démontrons un théorème d'annulation pour le premier terme de la p-filtration sur les anneaux des opérateurs différentiels sur ces quadriques. Une telle annulation est une condition nécessaire pour que ces variétés soient D-affines. Enfin, nous discutons des applications de ce résultat à des catégories dérivées des faisceaux cohérents. ***Pour citer cet article : A. Samokhin, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit X une variété lisse sur un corps k algébriquement clos de caractéristique quelconque, $\mathcal{O}_X$ le faisceau structural de X, et $\mathcal{D}_X$ le faisceau des opérateurs différentiels sur X. On note $\mathcal{M}(\mathcal{D})$ la catégorie des $\mathcal{D}$-modules (à gauche) qui sont quasi-cohérents comme des $\mathcal{O}_X$-modules. La variété X est dite D-affine si les deux conditions suivantes sont satisfaites : (i) pour chaque $\mathcal{F} \in \mathcal{M}(\mathcal{D})$ on a $H^k(X, \mathcal{F}) = 0$ pour $k > 0$, et (ii) $\mathcal{F}$ est engendré par ces sections globales sur $\mathcal{D}_X$.

Beilinson et Bernstein ont démontré dans [3] que les espaces homogènes des groupes de Lie étaient D-affines si le corps k est de caractéristique zéro. Par contre, comme l'ont découvert Kashiwara et Lauritzen dans [9], les espaces homogènes en caractéristique positive ne sont pas D-affines en général. Dans cet article nous étudions les quadriques de dimension $\leqslant 4$ en caractéristique positive (ce sont les espaces homogènes du groupe orthogonal). Nous prouvons qu'une condition nécessaire est satisfaite pour que de telles quadriques soient D-affines. Ce théorème, qui est le résultat

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principal de cet article, est similaire à celui d'Andersen et Kaneda de [1] où le cas de la variété de drapeaux du groupe du type $\mathbf{B}_2$ a été traité.

1. Introduction

Throughout we fix an algebraically closed field k of characteristic p. Let X be a smooth variety over k, and $\mathsf{F}: X \to X$ the absolute Frobenius morphism. Note that since X is smooth, the sheaf $\mathsf{F}_*\mathcal{O}_X$ is locally free (here F_* is the direct image functor under F). Recall that the sheaf of differential operators $\mathcal{D}_X$ on X admits the p-filtration defined as follows. For $r \geqslant 1$ denote $\mathcal{D}_r$ the endomorphism bundle $End_{\mathcal{O}_X}(\mathsf{F}^r_*\mathcal{O}_X)$ (here $\mathsf{F}^r = \mathsf{F} \circ \cdots \circ \mathsf{F}$ is the r-iteration of F). Then $\mathcal{D}_X = \bigcup \mathcal{D}_r$. Recall that X is said to be Frobenius split if the sheaf $\mathcal{O}_X$ is a direct summand in $\mathsf{F}_*\mathcal{O}_X$. Homogeneous spaces of linear algebraic groups are Frobenius split [11]. For a Frobenius split variety X and for all $i \in \mathbb{N}$ the following property holds (Proposition, Sec. 1, [1]):

$$H^i(X, \mathcal{D}_X) = 0 \Leftrightarrow H^i(X, \mathcal{D}_r) = 0 \text{ for any } r \in \mathbb{N}. \tag{1}$$

Below we consider the case of smooth quadrics of dimension $\leqslant 4$, which are homogeneous spaces of orthogonal groups, and prove a necessary condition for these quadrics to be D-affine (cf. the main theorem of [1]):

Theorem 1.1. *Let Q_n be a quadric of dimension $n \leqslant 4$. Then $H^i(\mathsf{Q}_n, \mathcal{D}_1) = 0$ for $i > 0$.*

2. Preliminaries

Let X be a smooth variety over k, and $\mathrm{Coh}(X)$ the category of coherent sheaves on X. The direct image functor F_* has a right adjoint functor $\mathsf{F}^!$ in $\mathrm{Coh}(X)$ [8]. The duality theory for the finite flat morphism F yields [8]:

Lemma 2.1. *The functor $\mathsf{F}^!$ is isomorphic to*

$$\mathsf{F}^!(?) = \mathsf{F}^*(?) \otimes \omega_X^{1-p}, \tag{2}$$

where ω_X is the canonical invertible sheaf on X.

