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Characterization of the kernel of the operator CURL CURL

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Abstract

In a simply-connected domain Ω in $\mathbb{R}^3$, the kernel of the operator **CURL CURL** acting on symmetric matrix fields from $\mathbb{L}^2_s(\Omega)$ to $\mathbb{H}^{-2}_s(\Omega)$ coincides with the space of linearized strain tensor fields. For not simply-connected domains, Volterra has characterized this kernel for smooth fields. Here we extend this result for domains with a Lipschitz-continuous boundary for fields in $\mathbb{L}^2_s(\Omega)$. ***To cite this article: P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Caractérisation du noyau de l'opérateur CURL CURL. Dans un domaine simplement connexe Ω de $\mathbb{R}^3$, le noyau de l'opérateur **CURL CURL** agissant sur des champs de matrices symétriques de $\mathbb{L}^2_s(\Omega)$ dans $\mathbb{H}^{-2}_s(\Omega)$, coïncide avec l'espace des champs de tenseurs de déformation linéarisés. Dans le cas de domaines non simplement connexes, Volterra a caractérisé ce noyau pour des champs réguliers. Dans cette Note, nous étendons ce résultat pour un domaine à frontière lipschitzienne et pour des champs dans $\mathbb{L}^2_s(\Omega)$. ***Pour citer cet article : P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

Let Ω be a domain in $\mathbb{R}^3$, i.e., an open, connected and bounded subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. The unit outward normal vector field to $\partial\Omega$ is denoted by $\mathbf{n}$. Latin indices range in the set $\{1, 2, 3\}$. The coordinates of a generic point $\mathbf{x} \in \overline{\Omega}$ are denoted by x_i, the components of a vector field $\mathbf{v}$ by v_i, and the components of a 3×3 matrix field $\mathbf{S}$ by S_{ij}. The summation convention with respect to repeated indices is used for Latin indices. Let $\mathbf{S}$ be a smooth symmetric matrix field. We denote by $\mathbf{CURL\,S}$ the tensor whose components are defined by $(\mathbf{CURL\,S})_{ij} = \epsilon_{ipk} S_{jk,p}$. The commas stand for partial derivatives and ϵ_{ipk} denote the components of the alternator tensor. Function spaces for scalar (respectively vector, or 3×3 matrix) fields are denoted with italic (respectively boldface, or capital boldface) characters. For the latter, the index $_s$ indicates symmetric fields.

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The operator **CURL CURL** is linear and continuous from $\mathbb{H}^2_s(\Omega)$ into $\mathbb{L}^2_s(\Omega)$. Beltrami's completeness theorem [8] provides a characterization of the ranges $\mathbf{CURL\,CURL}(\mathbb{H}^2_s(\Omega))$ and $\mathbf{CURL\,CURL}(\mathbb{H}^2_{0,s}(\Omega))$. Note that the characterization given in [8] for $\mathbf{CURL\,CURL}(\mathbb{H}^2_{0,s}(\Omega))$ is valid only for simply-connected domains. In the following, we study the operator **CURL CURL** from $\mathbb{L}^2_s(\Omega)$ into $\mathbb{H}^{-2}_s(\Omega)$, and in particular, we provide a direct characterization of its kernel. Let us remark that in the simply-connected case, the kernel is actually equal to $\nabla_s(\mathbf{H}^1(\Omega))$ according to [5] and [7]. Together, those results allow to characterize $\mathbf{CURL\,CURL}(\mathbb{H}^2_{0,s}(\Omega))$ for general, not simply-connected, domains.

2. Characterization of the kernel in the not simply-connected case

Since $\mathbf{CURL\,CURL}(\nabla_s \mathbf{v}) = 0$ in the distribution sense [3], it is clear that $\nabla_s(\mathbf{H}^1(\Omega))$ is a subset of the kernel. Therefore, we only have to study the intersection of the kernel with $\Sigma_{ad}(\Omega) = (\nabla_s(\mathbf{H}^1(\Omega)))^\perp$, where orthogonality is meant with respect to the usual $\mathbb{L}^2_s$ scalar product. By direct inspection, one finds that $\Sigma_{ad}(\Omega) = \{\mathbf{S} \in \mathbb{L}^2_s(\Omega);\ \mathbf{div}\,\mathbf{S} = 0 \text{ in } \Omega,\ \mathbf{Sn}_{|\partial\Omega} = \mathbf{0}\}$. It thus follows that the appropriate space is

$$\mathbb{K} = \left\{\mathbf{S} \in \mathbb{L}^2_s(\Omega)\colon\ \mathbf{CURL\,CURL\,S} = \mathbf{0} \text{ and } \mathbf{div}\,\mathbf{S} = \mathbf{0} \text{ in } \Omega,\ \mathbf{Sn}_{|\partial\Omega} = \mathbf{0}\right\}.$$

As noted above, one has $\mathbb{K} = \{\mathbf{0}\}$ in the simply-connected case.

