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Partial Differential Equations

On the controllability of linear parabolic equations with an arbitrary control location for stratified media

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Abstract

We prove a null controllability result with an arbitrary control location in dimension greater than or equal to two for a class of linear parabolic operators with non-smooth coefficients. The coefficients are assumed to be smooth in all but one directions. ***To cite this article: A. Benabdallah et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

De la contrôlabilité des équations paraboliques linéaires avec une localisation arbitraire du contrôle pour des milieux stratifiés. Nous prouvons un résultat de contrôlabilité à zéro avec une localisation arbitraire de la zone de contrôle en dimension plus grande que deux pour une classe d'opérateurs paraboliques avec des coefficients non réguliers. Les coefficients sont supposés singuliers dans une seule direction. ***Pour citer cet article : A. Benabdallah et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

La question de la contrôlabilité à zéro des équations paraboliques à coefficients réguliers a été résolue dans les années 1990 [9,8]. Le cas de coefficients discontinus dans la partie principale de l'opérateur a été abordé dans [6], pour des coefficients $\mathscr{C}^1$ par morceaux, au moyen d'une *inégalité de Carleman* en imposant à la zone d'observation d'être située dans la région où le coefficient est le plus petit. Récemment, une inégalité de Carleman sans restriction sur la zone d'observation a été démontrée en dimension 1 d'espace pour des coefficients $\mathscr{C}^1$ par morceaux [3,4], puis pour des coefficients à variations bornées (BV) [10]. De telles inégalités conduisent à des résultats de contrôlabilité pour des équations paraboliques semilinéaires et à des résultats de stabilité pour des problèmes inverses. La question de l'existence d'une inégalité de Carleman en dimension $n \geqslant 2$, sans contrainte sur la zone d'observation reste ouverte. Ici, nous résolvons le problème plus restreint de la contrôlabilité d'une classe particulière d'opérateurs. L'hypothèse principale est que les coefficients soient réguliers relativement à $(n-1)$ des variables d'espace, ce qui inclut les

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milieux stratifiés. La démonstration se fonde à la fois sur les estimations de Carleman en dimension 1 de [4,10] et la méthode de [9]. On ne traite donc que le cas linéaire.

Soit Ω un ouvert borné de $\mathbb{R}^n$, avec $\Omega = \Omega' \times (0,H)$, où Ω' est un ouvert non vide de $\mathbb{R}^{n-1}$ de frontière $\mathscr{C}^2$. On pose $x=(x',x_n)\in\Omega'\times(0,H)$. Soit $B(x)$, une matrice $n\times n$ qui possède la forme diagonale par blocs suivante : $B(x',x_n)=\operatorname{diag}(c_1(x_n)C_1(x'), c_2(x_n))$ où $c_1\in L^\infty(0,H)$, $c_2\in BV(0,H)$ et $C_1\in\mathscr{C}^1(\overline{\Omega'}, M_{n-1}(\mathbb{R}))$. La matrice $C_1(x')$ est symétrique. On suppose $0<c_{\min}\leqslant c_i(x_n)\leqslant c_{\max}$, $x_n\in(0,H)$, $i=1,2$, et $0<c_{\min}I_{n-1}\leqslant C_1(x')\leqslant c_{\max}I_{n-1}$, $x'\in\Omega'$, où I_k est la matrice identité d'ordre k, ce qui implique une ellipticité uniforme. Soit l'opérateur autoadjoint $A=-\nabla_x\cdot(B\nabla_x)$ dans $L^2(\Omega)$ de domaine $D(A)=\{u\in H_0^1(\Omega);\ \nabla_x\cdot(B\nabla_x u)\in L^2(\Omega)\}$. Soit $T>0$ et $Q_T=(0,T)\times\Omega$. On considère le système (2) où $q_0\in L^2(\Omega)$ et ω est un ouvert non vide de Ω tel que $\omega\Subset\Omega$. On choisit ω' un ouvert non vide de Ω' et ω_n un ouvert non vide de $(0,H)$ tels que $\omega'\times\omega_n\subset\omega$. On suppose de plus que le coefficient c_2 est de classe $\mathscr{C}^1$ dans un ouvert non vide de ω_n. Notre résultat principal de contrôlabilité à zéro est le suivant :

