Algebraic Geometry

Hironaka's characteristic polygon and effective resolution of surfaces

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Abstract

Hironaka's concept of a characteristic polyhedron of a singularity has been one of the most powerful and fruitful ideas of the last decades in singularity theory. In fact, since then, combinatorics have become a major tool in many important results. However, this seminal concept is still not enough to cope with some effective problems: for instance, giving a bound on the maximum number of blowing-ups to be performed on a surface before its multiplicity decreases. This short Note shows why such a bounding is not possible, with the original definitions. *To cite this article: R. Piedra, J.M. Tornero, C. R. Acad. Sci. Paris, Ser. I 344 (2007).*
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Résumé

Le polygone caractéristique d'Hironaka et la résolution effective des surfaces. Le concept, introduit par Hironaka, du polyèdre caractéristique d'une singularité a été une des idées les plus puissantes et profitables des dernières décennies dans la théorie des singularités. En fait, depuis son apparition les combinatoires sont devenues un outil central pour plusieurs résultats importants dans ce domaine. Pourtant, ce concept séminal n'est pas encore suffisant pour gérer quelques problèmes effectifs : par exemple, trouver une borne supérieure pour le nombre d'éclatements qu'on peut appliquer à une surface sans faire descendre sa multiplicité. Dans cette brève Note on montre pourquoi l'obtention d'une telle borne n'est pas possible, au moins avec les définitions originales. *Pour citer cet article : R. Piedra, J.M. Tornero, C. R. Acad. Sci. Paris, Ser. I 344 (2007).*
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1. Introduction

In this Note we will deal with embedded algebroid surfaces, that is, schemes given by the spectrum of a ring $R = K[[X, Y, Z]]/(F)$, where K is an algebraically closed field and F is a power series of order $n > 0$. F will be called an equation of the surface and n will be called the multiplicity of the surface.

Theorem 1 *(Levi–Zariski, algebroid version). Let $\mathcal{S}$ be an algebroid embedded surface with normal crossing singularities. Then, if we successively blow up smooth equimultiple subvarieties of maximal dimension, the multiplicity is dropped in a finite number of steps.*

The fact that a surface can be resolved by blowing up maximal centers was already investigated by Beppo Levi [7] and actually proved by Zariski [11] in characteristic zero as a part of his proof of the resolution for three-dimensional varieties. His results and techniques were paralleled by Abhyankar in positive characteristic [1].

It was Hironaka, however, who got a massive breakthrough by considering for the first time the use of combinatorial tools in singularity theory [5,6], a technique which has made it possible to tackle a number of problems (see for instance [3,9,4,8]).

As for surfaces concerned, a combinatorial approach to the Levi–Zariski theorem was pointed out by Hironaka in the introduction of [5], and partially developed in [6], although the arguments are not very clear in the positive characteristic case. The original purpose of this paper was sharing the combinatorial approach, using a somehow different induction argument to produce an upper bound for the number of blowing-ups that can be performed, following the Levi–Zariski procedure, before the multiplicity drops. In the case of curves, this was attached quite straightforwardly using the first characteristic exponent [5,10]. However, we have found that already for surfaces this bounding is not possible (at least using Hironaka's characteristic polygon).

2. Some technical set-up

For the sake of completeness, we recall here well-known technical results that will be of some help in the following. Let $\mathcal{S}$ be an embedded algebroid surface of multiplicity n, F an equation of $\mathcal{S}$. After a linear change of variables, using the Weierstrass Preparation Theorem, one can assume F to be in so-called Weierstrass form

$$F(X, Y, Z) = Z^n + \sum_{k=0}^{n-1} a_k Z^k, \quad \text{where } a_k(X, Y) = \sum_{i,j} a_{ijk} X^i Y^j.$$

To this situation a combinatorial object may be attached: note

$$N_{\{X,Y,Z\}}(F) = \left\{ (i, j, k) \in \mathbb{N}^3 \mid a_{ijk} \neq 0 \right\} \cup \left\{ (0, 0, n) \right\},$$

where we will omit the subscript whenever the variables are clear from the context. The Hironaka (or Newton, or Newton–Hironaka, ...) polygon of F is

$$\Delta_{\{X,Y,Z\}}(F)\big(\text{or } \Delta(F)\big) = \mathrm{CH}\left(\bigcup_{a_{ijk} \neq 0} \left[\left(\frac{i}{n-k}, \frac{j}{n-k} \right) + \mathbb{N}^2 \right] \right),$$

where CH stands for the convex hull. This object was already used in the famous Bowdoin notes lectures by Hironaka [6] and it appeared in printed form for the first time in the outstanding paper [5], where it covered the surface case of the much more general notion of characteristic polyhedron of a singularity.

If we allow Z to vary, using isomorphisms in $K[[X, Y, Z]]$ of the type $Z \mapsto Z + \alpha(X, Y)$, with $\alpha \in K[[X, Y]]$ not a unit, we obtain a collection of polygons which has a minimal element in the sense of inclusion (this is not obvious at all in positive characteristic). This object, noted $\Delta(\mathcal{S}, \{X, Y\})$, was called by Hironaka the characteristic polygon of the pair $(\mathcal{S}, \{X, Y\})$ [5].

For the case of characteristic zero or, more broadly, the case where n is not divided by the characteristic of K, it is customary to make the Tchirnhausen transformation, $Z \mapsto Z - a_{n-1}(X, Y)/n$, which is a change of variables of the type considered by Hironaka, for getting an equation with $a_{n-1} = 0$. Such an equation will be called a WT equation and it has many interesting properties, some of which we will recall. To begin with, in these equations, a permitted (that is, equimultiple and smooth) curve can be written in the form $\mathfrak{p} = (Z, G(X, Y))$.

