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C. R. Acad. Sci. Paris, Ser. I 344 (2007) 337–342
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Numerical Analysis

The Khokhlov–Zabolotskaya–Kuznetsov equation

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Received 22 July 2006; accepted 16 January 2007

Available online 20 February 2007

Presented by Olivier Pironneau

Abstract

For the KZK equation $(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0$ in the class of x-periodic and of zero mean value functions we have analysed the following: the derivation from Navier–Stokes system and the validity of its approximation, the existence, uniqueness and stability of the solution. The solution is proved to be global in time for sufficient small initial data with $\beta > 0$ and to have a blow-up if $\beta = 0$. ***To cite this article: A. Rozanova, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Équation de Khokhlov–Zabolotskaya–Kuznetsov. Pour l'équation KZK $(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0$ dans la classe des fonctions périodiques en x et de moyennes nulles, on a étudié la dérivation à partir du système de Navier–Stokes isentropique et la validation de son approximation, l'existence, l'unicité et la stabilité de la solution. On a prouvé que la solution est globale en temps pour des données initiales suffisement petites avec $\beta > 0$ et que la solution présente une onde de choc si $\beta = 0$. ***Pour citer cet article : A. Rozanova, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

On considère l'équation

$$(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0, \quad (x, y) \in \mathcal{R}_x/(L\mathcal{Z}) \times \mathcal{R}_y^{n-1}. \tag{1}$$

Elle porte le nom de KZK (Khokhlov–Zabolotskaya–Kuznetsov) et a été introduite pour étudier des problèmes d'acoustique non linéaire [2] comme approximation d'une équation de Navier–Stokes isentropique

$$\partial_t \rho_\epsilon + \nabla(\rho_\epsilon u_\epsilon) = 0, \qquad \rho\big[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla) u_\epsilon\big] = -\nabla p(\rho_\epsilon) + \epsilon \nu \Delta u_\epsilon, \tag{2}$$

avec équation d'état approchée $\rho_\epsilon = \rho_0 + \epsilon \tilde{\rho}_\epsilon$,

$$p = p(\rho_\epsilon) = c^2 \epsilon \tilde{\rho}_\epsilon + \frac{(\gamma - 1)c^2}{2\rho_0} \epsilon^2 \tilde{\rho}_\epsilon^2, \quad \text{et} \quad c = \sqrt{p'(\rho_0)} \text{ la vitesse du son.} \tag{3}$$

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doi:10.1016/j.crma.2007.01.010

Il s'agit de décrire des solutions qui se propagent comme des faisceaux (comme dans l'approximation paraxiale), mais dans un cadre non linéaire (les effets non linéaires et la viscosité sont du même ordre que les oscillations longitudinales).[1]

On établit des théorèmes de Cauchy (existence et stabilité par rapport aux données initiales) et on montre en quel sens les solutions de cette équation fournissent des approximations de certaines solutions du système initial.

On a confirmé les résultats numériques de [2] analytiquement.

1. Existence, stability, uniqueness and blow-up results

We consider the KZK equation (1) involving two positive constants β and γ with the condition of periodicity and of mean value zero (which corresponds to practical situations [2])

$$u(x+L,y,t)=u(x,y,t),\quad \int_0^L u(x,y,t)\,\mathrm{d}x=0. \tag{4}$$

We define the inverse of the derivative ∂_x^{-1} as an operator acting in the space of periodic functions with mean value zero

$$\partial_x^{-1}f=\int_0^x f(s)\,\mathrm{d}s+\int_0^L \frac{s}{L}f(s)\,\mathrm{d}s. \tag{5}$$

Therefore (1) is equivalent to the equation

$$u_t-uu_x-\beta u_{xx}-\gamma\partial_x^{-1}\Delta_y u=0\quad \text{in } \mathcal{R}_x/(L\mathcal{Z})\times\Omega. \tag{6}$$

Finally when $\gamma=0$ (1) reduces to the Burgers–Hopf equation for which existence smoothness and uniqueness of solution are well-known. For $\gamma=\beta=0$ it reduces to the Burgers equation which after a finite time exhibits singularities.[2]