Théorème 1. *Sous les hypothèses précédentes, pour tout $T>0$ et tout $q_0\in L^2(\Omega)$, il existe $u\in L^2((0,T)\times\Omega)$ tel que la solution associée, q, du système* (2) *vérifie $q(T)=0$ p.p. dans Ω.*

Un conséquence immédiate est l'observabilité pour le système homogène adjoint du système (2). Nous donnons maintenant un esquisse de la démonstration du Théorème 1.

On introduit les espaces $H_k=\overline{\operatorname{vect}\{\varphi_{k,p};\ p\geqslant 1\}}=\{\phi_k\otimes f;\ f\in L^2(0,H)\}$ où les ϕ_k, $k\in\mathbb{N}^*$, sont les fonctions propres de l'opérateur $A'=-\nabla_{x'}\cdot(C_1\nabla_{x'})$, de domaine $D(A')=\{u\in H_0^1(\Omega');\ \nabla_{x'}\cdot(C_1\nabla_{x'}u)\in L^2(\Omega')\}$, associées aux valeurs propres $\mu_1\leqslant\mu_2\leqslant\cdots$, que l'on choisit formant une base orthonormée de $L^2(\Omega')$. On définit les espaces $E_j=\bigoplus_{k\leqslant 2^j}H_k$ pour $j\in\mathbb{N}^*$, et on vérifie que $\overline{\bigcup_{j\in\mathbb{N}}E_j}=\bigoplus_{k\in\mathbb{N}^*}H_k=L^2(\Omega)$.

On décompose l'intervalle de temps $(0,T)$, $(0,T)=\bigcup_{j\in\mathbb{N}}[a_j,a_{j+1}]$, $a_{j+1}-a_j=2T_j$, avec $T_j=K\sigma_j^{-\rho}$, $\sigma_j=2^j$, avec K choisi tel que $2\sum_{j=0}^\infty T_j=T$ et $\rho\in(0,\frac{2}{3(n-1)})$. Comme dans [9], on construit une suite de contrôles u_j de $L^2((a_j,a_j+T_j)\times\Omega)$, $j\geqslant j_0$ pour un certain $j_0\in\mathbb{N}$, chargés de ramener à 0 la composante suivant E_j de la solution de (2) au temps a_j+T_j. Le contrôle est en revanche nul dans l'intervalle de temps $[a_j+T_j,a_{j+1}]$, afin de profiter de la décroissance exponentielle en temps de la solution (qui se trouve dans $E_j^\perp$ à l'instant $t=a_j+T_j$).

L'inégalité de Carleman pour l'opérateur ∂_t+A_k, où $A_k=-\partial_{x_n}(c_2(x_n)\partial_{x_n})+c_1(x_n)\mu_k$ de domaine $D(A_k)=\{u\in H_0^1(0,H);\ c_2\partial_{x_n}u\in H^1(0,H)\}$, prouvée dans [10] permet d'estimer en fonction de μ_{σ_j} le coût d'un contrôle agissant sur $\Omega'\times\omega_n$ pendant l'intervalle de temps $[a_j,a_j+T_j]$. Puis, en utilisant l'inégalité (8) relative aux fonctions propres de l'opérateur A' [9], on déduit une estimation du coût d'un contrôle, u_j, agissant sur $\omega'\times\omega_n$ pendant l'intervalle de temps $[a_j,a_j+T_j]$. Ce coût est de la forme :

$$\|u_j\|_{L^2((a_j,a_j+T_j)\times\Omega)}\leqslant C\mathrm{e}^{C\mu_{\sigma_j}^{2/3}}\|q_j\|_{L^2(\Omega)},\tag{1}$$

pour j suffisamment grand, où q_j est la valeur de la solution de (2) au temps $t=a_j$. On peut alors conclure comme dans [9].