In order to obtain such a characterization, we need to specify the geometry of Ω, as in [2]. We denote by Γ_q the connected components of $\partial\Omega$, $q = 0, \ldots, Q$. We assume that the domain Ω can be reduced to a simply-connected domain Ω^* by means of a finite number N of regular, non-intersecting, and oriented, cuts $\mathcal{C}_\alpha$, $\alpha = 1, \ldots, N$, such that the boundary of each cut $\mathcal{C}_\alpha$ is contained in $\partial\Omega$. We also assume that the cuts are such that the simply-connected domain $\Omega^* = \Omega \setminus \bigcup_{\alpha=1}^N \mathcal{C}_\alpha$ verifies the cone condition. Hence the usual Sobolev properties are satisfied [1,6].

Following an idea of Volterra [10], we introduce the following *space of Volterra's dislocations*:

$$\mathcal{V} = \left\{\mathbf{v} \in \mathbf{H}^1(\Omega^*)\colon\ [\![\mathbf{v}]\!]_{\mathcal{C}_\alpha} \text{ is an infinitesimal rigid displacement},\ \alpha = 1, \ldots, N\right\}, \tag{1}$$

where $[\![\mathbf{v}]\!]_{\mathcal{C}_\alpha}$ is the jump across the cut $\mathcal{C}_\alpha$. We recall that infinitesimal rigid displacements are of the form $\mathbf{a}^\alpha(\mathbf{v}) + \mathbf{b}^\alpha(\mathbf{v}) \wedge \mathbf{id}_\Omega$ where $\mathbf{a}^\alpha(\mathbf{v}) = a^\alpha_i(\mathbf{v})\mathbf{e}_i$ and $\mathbf{b}^\alpha(\mathbf{v}) \wedge \mathbf{id}_\Omega = b^\alpha_i(\mathbf{v})\mathbf{P}_i$, with $\mathbf{P}_i = -\epsilon_{ijk}x_k\mathbf{e}_j$.

We remark that, given $\mathbf{v} \in \mathcal{V}$, then by definition $\nabla_s \mathbf{v} \in \mathbb{L}^2_s(\Omega^*)$. Since $\text{meas}(\Omega) = \text{meas}(\Omega^*)$, $\mathbb{L}^2_s(\Omega^*)$ is isomorphic to $\mathbb{L}^2_s(\Omega)$. Hence one can associate with $\nabla_s \mathbf{v} \in \mathbb{L}^2_s(\Omega^*)$ its extension $\widetilde{\nabla_s \mathbf{v}} \in \mathbb{L}^2_s(\Omega)$ in a canonical way.

Proposition 2.1. *For every $\alpha = 1, \ldots, N$ and $i = 1, 2, 3$, there exist $\mathbf{u}^\alpha_i \in \mathcal{V}$ and $\mathbf{r}^\alpha_i \in \mathcal{V}$ such that*:

$$\int_{\Omega^*} \nabla_s \mathbf{u}^\alpha_i : \nabla_s \mathbf{v}\,\mathrm{d}\Omega - a^\alpha_i(\mathbf{v}) = 0 \quad \textit{for all } \mathbf{v} \in \mathcal{V}; \qquad \int_{\Omega^*} \nabla_s \mathbf{r}^\alpha_i : \nabla_s \mathbf{v}\,\mathrm{d}\Omega - b^\alpha_i(\mathbf{v}) = 0 \quad \textit{for all } \mathbf{v} \in \mathcal{V}. \tag{2}$$

Moreover, the vector fields $\mathbf{u}^\alpha_i$ and $\mathbf{r}^\alpha_i$ are uniquely determined modulo a global infinitesimal rigid displacement on Ω.

One notices that, according to (2), the $6N$ extensions $(\widetilde{\nabla_s \mathbf{u}^\alpha_i})_{\alpha,i}$ and $(\widetilde{\nabla_s \mathbf{r}^\alpha_i})_{\alpha,i}$ are linearly independent in $\mathbb{L}^2_s(\Omega)$.

Theorem 2.1. *The extensions $(\widetilde{\nabla_s \mathbf{u}^\alpha_i})_{\alpha,i}$ and $(\widetilde{\nabla_s \mathbf{r}^\alpha_i})_{\alpha,i}$ belong to the space $\mathbb{K}$.*

Proof. Since $\mathbf{D}(\Omega) \subset \mathcal{V}$ it follows from (2) that, in the distribution sense,

$$\mathbf{div}\big(\widetilde{\nabla_s \mathbf{u}^\alpha_i}\big) = \mathbf{0} \quad \text{in } \Omega. \tag{3}$$

Taking $\mathbf{v} \in \mathbf{H}^1(\Omega) \subset \mathcal{V}$, one then finds that:

$$\big(\widetilde{\nabla_s \mathbf{u}^\alpha_i}\big)\mathbf{n}_{|\partial\Omega} = \mathbf{0} \quad \text{in } \mathbf{H}^{-1/2}(\partial\Omega). \tag{4}$$