To prove the existence theorem we use a priori estimates for smooth solutions of the integrated KZK equation (6) (the L_2 norm and the H^s in $(\mathcal{R}_x^+/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1}$) are denoted by $|u|$ and by $\|u\|_s$)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|u(\cdot,\cdot,t)\big|^2+\beta\big|\partial_x u(\cdot,\cdot,t)\big|^2=0, \tag{7}$$

$$\text{for } s>\left[\frac{n}{2}\right]+1\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_s^2+\beta\|\partial_x u\|_s^2\leqslant C(s)\|u\|_s^3 \tag{8}$$

$$\text{and}\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_s^2+\beta C(L)\|u\|_s^2\leqslant C(s)\|u\|_s^3. \tag{9}$$

The estimates (8), (9) are valid for $s>[\frac{n}{2}]+1$ which is necessary for the use of the Sobolev theorem. The estimate (7) shows that we have the conservation of L_2-norm if $\beta=0$. We obtain the following:

Theorem 1.1. *For the Cauchy problem*

$$u_t-uu_x-\beta u_{xx}-\gamma\partial_x^{-1}(\Delta_y u)=0,\quad u(x,y,0)=u_0 \tag{10}$$

considered in $(\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1}$ *with* (4), *and with* $\beta\geqslant 0$ *one has the following results*:

(i) *For* $s>[\frac{n}{2}]+1$ *there exists a constant* $C(s,L)$ *such that for any initial data* $u_0\in H^s$ *the problem* (10) *has on an interval* $[0,T[$ *with* $T\geqslant\frac{1}{C(s,L)\|u_0\|_{H^s}}$ *a solution in* $C([0,T[,H^s)\cap C^1([0,T[,H^{s-2})$.

[1] Ce travail a été motivé par l'usage fait au Laboratoire d'Ondes Acoustique de cette équation dans les problèmes d'écho-grraphie et de retournement temporel. Il a été réalisé dans le cadre de l'ACI « Retournement Temporel », convention A020 de Ministère de la Recherche Français. Cette équation est aussi utilisé en magnitisme [7]. Le modèle régularisé de l'équation a été envisagé dans [5].

[2] The J. Bourgain-type method with introduction of the Bourgain spaces cannot be used here because of the absence of the terms with an odd derivative as for example u_{xxx} in (6). The KZK equation is not integrable at variance with the Kadomtsev–Petviashvili (KP).

(ii) *Let T^* be the biggest time on which such solution is defined then one has*

$$\int_0^{T^*} \sup_{x,y}\left(\left|\partial_x u(x,y,t)\right| + \left|\nabla_y u(x,y,t)\right|\right) \mathrm{d}t = \infty. \tag{11}$$

(iii) *If $\beta > 0$ there exists a constant C_1 such that $\|u_0\|_s \leqslant C_1 \Rightarrow T^* = \infty$.*

(iv) *For two solutions u and v of KZK equation, with $u \in L_\infty([0,T[; H^s)$ and $v \in L^2([0,T[; L_2)$, one has the following stability uniqueness result*:

$$\left|u(,t) - v(\cdot,t)\right|_{L^2} \leqslant \mathrm{e}^{\int_0^t \sup_{x,y} |\partial_x u(x,y,s)|\,\mathrm{d}s}\left|u(\cdot,0) - v(\cdot,0)\right|_{L^2}. \tag{12}$$

Remark 1. The estimate (12) is of strong-weak form: as in [3] only the L_∞ norm of u_x is needed.

Remark 2. When there is no viscosity all the corresponding statements of Theorem 1.1 remain valid for $0 > t > -C$ with a convenient C.

Theorem 1.2. *The equation*

$$(u_t - uu_x)_x - \gamma \Delta_y u = 0 \quad \textit{in } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{13}$$

with Neumann boundary condition on $\partial\Omega$ has no global in time smooth solution if $\sup_{x,y} \partial_x u(x,y,0)$ *is large enough with respect to γ.*

Remark 3. As we can see from [2] the result of the theorem confirms the numerical results. One observes that for $\beta \to 0$ the solution of the KZK equation has a quasi-shock approaching the shock wave, into which it degenerates for $\beta = 0$.