Definition 2. A vertex (P_1, P_2) of $\Delta(F)$ is called *contractible* if there exists a change of variables φ, given by $Z \mapsto Z + \alpha X^a Y^b$, with $\alpha \in K$, such that $\Delta(\varphi(F)) \subset \Delta(F) \setminus \{(P_1, P_2)\}$. Were this the case, φ is called the contraction of the vertex (P_1, P_2).

If we can apply the Tchirnhausen transformation, the resulting equation has no contractible vertices. In fact, a vertex (a, b) is contractible if and only if it represents *all* the monomials from $(Z + \alpha X^a Y^b)^n$ and this cannot happen since $a_{n-1} = 0$. As it becomes obvious from the equations associated to the different blowing-ups this situation will remain during the resolution process (at least, until a multiplicity decrease happens). In classical terms, $Z = 0$ is a linear hypersurface with permanent maximal contact with the surface $\mathcal{S}$.

Hironaka proved in [5] (for arbitrary characteristic) that all the vertices of $\Delta(F)$ are not contractible if and only if $\Delta(F) = \Delta(\mathcal{S}, \{X, Y\})$. From the previous remark this is obvious in the case of WT equations.

3. Interesting examples

We will first prove that Hironaka's characteristic polygon does not contain enough information in order to bound the resolution process, even if we are interested only in a first multiplicity decreasing.

To see that, assume that K has characteristic other than 3 and consider the surface $\mathcal{S}$ defined by the equation

$$F = Z^3 + X^m Z + (X - Y)^4, \quad \text{with } m \geqslant 19;$$

which is, obviously, a WT equation and, hence, the characteristic polygon of $\mathcal{S}$ has exactly two vertices, at $(4/3, 0)$ and at $(0, 4/3)$, regardless of m.

We will first make, in absence of permitted curves, a quadratic transformation (that is, blowing-up the origin) on the direction $(1 : 1 : 0)$, giving as a result a surface $\mathcal{S}_1$. Next we will make two quadratic transformations on the direction $(1 : 0 : 0)$ (no permitted curves in either surfaces) after which we get a new surface $\mathcal{S}_3$. We are forced now to perform a monoidal transformation (blowing-up of a curve) centered on (Z, X), as it is now permitted. We get then $\mathcal{S}_4$, defined by

$$F_4 = Z^3 + X^{m-8} Z + Y^4.$$

Now it is straightforward that, after three quadratic transformations centered on $(1 : 0 : 0)$ and a monoidal transformation center on (Z, X), we should get a surface $\mathcal{S}_8$ defined by

$$F_8 = Z^3 + X^{m-16} Z + Y^4.$$

Obviously this implies that it is not possible to get a bound for the number of blowing-ups needed for decreasing the multiplicity of $\mathcal{S}$: as we change m we get a family of surfaces with the same Hironaka polygon but needing an arbitrarily large number of blowing-ups to get a multiplicity loss.

The key for this counterexample is the first quadratic transformation; in fact it is easy to bound a resolution process using uniquely monoidal transformations or quadratic transformations centered in $(1 : 0 : 0)$ and/or $(0 : 1 : 0)$ (this is connected with the Weak Hironaka's Polyhedra Game, solved by Spivakovsky in [9]).

However, as this example makes apparent, if we ever want to bound the resolution process we need to be able to track much more complicated relations between our parameters than the ones considered by Hironaka. In this spirit, one may think of 'sufficiently' general polygons, which remain fixed by isomorphisms inside $K[[X, Y]]$ or may think of all the possible polygons obtained by looking at all the systems of parameters in $K[[X, Y, Z]]$. The following example, however, shows that this is not enough, if we want a complete account of the resolution process.

Consider the following equations (we thank the referee for this inspiring example) with $\mathrm{ch}(K) \neq 2$:

$$F = Z^2 + \left(Y^2 - X^3\right)^3, \qquad G = Z^2 + \left(Y^2 - X^3\right)^3 + Y^\lambda, \quad \lambda \gg 0;$$

which have in fact the same Newton polygon, regardless of variable changes. If one tries to resolve F, one has to blow-up the origin and can either:

(a) Choose the direction $(1 : 0 : 0)$, and then blow-up twice the former exceptional divisor. Finally, one should blow-up the exceptional curve $(Z, Y - X^2)$. This solves the singularity in four blowing-ups.
(b) Choose $(0 : 1 : 0)$ which leads to a similar argument (only three blowing-ups to be performed here).
(c) Choose $(1 : \alpha : 0)$, for $\alpha \neq 0$, which also needs three blowing-ups to be solved.

Moving now to G, the *perturbed example*, as there are no permitted curves, we also must blow up the origin. In this case, if we start with $(1 : 0 : 0)$, we can blow up twice of the exceptional divisor. Now there is no permitted

curve to blow up. So we choose the origin again as center. In any direction other than $(0 : 1 : 0)$ this also resolves the singularity. Finally, choosing $(0 : 1 : 0)$, we get a surface which is now solved by a single blowing-up. All in all, up to five transformations are needed to decrease the multiplicity.

Henceforth, same Hironaka polygon does not mean isomorphic resolution trees, which is the combinatorial object encoding then all possible resolution processes [8]. However, whether this tree, or at least the significant part à la Levi–Zariski, can be bounded from a sufficiently generic polygon is a still open question.

Remark 3. We should finally note that for the Artin–Schreier case much more can be known about bounding the resolution process, as it is shown in [2], unfortunately unpublished. The authors thank the referee for pointing out this valuable reference which may lead the way to a more general result.

Acknowledgements

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