Remarque 1. On peut aussi considérer d'autres situations, comme par exemple le cas d'opérateurs elliptiques en coordonnées cylindriques. Dans ce cas, les interfaces (pour des coefficients continues par morceaux) *ne rencontrent pas* le bord du domaine Ω. Les coefficients peuvent être BV dans la direction radiale. On peut aussi considérer le cas de $\mathcal{M}\times(0,H)$ où $\mathcal{M}$ est une variété riemannienne $\mathscr{C}^\infty$ compacte (avec ou sans bord) de dimension $n-1$ et l'opérateur A est donné par $A=c_1(x_n)A'-\partial_{x_n}c_2(x_n)\partial_{x_n}$ avec $c_1\in L^\infty(0,H)$, $c_2\in BV(0,H)$ (vérifiant les mêmes hypothèses que celles données plus haut), et par exemple $A'=-\Delta$, avec Δ le Laplacien sur $\mathcal{M}$.

Remarque 2. Comme dans [8], le résultat obtenu pour un contrôle distribué permet d'obtenir un résultat pour un contrôle frontière (ici sur une partie de la frontière $\Omega'\times\{0\}$ ou $\Omega'\times\{H\}$).

1. Introduction

The question of the null controllability of linear parabolic partial differential equations with smooth coefficients was solved in the 1990s [9,8]. In the case of discontinuous coefficients in the principal part of the parabolic operator, the controllability issue and its dual counterpart, observability, are not fully solved yet. A result of controllability for a semilinear parabolic equation with a discontinuous coefficient was proven in [6] by means of a Carleman observability estimate. Roughly speaking, as in the case of hyperbolic systems (see e.g. [11, page 357]), the authors of [6] proved their controllability result in the case where the control is supported in the region where the diffusion coefficient is the 'lowest'. In both cases, however, the approximate controllability, and its dual counterpart, uniqueness, are true without any restriction on the monotonicity of the coefficients. It is then natural to question whether or not an observability estimate holds in the case of non-smooth coefficients and arbitrary observation location.

Recently, in the one-dimensional case, the controllability result for parabolic equations was proven for general piecewise $\mathscr{C}^1$ coefficients in [4], and for coefficients with bounded variations (BV) in [10]. The proof relies on global *Carleman estimates*, which moreover allows to treat semilinear equations. Such global Carleman estimates are also of interest to prove stability results for some inverse problems. A controllability result for parabolic equations with general bounded coefficients was independently proven in [2]. The method used there to achieve null controllability is that of [9], which limits the field of applications to linear equations.

In the n-dimensional case, $n \geqslant 2$, the controllability with an arbitrary control location is still open. In particular, an extension based on the proof of the Carleman estimate in the one-dimensional case, leads to uncontrolled tangential terms at the interfaces of discontinuities of the coefficient. This work provides a positive answer to the controllability question for a class of discontinuous coefficients: the main assumption we make is that the coefficients are smooth w.r.t. to all but one variables, which includes the case of stratified media. The proof relies both on the Carleman estimates of [4,10] in the one-dimensional case and the method of [9]. We thus only treat linear equations.