The proof of the blow-up follows the ideas of S. Alinhac [1]. First the blow-up is observed for $\gamma = 0$ and related to a singularity in the projection of an unfolded 'blow-up system'. Second the properties of this unfolded blow-up system are shown to be stable under small perturbations. One uses a Nash–Moser theorem with tamed estimates and this is the reason why will exists a T^* such that:

$$\lim_{t \to T^*} (T^* - t) \sup_{x,y} \partial_x u(x,y,t) > 0.$$

2. Approximation results

One finds in physical works [2,8] a formal derivation of the KZK equation as a second order approximation of the isentropic Navier–Stokes system (2) with the approximate state equation (3). We offer here to give a strict proof and orders of magnitude of the approximation.

We study solutions propagating along the axis x_1 direction. Therefore it is assumed that its variation in the direction $x' = (x_2, x_3, \ldots, x_n)$ perpendicular to the x_1 axis is much larger that its variation along the axis x_1. We use the ansatz

$$\rho_\epsilon = \rho_0 + \epsilon\tilde{\rho}_\epsilon = \rho_0 + \epsilon I\left(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}\, x'\right) = \rho_0 + \epsilon I(\tau, z, y), \tag{14}$$

$$u_\epsilon = \epsilon\tilde{u}_\epsilon = \epsilon(u_{\epsilon,1}, u'_\epsilon) = \epsilon(v + \epsilon v_1, \sqrt{\epsilon}\,\vec{w})\left(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}\, x'\right) = \epsilon(v + \epsilon v_1, \sqrt{\epsilon}\,\vec{w})(\tau, z, y), \tag{15}$$

here ϵ is a dimensionless parameter which characterizes the smallness of the perturbation. For instance in water with a initial power of the order of 0.3 Vt/cm^2 $\epsilon = 10^{-5}$. We obtain

$$\epsilon(\rho_0 v - cI) = 0 \Rightarrow v(\tau, z, y) = \frac{c}{\rho_0} I(\tau, z, y), \tag{16}$$

$$\epsilon^{3/2}(\rho_0 \partial_\tau \vec{w} + c^2 \nabla_y I) = 0 \Rightarrow \vec{w}(\tau, z, y) = \frac{c^2}{\rho_0}\left(\int_0^\tau \nabla_y I(s, z, y)\,\mathrm{d}s + \int_0^L \frac{s}{L} \nabla_y I(s, z, y)\,\mathrm{d}s\right), \tag{17}$$

$$\epsilon^2\left(\rho_0\partial_\tau v_1 + c^2\partial_z I - \frac{\gamma-1}{2\rho_0}c\partial_\tau I^2 - \frac{\nu}{c\rho_0}\partial_\tau^2 I\right) = 0 \Rightarrow$$

$$v_1(\tau,z,y) = \frac{c^3}{2\rho_0}\partial_\tau^{-2}\Delta_y I + \frac{\gamma-1}{4\rho_0^2}cI^2 - \frac{c(\gamma-1)}{4L\rho_0^2}\int_0^L I^2\,\mathrm{d}\tau + \frac{3\nu}{2c\rho_0^2}\partial_\tau I, \tag{18}$$

$$\epsilon^2\left(c\partial_{\tau z}^2 I - \frac{\gamma+1}{4\rho_0}\partial_\tau^2 I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^3 I - \frac{c^2}{2}\Delta_y I\right) = 0 \quad \text{which is the KZK equation.} \tag{19}$$

We consider the exact system (2) denoted $\Phi(\rho_\epsilon, u_\epsilon, \nu) = 0$ and the KZK solution $I(\tau,z,y) = I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}\,x')$ of (19).

The viscosity ν introduces some difference in the construction.