For any $i \geqslant 0$ one has an isomorphism, the sheaf $\mathsf{F}_*\mathcal{O}_X$ being locally free:

$$H^i\big(X, End_{\mathcal{O}_X}(\mathsf{F}_*\mathcal{O}_X)\big) = \mathrm{Ext}^i(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X). \tag{3}$$

From Lemma 2.1 we obtain:

$$\mathrm{Ext}^i_X(\mathsf{F}_*\mathcal{O}_X, \mathsf{F}_*\mathcal{O}_X) = \mathrm{Ext}^i\big(\mathcal{O}_X, \mathsf{F}^!\mathsf{F}_*(\mathcal{O}_X)\big) = H^i\big(X, \mathsf{F}^*\mathsf{F}_*\mathcal{O}_X \otimes \omega_X^{1-p}\big). \tag{4}$$

Consider the fibered square:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "\mathsf{F}"] \\ X \arrow[r, "\mathsf{F}"] & X \end{tikzcd}$$

Let $i : \Delta \hookrightarrow X \times X$ be the diagonal embedding, and $\tilde{i}$ the embedding $\tilde{X} \hookrightarrow X \times X$ obtained from the above fibered square.

Lemma 2.2. *One has an isomorphism of sheaves*:

$$\tilde{i}_*\mathcal{O}_{\tilde{X}} = (\mathsf{F} \times \mathsf{F})^*(i_*\mathcal{O}_\Delta). \tag{5}$$

Here $\mathsf{F} \times \mathsf{F}$ is the Frobenius morphism on $X \times X$. The lemma is equivalent to saying that the fibered product $\tilde{X}$ is isomorphic to the Frobenius neighbourhood of the diagonal $\Delta \subset X \times X$ (cf. [4]).

Lemma 2.3. *There is an isomorphism of cohomology groups*:

$$H^i\big(X, \mathsf{F}^*\mathsf{F}_*(\mathcal{O}_X)\otimes\omega_X^{1-p}\big) = H^i\big(X\times X, (\mathsf{F}\times\mathsf{F})^*(i_*\mathcal{O}_\Delta)\otimes\big(\omega_X^{1-p}\boxtimes\mathcal{O}_X\big)\big). \tag{6}$$

Proof. Recall that the sign $\boxtimes$ in the right-hand side of (6) denotes the external tensor product. Applying the flat base change to the above fibered square, we get an isomorphism of functors, the morphism F being flat:

$$\mathsf{F}^*\mathsf{F}_* = \pi_{1*}\pi_2^*. \tag{7}$$

Note that all the functors F_*, F^*, π_{1*}, and π_2^* are exact, the morphism F being affine. The isomorphism (7) implies an isomorphism of cohomology groups

$$H^i\big(X, \mathsf{F}^*\mathsf{F}_*(\mathcal{O}_X)\otimes\omega_X^{1-p}\big) = H^i\big(X, \pi_{1*}\pi_2{}^*(\mathcal{O}_X)\otimes\omega_X^{1-p}\big). \tag{8}$$

By the projection formula the right-hand side group in (8) is isomorphic to $H^i(\tilde{X}, \pi_2{}^*\mathcal{O}_X\otimes\pi_1{}^*\omega_X^{1-p})$. Let p_1 and p_2 be the projections of $X\times X$ onto the first and the second component respectively. One has $\pi_1 = p_1\circ\tilde{i}$ and $\pi_2 = p_2\circ\tilde{i}$. Hence an isomorphism of sheaves

$$\pi_2^*\mathcal{O}_X\otimes\pi_1^*\omega_X^{1-p} = \tilde{i}^*\big(p_2^*\mathcal{O}_X\otimes p_1^*\omega_X^{1-p}\big) = \tilde{i}^*\big(\omega_X^{1-p}\boxtimes\mathcal{O}_X\big). \tag{9}$$