We let Ω be an open subset in $\mathbb{R}^n$, with $\Omega = \Omega' \times (0, H)$, where Ω' is an nonempty regular bounded open subset of $\mathbb{R}^{n-1}$ with $\mathscr{C}^2$ boundary. We shall use the notation $x = (x', x_n) \in \Omega' \times (0, H)$. For a real Hilbert space X, $\|\cdot\|_X$ (resp. $(\cdot\,,\cdot)_X$) will denote the norm (resp. the real scalar product) in X. Let $B(x)$, $x \in \Omega$, be with values in $M_n(\mathbb{R})$, the space of square matrices with real coefficients of order n. We make the following assumption:

Assumption 1.1. The matrix diffusion coefficient $B(x', x_n)$ has the following block-diagonal form

$$B(x', x_n) = \begin{pmatrix} c_1(x_n)C_1(x') & 0 \\ 0 & c_2(x_n) \end{pmatrix},$$

where $c_1 \in L^\infty(0, H)$, $c_2 \in BV(0, H)$ and $C_1 \in \mathscr{C}^1(\overline{\Omega'}, M_{n-1}(\mathbb{R}))$. The matrix $C_1(x')$ is symmetric. We further assume $0 < c_{\min} \leqslant c_i(x_n) \leqslant c_{\max}$, $x_n \in (0, H)$, $i = 1, 2$, and $0 < c_{\min} I_{n-1} \leqslant C_1(x') \leqslant c_{\max} I_{n-1}$, $x' \in \Omega'$, where I_k is the identity matrix of order k, which implies uniform ellipticity.

We consider the selfadjoint operator $A = -\nabla_x \cdot (B\nabla_x)$ in $L^2(\Omega)$ with domain $D(A) = \{u \in H_0^1(\Omega);\ \nabla_x \cdot (B\nabla_x u) \in L^2(\Omega)\}$. Let $T > 0$. We shall use the notation $Q_T = (0, T) \times \Omega$. We consider the following parabolic system

$$\begin{cases} \partial_t q - \nabla_x \cdot (B\nabla_x q) = 1_\omega u & \text{in } Q_T, \\ q(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \\ q(0, x) = q_0(x) & \text{in } \Omega \end{cases} \tag{2}$$

(coefficients and solutions are real valued) where $q_0 \in L^2(\Omega)$ and ω is a nonempty open subset of Ω such that $\omega \Subset \Omega$. We choose ω' a nonempty open subset of Ω' and ω_n a nonempty open subset of $(0, H)$ such that $\omega' \times \omega_n \subset \omega$.

Assumption 1.2. The coefficient c_2 is of class $\mathscr{C}^1$ in some nonempty open subset of ω_n.

We analyze the null controllability of system (2), or equivalently its exact controllability to the trajectories, when a distributed control $u \in L^2((0, T) \times \Omega)$ acts on the system. The main result is the following theorem:

Theorem 1.3. *Under Assumption* 1.1*, for arbitrary time $T > 0$ and initial condition $q_0 \in L^2(\Omega)$, there exists $u \in L^2((0, T) \times \Omega)$ such that the corresponding solution q of system* (2) *satisfies $q(T) = 0$ a.e. in Ω.*

Corollary 1.4. *There exists an observability inequality for the homogeneous adjoint system of system* (2).

The proof makes use of the technique introduced by G. Lebeau and L. Robbiano [9], as well as the one-dimensional Carleman estimates of [4,10].

In this Note, when the constants C or C', etc, are used, their values may change from one line to the other. If we want to keep track of the value of a constant we shall use another letter.

2. Spectral properties

Similarly to A, we define the selfadjoint operator $A' = -\nabla_{x'} \cdot (C_1 \nabla_{x'})$, in $L^2(\Omega')$, with domain $D(A') = \{u \in H_0^1(\Omega');\ \nabla_{x'} \cdot (C_1 \nabla_{x'} u) \in L^2(\Omega')\}$. With orthonormal eigenfunctions $(\phi_k)_{k\geqslant 1}$, associated to the eigenvalues, with finite multiplicities, $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k \leqslant \mu_{k+1} \leqslant \cdots$, we construct a Hilbert basis of $L^2(\Omega')$.