With no viscosity both the nonlinear system of elasticity [3] and the KZK equation are well posed for positive and negative but finite time, so, and since $z = \epsilon x_1$ (as soon as z becomes the time variable according to the KZK derivation), the problem of approximation is considered in a cone

$$C(t) = \{0 < s < t\} \times Q_\epsilon(s) = \left\{x = (x_1, x')\colon\ |x_1| \leqslant \frac{R}{\epsilon} - Ms,\ M \geqslant c,\ x' \in \mathcal{R}^{n-1}\right\}. \tag{20}$$

With viscosity both problems are well posed only for positive time but under a smallness hypothesis of initial data up to infinity (we prove this for the Navier–Stokes system). So the approximation is considered in the half space $x_1 > 0$, $t > 0$, $x' \in \mathcal{R}^{n-1}$. This time we have also to solve the boundary-valued problem for Navier–Stokes system (cf. [4,6] for the related questions for Navier–Stokes system in the half space).

With the sufficiently smooth initial data $I_0(t,y)$ we obtain a smooth solution of the KZK equation $I(\tau,z,y)$.

In the nonviscous case $\nu = 0$ we consider the solution I of the KZK equation (19) with $\nu = 0$ and introduce the correctors v, w and v_1 given by (16)–(18) then we have the following:

Theorem 2.1. *We take the initial data for the KZK equation with $\nu = 0$ from H^s, $s > \max\{4, [\frac{n}{2}]+1\}$. Consider now the Euler system $\Phi(U_\epsilon, 0) = 0$, $U_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon)^{\mathrm{T}}$ in a cone* (20) *with the initial data*

$$(\bar\rho_\epsilon - \rho_\epsilon)|_{t=0} = 0, \qquad (\bar u_\epsilon - u_\epsilon)|_{t=0} = 0. \tag{21}$$

Then (*see* [3, *p.* 62]) *there exists T_0 such that for the time interval $0 \leqslant t \leqslant \frac{T_0}{\epsilon}$ there exists the classical solution $U_\epsilon = (\rho_\epsilon, u_\epsilon)$ of the Euler system in a cone $C(T) = \{0 < t < T \mid T < \frac{T_0}{\epsilon}\} \times Q_\epsilon(t)$ with*

$$\|\nabla\cdot U_\epsilon\|_{L_\infty([0,\frac{T_0}{\epsilon}[;H^{s-1})} < \epsilon C \quad \textit{for } s > \left[\frac{n}{2}\right] + 1.$$

Then there exists a constant C such that for any ϵ small enough the solutions U_ϵ of the exact system and $\overline{U}_\epsilon = (\bar\rho_\epsilon, \bar\rho_\epsilon\bar u_\epsilon)^{\mathrm{T}}$ of the approximate system $\Phi(\overline{U}_\epsilon, 0) = \epsilon^{5/2}R$ (the functions $\bar u_\epsilon$, $\bar\rho_\epsilon$ are constructed according to formulas (14), (15) *and the rest R is bounded), which have been determinate as above in the cone $C(T)$ with the same initial data* (21), *satisfy the estimate*

$$\|\overline{U}_\epsilon - U_\epsilon\|^2_{L_2(Q_\epsilon(t))} \leqslant \epsilon^5 \mathrm{e}^{C\|\nabla\cdot\overline{U}_\epsilon\|_{L_\infty(C(T))}t} \leqslant \epsilon^5\mathrm{e}^{C\epsilon t}. \tag{22}$$

The estimate (22) *also holds in the norm of $H^{s'}$ for $s' = s - 4$ as soon as in the cone the booth solutions $\overline{U}_\epsilon$ and $\widetilde{U}_\epsilon$ are in $C([0, \frac{T_0}{\epsilon}[; H^s)$.*

For the viscous case we consider the solution of the KZK equation for $\nu > 0$ and the correctors v, w and v_1 given by (16)–(18). First we observe

Proposition 2.2. *Suppose that the initial data of the KZK Cauchy problem $I_0(t,y) = I_0(t, \sqrt{\epsilon}\,x')$ is such that*