From (9) and the projection formula one obtains

$$H^i\big(\tilde{X}, \pi_2^*\mathcal{O}_X\otimes\pi_1^*\omega_X^{1-p}\big) = H^i\big(\tilde{X}, \tilde{i}^*\big(\omega_X^{1-p}\boxtimes\mathcal{O}_X\big)\big) = H^i\big(X\times X, \tilde{i}_*\mathcal{O}_{\tilde{X}}\otimes\big(\omega_X^{1-p}\boxtimes\mathcal{O}_X\big)\big). \tag{10}$$

Using Lemma 2.2 we get the statement. □

Corollary 2.4. *One has as well an isomorphism*:

$$H^i\big(X, \mathsf{F}^*\mathsf{F}_*(\mathcal{O}_X)\otimes\omega_X^{1-p}\big) = H^i\big(X\times X, (\mathsf{F}\times\mathsf{F})^*(i_*\mathcal{O}_\Delta)\otimes\big(\mathcal{O}_X\boxtimes\omega_X{}^{1-p}\big)\big). \tag{11}$$

Finally, we need a well-known lemma (e.g., SGA3):

Lemma 2.5. *If a sheaf $\mathcal{F}$ on a variety X is quasi-isomorphic to a bounded complex $\mathcal{F}^\bullet$ then $H^i(X,\mathcal{F}) = 0$ provided that $H^p(X,\mathcal{F}^q) = 0$ for all $p+q=i$.*

3. Vanishing

In this section we prove Theorem 1.1. Let Q_n be a smooth quadric of dimension $n\leqslant 4$. Note that Q_1 is isomorphic to $\mathbb{P}^1$, and Q_2 is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$. For projective spaces in positive characteristic the D-affinity was proved in [6] by B. Haastert. We need, therefore, to consider the case $n=3$ and $n=4$. We treat the case of three-dimensional quadrics, the four-dimensional case being similar. Let G be a simply connected simple algebraic group over k of type $\mathbf{B}_2$, B a Borel subgroup of G, and $\mathrm{P}\subset\mathrm{G}$ a parabolic subgroup such that G/P is isomorphic to a quadric Q_3. Denote $\pi:\mathrm{G/B}\to\mathrm{G/P}$ the projection. There exists a line bundle $\mathcal{L}$ over G/B such that $\mathrm{R}^0\pi_*\mathcal{L}$ is a rank two vector bundle over Q_3, the spinor bundle. Denote $\mathcal{U}$ the dual bundle to $\mathrm{R}^0\pi_*\mathcal{L}$. There is a short exact sequence:

$$0\to\mathcal{U}\to V\otimes\mathcal{O}_{\mathsf{Q}_3}\to\mathcal{U}^*\to 0, \tag{12}$$

where V is a symplectic k-vector space of dimension 4, i.e. a space equipped with a non-degenerate skew form $\omega\in\bigwedge^2 V^*$.

Lemma 3.1. *Let Q_3 be a smooth quadric of dimension 3, and $i:\Delta\subset\mathsf{Q}_3\times\mathsf{Q}_3$ the diagonal embedding. Then the following complex is exact*:

$$0\to\mathcal{U}\boxtimes\mathcal{U}(-2)\to\Psi_2\boxtimes\mathcal{O}_{\mathsf{Q}_3}(-2)\to\Psi_1\boxtimes\mathcal{O}_{\mathsf{Q}_3}(-1)\to\mathcal{O}_{\mathsf{Q}_3}\boxtimes\mathcal{O}_{\mathsf{Q}_3}\to i_*\mathcal{O}_\Delta\to 0. \tag{13}$$

Here Ψ_i for $i=1,2$ are some vector bundles on Q_3, which have right resolutions

$$0\to\Psi_i\to B_i\otimes_k\mathcal{O}_{\mathsf{Q}_3}\to\cdots\to B_1\otimes_k\mathcal{O}_{\mathsf{Q}_3}(i-1)\to\mathcal{O}_{\mathsf{Q}_3}(i)\to 0, \tag{14}$$

and B_i are k-vector spaces.