We also define the selfadjoint operators A_k, $k \in \mathbb{N}^*$, on $L^2(0,H)$ by $A_k = -\partial_{x_n}(c_2(x_n)\partial_{x_n}) + c_1(x_n)\mu_k$ with domain $D(A_k) = \{u \in H_0^1(0,H);\ c_2\partial_{x_n} u \in H^1(0,H)\}$. We denote by $\psi_{k,p}(x_n)$, $p \in \mathbb{N}^*$, orthonormal eigenfunctions with associated eigenvalues $\lambda_{k,1} \leqslant \lambda_{k,2} \leqslant \cdots \leqslant \lambda_{k,p} \leqslant \lambda_{k,p+1} \leqslant \cdots$. Note that we have $\lambda_{k,p} > c_{\min}\,\mu_k$.

From the separation of variables in the coefficients of the matrix B, we have the following proposition:

Proposition 2.1. *The eigenfunctions of the operator A given by $\varphi_{k,p}(x',x_n) = (\phi_k \otimes \psi_{k,p})(x',x_n) = \phi_k(x')\psi_{k,p}(x_n)$ with associated eigenvalue $\lambda_{k,p}$, $k,p \in \mathbb{N}^*$, form a Hilbert basis of $L^2(\Omega)$.*

Let us denote by H_k the following closed infinite dimensional subspace of $L^2(\Omega)$: $H_k = \overline{\operatorname{span}\{\varphi_{k,p};\ p \geqslant 1\}} = \{\phi_k \otimes f;\ f \in L^2(0,H)\}$ and let us set $E_j = \bigoplus_{k\leqslant 2^j} H_k$ for $j \in \mathbb{N}^*$. In the sequel we shall denote by Π_{E_j} the orthogonal projection onto E_j in $L^2(\Omega)$. We have the following proposition:

Proposition 2.2. *The following properties hold*:

(i) $E_j \subset E_{j+1}$, $j \in \mathbb{N}$, *and* $\overline{\bigcup_{j\in\mathbb{N}} E_j} = \bigoplus_{k\in\mathbb{N}^*} H_k = L^2(\Omega)$;
(ii) *The operator* $(-A, D(A))$ *generates a* $\mathscr{C}^0$*-semigroup of contraction,* $S(t) = \mathrm{e}^{-tA}$ *for* $t \geqslant 0$, *and for all* $f \in L^2(\Omega)$ *we have* $S(t)f = \sum_{k,p\geqslant 1} \mathrm{e}^{-t\lambda_{k,p}}(f,\varphi_{k,p})\varphi_{k,p}$;
(iii) *For all* $k \geqslant 1$, $S(t)$ *is reduced by the space* H_k.

See for instance [5] or [12].

3. Existence and estimation of a control acting on E_j

Following [9], for $\rho \in (0, \frac{2}{3(n-1)})$, we set $T_j = K\sigma_j^{-\rho}$, with $\sigma_j = 2^j$, for all $j \in \mathbb{N}$. The constant K is adjusted so that $2\sum_{j=0}^{\infty} T_j = T$. Then, we set $a_0 = 0$, $a_{j+1} = a_j + 2T_j$, for $j \geqslant 0$.

We show that for all $q_j \in L^2(\Omega)$, $j \geqslant j_0$, for some $j_0 \in \mathbb{N}$, there exists $u_j \in L^2(a_j, a_j + T_j; L^2(\Omega))$ such that the solution q to

$$\begin{cases} \partial_t q - \nabla_x \cdot (B\nabla_x q) = 1_\omega u_j & \text{in } (a_j, a_j+T_j) \times \Omega, \\ q(t,x) = 0 & \text{on } (a_j, a_j+T_j) \times \partial\Omega, \\ q(a_j, x) = q_j(x) & \text{in } \Omega, \end{cases} \tag{3}$$

satisfies $\Pi_{E_j} q(a_j + T_j, x) = 0$. Since $S(t)$ and Π_{E_j} commute, this is equivalent to the observability inequality [7]

$$\|y(a_j,\cdot)\|^2_{L^2(\Omega)} \leqslant C^2_{T_j} \int_{a_j}^{a_j+T_j} \int_{\omega} |y(t)|^2 \,\mathrm{d}t\,\mathrm{d}x, \tag{4}$$