(i) *it is periodic on t with the period L and of mean value zero,*
(ii) *for fixed t it has the same sign for all $y \in \mathcal{R}^{n-1}$, and for $t \in\,]0, L[$ change the sign, i.e., $I_0 = 0$, only finite number times,*

(iii) $I_0(t, y) \in H^{s'}(\{t \geqslant 0\} \times \mathcal{R}^{n-1})$ *for* $s' > \max\{6, [\frac{n}{2}] + 1\}$,
(iv) I_0 *is sufficiently small in the sense of Theorem* 1.1 *and* $I_0 = \epsilon^p \tilde{I}_0$, $p \geqslant 0$.

Then there exits a unique global in time solution $I(\tau, z, y)$ *of* (19) *for* $z = \epsilon x_1 > 0$. *Therefore the functions* $\bar{\rho}_\epsilon$, $\bar{u}_\epsilon$, *defined by* (14) *and* (15) *in the half space*

$$\{x_1 > 0,\ x' \in \mathcal{R}^{n-1},\ t > 0\}, \tag{23}$$

are smooth:

$$\bar{\rho}_\epsilon \in C\big([0, \infty[, H^{s'}\big(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1}\big)\big) \cap C^1\big([0, \infty[; H^{s'-2}\big(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1}\big)\big), \tag{24}$$

$$\bar{u}_\epsilon \in C\big([0, \infty[; H^{s'-2}\big(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1}\big)\big) \cap C^1\big([0, \infty[; H^{s'-4}\big(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1}\big)\big). \tag{25}$$

The Navier–Stokes system (2) *in the half space with the initial data* (21) *and following boundary conditions*

$$(\bar{u}_\epsilon - u_\epsilon)|_{x_1=0} = 0,$$

and when the first component of the velocity is positive $u_{\epsilon,1}|_{x_1=0} > 0$ (*i.e. at points where the fluid enters the domain*) *the additional boundary condition*

$$(\bar{\rho}_\epsilon - \rho_\epsilon)|_{x_1=0} = 0.$$

When $u_{\epsilon,1}|_{x_1=0} \leqslant 0$ *there is not any boundary condition for* ρ_ϵ.

Suppose also that $u_\epsilon \to 0$, $\rho_\epsilon \to \rho_0$ *as* $|x| \to \infty$.

Then there exists a constant $T_0 > 0$ *such that for all* $t < T_0/\epsilon^{2+p}$ *there exists a unique solution* $U_\epsilon = (\rho_\epsilon, u_\epsilon)$ *of Navier–Stokes system* (2) *with the same smoothness as* (24), (25).

Remark 4. Since the boundary conditions for the Navier–Stokes system are periodic and of mean value zero on t, $u_1|_{x_1=0}$ changes the sign and the inflow part of the boundary goes after the inflow one and so on. This avoids the pathology that may result from a change of type of the boundary condition in the tangential variables. This hypothesis follows from the physical works of Zabolotskaya (see [2]), where one takes as the initial conditions for the KZK equation (which correspond to the boundary condition for u_1) the expression $I(\tau, 0, y) = -F(y)\sin\tau$. The amplitude distribution $F(y)$ is taken two types: for a Gaussian beam $F(y) = \mathrm{e}^{-y^2}$, and for a beam with a polynomial amplitude

$$F(y) = \begin{cases} (1 - y^2)^2, & y \leqslant 1, \\ 0, & y > 1. \end{cases}$$