Proof. One can show, without recurrence to Kapranov's theorem for quadrics ([7], Theorem 4.10), that the collection of bundles $\mathcal{U}(-2), \mathcal{O}_{\mathsf{Q}_3}(-2), \mathcal{O}_{\mathsf{Q}_3}(-1), \mathcal{O}_{\mathsf{Q}_3}$ is a complete exceptional collection [12] in $\mathrm{D}^b(\mathsf{Q}_3)$, the bounded derived category of coherent sheaves on Q_3, and then use a purely categorical construction of a resolution of the diagonal [12] that can be performed over fields of arbitrary characteristic. The bundles $\mathcal{U}, \Psi_2, \Psi_1, \mathcal{O}_{\mathsf{Q}_3}$ are terms of the so-called right dual collection. Otherwise, one can suitably modify Kapranov's argument so that it will hold over fields of positive characteristic (for more details see [13]). $\square$

Theorem 3.2. *One has* $\mathrm{Ext}^i(\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_3}, \mathsf{F}_*\mathcal{O}_{\mathsf{Q}_3}) = 0$ *for* $i > 0$.

Proof. By Corollary 2.4 we need to show that $H^i(\mathsf{Q}_3 \times \mathsf{Q}_3, (\mathsf{F}\times\mathsf{F})^*(i_*\mathcal{O}_\Delta)\otimes(\mathcal{O}_{\mathsf{Q}_3}\boxtimes\omega_{\mathsf{Q}_3}^{1-p})) = 0$ for $i > 0$. Recall that $\omega_{\mathsf{Q}_3} = \mathcal{O}_{\mathsf{Q}_3}(-3)$. For a line bundle $\mathcal{L}$ one has $\mathsf{F}^*\mathcal{L} = \mathcal{L}^p$. Taking the pull-back under $(\mathsf{F}\times\mathsf{F})^*$ of the resolution (13), we get a complex of coherent sheaves in degrees $-3, \ldots, 0$:

$$0 \to \mathsf{F}^*\mathcal{U}\boxtimes\mathsf{F}^*\big(\mathcal{U}(-2)\big) \to \mathsf{F}^*\Psi_2\boxtimes\mathcal{O}_{\mathsf{Q}_3}(-2p) \to \mathsf{F}^*\Psi_1\boxtimes\mathcal{O}_{\mathsf{Q}_3}(-p) \to \mathcal{O}_{\mathsf{Q}_3}\boxtimes\mathcal{O}_{\mathsf{Q}_3} \to 0. \tag{15}$$

Denote $C^\bullet$ the complex (15), and let $\tilde{C}^\bullet$ be the tensor product of $C^\bullet$ with the invertible sheaf $\mathcal{O}_{\mathsf{Q}_3}\boxtimes\omega_{\mathsf{Q}_3}^{1-p}$. Then $\tilde{C}^\bullet$ is quasiisomorphic to the sheaf $(\mathsf{F}\times\mathsf{F})^*(i_*\mathcal{O}_\Delta)\otimes(\mathcal{O}_{\mathsf{Q}_3}\boxtimes\omega_{\mathsf{Q}_3}^{1-p})$. We thus have to compute the hypercohomology of $\tilde{C}^\bullet$. There is a distinguished triangle in $\mathrm{D}^b(\mathsf{Q}_3)$:

$$\cdots \longrightarrow \mathsf{F}^*\mathcal{U}\boxtimes(\mathsf{F}^*\mathcal{U}^*\otimes\omega_{\mathsf{Q}_3})[3] \longrightarrow \tilde{C}^\bullet \longrightarrow \sigma_{\geqslant -2}(\tilde{C}^\bullet) \xrightarrow{[1]} \cdots. \tag{16}$$

Here $\sigma_{\geqslant -2}$ is the stupid truncation, and [1] is a shift functor in $\mathrm{D}^b(\mathsf{Q}_3)$. Let us first look at the truncated complex $\sigma_{\geqslant -2}(\tilde{C}^\bullet)$, which is quasiisomorphic to

$$0 \to \mathsf{F}^*\Psi_2\boxtimes\mathcal{O}_{\mathsf{Q}_3}(p-3) \to \mathsf{F}^*\Psi_1\boxtimes\mathcal{O}_{\mathsf{Q}_3}(2p-3) \to \mathcal{O}_{\mathsf{Q}_3}\boxtimes\mathcal{O}_{\mathsf{Q}_3}(3p-3) \to 0. \tag{17}$$