for the solution $y \in \mathscr{C}([a_j, a_j+T_j]; E_j)$ of the adjoint system

$$-\partial_t y - \nabla_x \cdot (B\nabla_x y) = 0 \quad \text{in } (a_j, a_j+T_j) \times \Omega, \quad \text{and} \quad y(a_j+T_j) = y_0 \quad \text{in } \Omega,\ y_0 \in E_j, \tag{5}$$

which moreover yields the existence of a control $u_j \in L^2((a_j, a_j+T_j); L^2(\Omega))$ such that

$$\|u_j\|_{L^2((a_j,a_j+T_j)\times\Omega)} \leqslant C_{T_j}\|\Pi_{E_j} q_j\|_{E_j} \leqslant C_{T_j}\|q_j\|_{L^2(\Omega)}. \tag{6}$$

We now prove (4). We first recall a Carleman estimate for a parabolic operator with a BV diffusion coefficient proven in [10]. For a positive function $\tilde{\beta}$, we introduce $\beta = \tilde{\beta} + K$ with $K = m\|\tilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and

$t \in (a_j, a_j + T_j)$, we define the following weight functions [8] $\varphi(t,x) = \frac{e^{\lambda\beta(x)}}{(t-a_j)(a_j+T_j-t)}$, $\eta(t,x) = \frac{e^{\lambda\bar\beta} - e^{\lambda\beta(x)}}{(t-a_j)(a_j+T_j-t)}$, with $\bar\beta = 2m\|\tilde\beta\|_\infty$.

Theorem 3.1. *Let $\mathcal{O} \Subset (0,H)$ be a nonempty open set and $\gamma \in BV(0,H)$ with $0 < c_{\min} \leqslant \gamma \leqslant c_{\max}$ and γ of class $\mathscr{C}^1$ in $\mathcal{O}$. There exist a positive continuous function $\tilde\beta$, and $\lambda_0 = \lambda_0(H, \mathcal{O}, c_{\min}, c_{\max}) > 0$, $s_0 = \tilde s_0(H, \mathcal{O}, c_{\min}, c_{\max})(T_j + T_j^2) > 0$ and a positive constant $C = C(H, \mathcal{O}, c_{\min}, c_{\max})$ so that*

$$s^{-1}\int_{a_j}^{a_j+T_j}\int_0^H e^{-2s\eta}\varphi^{-1}\bigl(|\partial_t z|^2 + \bigl|\partial_x(\gamma\partial_x z)\bigr|^2\bigr)\,dx\,dt + s\lambda^2\int_{a_j}^{a_j+T_j}\int_0^H e^{-2s\eta}\varphi|\partial_x z|^2\,dx\,dt$$
$$+ s^3\lambda^4\int_{a_j}^{a_j+T_j}\int_0^H e^{-2s\eta}\varphi^3|z|^2\,dx\,dt \leqslant C\left[s^3\lambda^4\int_{a_j}^{a_j+T_j}\int_{\mathcal{O}} e^{-2s\eta}\varphi^3|z|^2\,dx\,dt + \int_{a_j}^{a_j+T_j}\int_0^H e^{-2s\eta}|f|^2\,dx\,dt\right],$$

for $s \geqslant s_0$, $\lambda \geqslant \lambda_0$ and for all z (weak) solution of $\partial_t z + \partial_x(\gamma\partial_x z) = f$ in $(a_j, a_j+T_j) \times (0,H)$, $z(t,0) = z(t,H) = 0$, and $z(a_j+T_j, x) = z_0(x)$ in $(0,H)$, with $z_0 \in L^2(0,H)$ and $f \in L^2((a_j, a_j+T_j) \times (0,H))$.

With the previous Carleman estimate we can prove an observability inequality for the parabolic operator $-\partial_t + A_k$.