Then we prove

Theorem 2.3. *Suppose the assumptions of Proposition* 2.2. *Then there exists a unique global in time solution* $\overline{U}_\epsilon = (\bar{\rho}_\epsilon, \bar{u}_\epsilon)$ *of the approximate system* $\Phi(\overline{U}_\epsilon, \nu) = \epsilon^{5/2} R$ (*with a bounded rest* R) *deduced from a solution of the KZK equation with the help of* (16)–(18). *The function* $\overline{U}_\epsilon(x_1, x', t) = \overline{U}_\epsilon(x_1 - ct, \epsilon x_1, \sqrt{\epsilon}\, x')$ *is defined in the half space* (23). *Moreover, according to its definition, we have* (24), (25). *Then there exists a constant* C *such that for all rather small* ϵ *the solutions* $(\rho_\epsilon, u_\epsilon)$ *of* (2) *and* $(\bar{\rho}_\epsilon, \bar{u}_\epsilon)$ *of the approximate system satisfy the following stability estimate*

$$\|\rho_\epsilon - \bar{\rho}_\epsilon\|_{L_2} + \|\rho_\epsilon u_\epsilon - \bar{\rho}_\epsilon \bar{u}_\epsilon\|_{L_2} \leqslant \epsilon^{5/2} e^{C\|\nabla \cdot \overline{U}_\epsilon\|_{L_\infty} t} \leqslant \epsilon^{5/2} \mathrm{e}^{C\epsilon t} \tag{26}$$

which remains any finite time $0 < t < \frac{T}{\epsilon} \ln \frac{1}{\epsilon}$ *smaller than the order* ϵ (*here* T *is a positive constant and* $T = \mathrm{O}(1)$).

Remark 5. To have the estimate (26) it is sufficient to have an admissible weak solution of the Navier–Stokes system (2) satisfying the boundary conditions in the half space.

Definition 2.4. The pair of functions (ρ, u) is called an admissible weak solution of Navier–Stokes system (2) satisfying the boundary conditions in the half space if it satisfies the following properties:

(i) it is a weak solution of (2), and it satisfies in the sense of distributions (see [3, p.52])

$$\partial_t \eta(U_\epsilon) + \nabla\cdot q(U_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ u_\epsilon \triangle u_\epsilon \end{bmatrix} \leqslant 0,$$

(ii) it satisfies the equality

$$-\int\limits_{x_1>0} \frac{U_\epsilon^2}{2}\,\mathrm{d}x + \int\limits_0^t \int\limits_{x_1>0} \left(\nabla\cdot U_\epsilon F(U_\epsilon) + \epsilon\nu \begin{bmatrix} 0 \\ |\nabla\cdot u_\epsilon|^2 \end{bmatrix}\right) \mathrm{d}x\,\mathrm{d}s + \int\limits_{x_1>0} U_0^2(x)\,\mathrm{d}x$$
$$+ \int\limits_0^t \int\limits_{\mathcal{R}^{n-1}} U_\epsilon \left(F(U_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ \nabla u_\epsilon \end{bmatrix}\right)\Bigg|_{x_1=0} \mathrm{d}x'\,\mathrm{d}t = 0.$$

References

[1] S. Alinhac, Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions, II, Acta Math. 182 (1999) 1–23.

[2] N.S. Bakhvalov, Ya.M. Zhileikin, E.A. Zabolotskaya, Nonlinear Theory of Sound Beams, American Institute of Physics, New York, 1987. Nelineinaya Teoriya Zvukovih Puchkov, Nauka, Moscow, 1982.

[3] C. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Grundlehren der Mathematischen Wissenschaften, vol. 325, Springer-Verlag, 2000.

[4] B. Gustafsson, A. Sundström, Incompletely parabolic problems in fluid dynamics, SIAM J. Appl. Math. 35 (2) (1978) 343–357.

[5] I. Kostin, G. Panasenko, Analysis and homogenization of the Khokhlov–Zabolotskaya–Kuznetsov type equation, Oral communication in the International Conference on Advanced Problems in Mechanics, St Petersburg, June 2005.

[6] S. Novo, Compressible Navier–Stokes model with inflow–outflow boundary conditions, J. Math. Fluid Mech. 7 (2005) 485–514.

[7] D. Sanchez, Long waves in ferromagnetic media, Zabolotskaya–Khokhlov equation, J. Differential Equations 210 (2005) 263–289.

[8] E.A. Zabolotskaya, R.V. Khokhlov, Quasi-planes waves in the nonlinear acoustic of confined beams, Sov. Phys. Acoust. 15 (1) (1969) 35–40.