Let us show that $H^k(\mathsf{Q}_3, \mathsf{F}^*\Psi_i) = 0$ for $k > i$ and $i = 1, 2$. Indeed, the sheaves Ψ_1 and Ψ_2 have resolutions as in (14). By the Kempf vanishing theorem [10], effective line bundles on homogeneous spaces have no higher cohomology. The terms of resolutions (14) for $n = 3$ and $i = 1, 2$ consist of direct sums of effective line bundles and of direct sums of the sheaf $\mathcal{O}_{\mathsf{Q}_3}$. Positivity of a line bundle is preserved under the Frobenius pullback, hence the terms of the resolutions of sheaves $\mathsf{F}^*\Psi_i$ have only zero cohomology. Applying Lemma 2.5, we obtain the above vanishing. Further, line bundles that occur in the second argument of the terms of the complex (17) are effective for $p > 3$. For $p = 2$ and $p = 3$ the line bundle occurring in the leftmost term of (17) is isomorphic to $\mathcal{O}_{\mathsf{Q}_3}$ and $\mathcal{O}_{\mathsf{Q}_3}(-1)$, respectively. For all p these line bundles have no higher cohomology. Using again Lemma 2.5, we get $\mathbb{H}^i(\sigma_{\geqslant -2}(\tilde{C}^\bullet)) = 0$ for $i > 0$.

Consider now the bundle $\mathsf{F}^*\mathcal{U}\boxtimes(\mathsf{F}^*\mathcal{U}^*\otimes\omega_{\mathsf{Q}_3})$. The proof of Theorem 3.2 will be completed if we show that $H^i(\mathsf{Q}_3\times\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}\boxtimes(\mathsf{F}^*\mathcal{U}^*\otimes\omega_{\mathsf{Q}_3})) = 0$ for $i \neq 3$. Indeed, taking cohomology of the triangle (16), we see that $\mathbb{H}^i(\tilde{C}^\bullet) = 0$ for $i > 0$. $\square$

Lemma 3.3. *One has* $H^i(\mathsf{Q}_3\times\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}\boxtimes(\mathsf{F}^*\mathcal{U}^*\otimes\omega_{\mathsf{Q}_3})) = 0$ *for* $i \neq 3$.

Proof. By the Serre duality one has $H^k(\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}) = H^{3-k}(\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}^*\otimes\omega_{\mathsf{Q}_3})$, for $0 \leqslant k \leqslant 3$. Using the Künneth formula, we see that it is sufficient to show that $H^k(\mathsf{Q}_3, \mathsf{F}^*\mathcal{U})$ is non-zero for just one value of k. In fact, $H^k(\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}) = 0$ if $k \neq 2$. To prove this, apply the Frobenius pullback F^* to (12). We get

$$0 \to \mathsf{F}^*\mathcal{U} \to \mathsf{F}^*V\otimes\mathcal{O}_{\mathsf{Q}_3} \to \mathsf{F}^*\mathcal{U}^* \to 0. \tag{18}$$

We need a particular case of the following theorem due to Carter and Lusztig ([2], Theorem 6.2):

Theorem 3.4. *Let k be a field of characteristic p and let E be a vector space of dimension r over k. Then there exists an exact sequence of $GL_r(k)$-modules*

$$0 \to \mathsf{F}^*E \to \mathsf{S}^p(E) \to \Sigma^{(p-1,1)}(E) \to \cdots \to \Sigma^\lambda(E) \to 0 \tag{19}$$

where $\lambda = (p - \min(p-1, r-1), 1, 1, \ldots)$. Here Σ^λ are Schur functors. In particular, if a partition λ is equal to $(p, 0, \ldots)$ then Σ^λ is equal to p-th symmetric power functor S^p. This construction globalizes to produce a resolution of the Frobenius pull-back of a vector bundle [2].