Proposition 3.2. *There exist positive constants $C = C(H, \omega_n, c_{\min}, c_{\max})$, and $C' = C'(H, \omega_n, c_{\min}, c_{\max})$ such that, for all $k \in \mathbb{N}^*$ and for $s_k = \max(C' T_j^2 \mu_k^{2/3}, \tilde s_0(T_j + T_j^2))$, the solutions to*

$$\begin{cases} -\partial_t z + A_k z = 0 & \text{in } (a_j, a_j+T_j) \times \Omega, \\ z(t) = 0 & \text{on } (a_j, a_j+T_j) \times \partial\Omega, \end{cases} \tag{7}$$

satisfy $\|z(a_j,\cdot)\|^2_{L^2(0,H)} \leqslant \frac{1}{T_j} C e^{C s_k T_j^{-2}} \int_{a_j}^{a_j+T_j}\int_{\omega_n} |z(t,x)|^2\,dx\,dt$.

Proof. We apply Theorem 3.1, with $f = -c_1\mu_k z$, $\gamma = c_2$, $\lambda = \lambda_0$ and $s \geqslant \tilde s_0(T_j + T_J^2)$, and obtain

$$\int_{a_j}^{a_j+T_j}\int_0^H e^{-2s\eta}\varphi^3\bigl(s^3 - C\varphi^{-3}\mu_k^2\bigr)|z|^2\,dx_n\,dt \leqslant C'' s^3\int_{a_j}^{a_j+T_j}\int_{\omega_n} e^{-2s\eta}\varphi^3|z|^2\,dx_n\,dt.$$

Noting that $\varphi^{-1} \leqslant C T_j^2$, the coefficient $s^3 - C\varphi^{-3}\mu_k^2$ will be positive for $s \geqslant C' T_j^2 \mu_k^{2/3}$. Setting $s = s_k = \max(C' T_j^2 \mu_k^{2/3}, \tilde s_0(T_j + T_j^2))$, we obtain $\int_{a_j+T_j/4}^{a_j+3T_j/4}\int_0^H |z|^2\,dx_n\,dt \leqslant C e^{C s_k T_j^{-2}} \int_{a_j}^{a_j+T_j}\int_{\omega_n} |z|^2\,dx_n\,dt$. Making use of the parabolic 'dissipation effect', i.e. $\frac{d}{dt}|z(t)|^2 \geqslant 0$ here, we obtain the desired inequality. □

We can now obtain the observability in the space E_j:

Proposition 3.3. *Let ω' be a nonempty open subset of Ω' with $\omega' \Subset \Omega'$. There exists $j_0 \in \mathbb{N}$ such that, if $j \geqslant j_0$, solutions of* (5) *satisfy* (4) *with $C_{T_j} = C e^{C\mu_{\sigma_j}^{2/3}}$.*

To prove Proposition 3.3, we shall need the following result which was first proven in [9]:

Theorem 3.4. *There exists $C = C(\Omega', \omega', c_{\min}, c_{\max}) > 0$ such that for all $l \in \mathbb{N}^*$,*

$$\sum_{k\leqslant l} |b_k|^2 \leqslant C e^{C\sqrt{\mu_l}} \int_{\omega'} \Bigl|\sum_{k\leqslant l} b_k \phi_k(x')\Bigr|^2\,dx', \quad (b_1,\ldots,b_l) \in \mathbb{R}^l. \tag{8}$$

Proof of Proposition 3.3. The definition of E_j implies that $y(t,x',x_n) = \sum_{k\leqslant\sigma_j} \phi_k(x') y_k(t,x_n)$ and one sees that y is solution in $\mathscr{C}([a_j, a_j+T_j]; E_j)$ of (5), if and only if each function y_k, $1 \leqslant k \leqslant \sigma_j$, is solution in

$\mathscr{C}([a_j, a_j + T_j]; L^2(0,H))$ of (7) and $y_k(a_j + T_j) = y_{0,k}$, where $y_0(x', x_n) = \sum_{k \leqslant \sigma_j} \phi_k(x') y_{0,k}(x_n)$. We then have $\|y(t,\cdot)\|^2_{L^2(\Omega)} = \sum_{k \leqslant \sigma_j} \|y_k(t,\cdot)\|^2_{L^2(0,H)}$ and similarly $\|y(t,\cdot)\|^2_{L^2(\Omega' \times \omega_n)} = \sum_{k \leqslant \sigma_j} \|y_k(t,\cdot)\|^2_{L^2(\omega_n)}$ for $t \in [a_j, a_j + T]$. According to the Weyl formula [1, Theorem 14.6, p. 250], $\mu_{\sigma_j} \sim C(\Omega')(\sigma_j)^{2/(n-1)}$, we see that, for j sufficiently large, we have $s_{\sigma_j} = C T_j^2 \mu_{\sigma_j}^{2/3}$. We then have $T_j^{-2} \max_{1 \leqslant k \leqslant \sigma_j} s_k = T_j^{-2} s_{\sigma_j} = C \mu_{\sigma_j}^{2/3}$. From Proposition 3.2 we obtain

$$\|y(a_j, \cdot)\|^2_{L^2(\Omega)} \leqslant C \mathrm{e}^{C \mu_{\sigma_j}^{2/3}} \int_{a_j}^{a_j + T_j} \int_{\omega_n} \sum_{k \leqslant \sigma_j} |y_k(t, x_n)|^2 \,\mathrm{d}t \,\mathrm{d}x_n. \tag{9}$$

If we use $y_k(t, x_n)$ in place of b_k in (8), we deduce

$$\|y(a_j, \cdot)\|^2_{L^2(\Omega)} \leqslant C \mathrm{e}^{C \mu_{\sigma_j}^{2/3}} \int_{a_j}^{a_j + T_j} \int_{\omega_n} C' \mathrm{e}^{C' \sqrt{\mu_{\sigma_j}}} \int_{\omega'} \left| \sum_{k \leqslant \sigma_j} \phi_k(x') y_k(t, x_n) \right|^2 \mathrm{d}x' \,\mathrm{d}x_n \,\mathrm{d}t. \qquad \square \tag{10}$$

Following the method of [9], with the value of C_{T_j} we have obtained and the choice made for T_j, the result of Theorem 1.3 follows.

Remark 1. For the sake of presentation, we chose to take the open set Ω of the form $\Omega' \times (0, H)$ with Ω' a bounded open subset of $\mathbb{R}^{n-1}$. There are other situations that can be handled by the method we have presented. We could for instance consider an uniformly elliptic operator in cylindrical coordinates and address the case of a ring with variations of the medium in the radial direction. In this case, the interfaces that locate the jumps of the coefficients of the diffusion matrix (in the case of piecewise continuous coefficients) *do not reach* the boundary of the domain Ω. As before we can address the case of a BV-type regularity in the radial direction.

One other natural extension would be the case of a domain of the form $\mathcal{M} \times (0, H)$ where $\mathcal{M}$ is a smooth $(n-1)$-dimensional Riemannian compact manifold with or without a boundary. The parabolic operators under consideration would then be of the form $\partial_t + A$, where $A = c_1(x_n) A' - \partial_{x_n} c_2(x_n) \partial_{x_n}$ with $c_1 \in L^\infty(0, H)$, $c_2 \in BV(0, H)$ (satisfying the same assumptions as those given above), and say $A' = -\Delta$, with Δ the Laplace operator on $\mathcal{M}$. In such a case, estimate (8) can be found in [9].

Remark 2. As usual, as in [8], the result obtained on distributed controls yields a boundary control result. Here the control function could act in a nonempty open region of $\Omega' \times \{0\}$ or $\Omega' \times \{H\}$ as a boundary condition for the parabolic system under consideration. This is of particular interest for geometrical situations like that described in the previous remark.

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