Theorem 3.4, applied to the bundle $\mathcal{U}^*$, furnishes a short exact sequence, the bundle $\mathcal{U}^*$ having rank 2:

$$0 \to \mathsf{F}^*\mathcal{U}^* \to \mathsf{S}^p\mathcal{U}^* \to \mathsf{S}^{p-2}\mathcal{U}^* \otimes \mathcal{O}_{\mathsf{Q}_3}(1) \to 0. \tag{20}$$

The vector bundles $\mathsf{S}^p\mathcal{U}^*$ and $\mathsf{S}^{p-2}\mathcal{U}^* \otimes \mathcal{O}_{\mathsf{Q}_3}(1)$ are pushforwards onto Q_3 of effective line bundles $\mathcal{L}_{\chi_1}$ and $\mathcal{L}_{\chi_1}$ over G/B that correspond to the weights $\chi_1 = (p,0)$ and $\chi_2 = (p-1,1)$, respectively. Moreover, the restrictions of $\mathcal{L}_{\chi_1}$ and $\mathcal{L}_{\chi_1}$ to the fibers of π have positive degrees. Considering the Leray spectral sequence for the morphism π and using the Kempf vanishing theorem, we obtain that the bundles $\mathsf{S}^p\mathcal{U}^*$ and $\mathsf{S}^{p-2}\mathcal{U}^* \otimes \mathcal{O}_{\mathsf{Q}_3}(1)$ both have no higher cohomology. By Lemma 2.5 one then has $H^i(\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}^*) = 0$ for $i > 1$. Moreover, there is an isomorphism $H^0(\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}^*) = V$. Considering the long exact cohomology sequence associated to (18), we get the statement of Lemma 3.3. Hence, it remains to prove the isomorphism $H^0(\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}^*) = V$. Indeed, taking cohomology of (20), we get a short exact sequence:

$$0 \to H^0(\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}^*) \to H^0(\mathsf{Q}_3, \mathsf{S}^p\mathcal{U}^*) \to H^0\big(\mathsf{Q}_3, \mathsf{S}^{p-2}\mathcal{U}^* \otimes \mathcal{O}_{\mathsf{Q}_3}(1)\big). \tag{21}$$

Using again the Kempf vanishing theorem, we obtain $H^0(\mathsf{Q}_3,\mathsf{S}^p\mathcal{U}^*) = \mathsf{S}^p V^*$ and $H^0(\mathsf{Q}_3, \mathsf{S}^{p-2}\mathcal{U}^* \otimes \mathcal{O}_{\mathsf{Q}_3}(1)) = \Sigma^{(p-1,1)}V^*$. On the other hand, the resolution (19), applied to the vector space V^*, furnishes a short exact sequence:

$$0 \to \mathsf{F}^*V^* \to \mathsf{S}^p V^* \to \Sigma^{(p-1,1)}V^*. \tag{22}$$

Comparing (21) and (22), and taking into account the above isomorphisms, we get the isomorphism

$$H^0(\mathsf{Q}_3, \mathsf{F}^*\mathcal{U}^*) = V.$$

Finally, one has an isomorphism $V = V^*$ since V is symplectic. This implies Lemma 3.3. □

The proof of Theorem 1.1 in the case of four-dimensional quadrics is essentially the same, the only difference being that there are two spinor bundles in this case.

4. Tilting bundles

This paper grew out of an attempt to understand possible applications of the work [4] to the derived categories of coherent sheaves on homogeneous spaces. It follows from one of the main results of [4] that for a homogeneous space $\mathbf{G}/\mathbf{P}$ over k (here $\operatorname{char} k = p > h$, where h is the Coxeter number of a semisimple algebraic group $\mathbf{G}$ over k) the bundle $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{P}}$ is a generator [5] in $\mathrm{D}^b(\mathbf{G}/\mathbf{P})$, the derived category of coherent sheaves on $\mathbf{G}/\mathbf{P}$. Theorem 1.1 in this framework can be rephrased as saying that the bundle $\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_n}$ is a tilting bundle on Q_n for $n \leqslant 4$. By [5] one then has an equivalence of categories:

$$\mathrm{D}^b(\mathsf{Q}_n) \simeq \mathrm{D}^b\big(End(\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_n}) - \mathrm{mod}\big), \tag{23}$$

for $n \leqslant 4$. It is this kind of equivalence that we would like to get for all homogeneous spaces. In a forthcoming paper [13] we show, in particular, an equivalence of the type (23) for quadrics of arbitrary dimension, and for some other homogeneous spaces (including the flag variety in type $\mathbf{B}_2$, cf. [1]). This gives a hope that for a homogeneous space $\mathbf{G}/\mathbf{P}$ the bundle $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{P}}$ may have some relevance to tilting bundles on such a variety (such bundles are conjectured to exist on all homogeneous spaces).

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