Hence we conclude that $\widetilde{\nabla_s \mathbf{u}^\alpha_i}$ belongs to $\Sigma_{ad}(\Omega)$. We only have to prove that, for all $\mathbf{S} \in \mathbb{D}_s(\Omega)$, one has

$$\big\langle \mathbf{CURL\,CURL}\big(\widetilde{\nabla_s \mathbf{u}^\alpha_i}\big), \mathbf{S}\big\rangle = 0, \tag{5}$$

where $\langle \cdot\,, \cdot \rangle$ denotes the duality pairing between $\mathbb{D}'_s(\Omega)$ and $\mathbb{D}_s(\Omega)$. This result is a consequence of the relation:

$$\begin{aligned}\langle \mathbf{CURL\,CURL}(\widetilde{\nabla_s \mathbf{u}_i^\alpha}), \mathbf{S}\rangle &= \int_\Omega \widetilde{\nabla_s \mathbf{u}_i^\alpha} : \mathbf{CURL\,CURL\,S}\,\mathrm{d}\Omega = \int_{\Omega^*} \nabla_s \mathbf{u}_i^\alpha : \mathbf{CURL\,CURL\,S}\,\mathrm{d}\Omega \\ &= \int_{\partial\Omega^*} \mathbf{u}_i^\alpha \cdot (\mathbf{CURL\,CURL\,S})\mathbf{n}\,\mathrm{d}\Gamma = \sum_\alpha \int_{\mathcal{C}_\alpha} [\![\mathbf{u}_i^\alpha]\!] \cdot (\mathbf{CURL\,CURL\,S})\mathbf{n}\,\mathrm{d}\mathcal{C} \\ &= \sum_\alpha \int_{\mathcal{C}_\alpha} \big(\mathbf{a}^\alpha(\mathbf{u}_i^\alpha) + \mathbf{b}^\alpha(\mathbf{u}_i^\alpha) \wedge \mathbf{id}_\Omega\big) \cdot (\mathbf{CURL\,CURL\,S})\mathbf{n}\,\mathrm{d}\mathcal{C} = 0.\end{aligned}$$

The last equality follows from a localization argument around each cut. This allows one to consider each term separately, and one can perform the standard integration by parts. This expression vanishes since

$$\mathbf{div}(\mathbf{CURL\,CURL\,S}) = \mathbf{0} \quad \text{and} \quad \nabla_s\big(\mathbf{a}^\alpha(\mathbf{u}_i^\alpha) + \mathbf{b}^\alpha(\mathbf{u}_i^\alpha) \wedge \mathbf{id}_\Omega\big) = \mathbf{0}.$$

The same proof holds for $\widetilde{\nabla_s \mathbf{r}_i^\alpha}$. □

We can now state the announced characterization, at least for a specific class of cuts:

Theorem 2.2. *Assume that all the cuts $\mathcal{C}_\alpha$ are planar. Then the space $\mathbb{K}$ is spanned by the matrix fields $\widetilde{\nabla_s \mathbf{u}_i^\alpha}$ and $\widetilde{\nabla_s \mathbf{r}_i^\alpha}$, $\alpha = 1, \ldots, N$, $i = 1, 2, 3$.*

Proof. Given $\mathbf{W} \in \mathbb{K}$, let $\mathbf{Z}$ be defined by:

$$\mathbf{Z} = \mathbf{W} - \sum_{\alpha=1}^N \big\{\langle \mathbf{Wn}, \mathbf{e}_i\rangle_{\mathcal{C}_\alpha} \widetilde{\nabla_s \mathbf{u}_i^\alpha}\big\} - \sum_{\alpha=1}^N \big\{\langle \mathbf{Wn}, \mathbf{P}_i\rangle_{\mathcal{C}_\alpha} \widetilde{\nabla_s \mathbf{r}_i^\alpha}\big\},$$

where $\langle\cdot,\cdot\rangle_{\mathcal{C}_\alpha}$ denotes the duality pairing between $\mathbf{H}^{-1/2}(\mathcal{C}_\alpha)$ and $\mathbf{H}^{1/2}(\mathcal{C}_\alpha)$. The assumption on $\mathbf{W}$ implies that $\mathbf{CURL\,CURL}(\mathbf{Z}_{|\Omega^*}) = 0$. Because Ω^* is simply-connected, there exists $\hat{\mathbf{u}} \in \mathbf{H}^1(\Omega^*)$ such that $\mathbf{Z}_{|\Omega^*} = \nabla_s \hat{\mathbf{u}}$ (see [5,8]). Using Green's formula in Ω^* and Eqs. (2), one can prove that $\int_{\Omega^*} \nabla_s \hat{\mathbf{u}} : \nabla_s \mathbf{v}\,\mathrm{d}\Omega = 0$ for all $\mathbf{v} \in \mathcal{V}$. When the cuts are planar, one can prove directly, using integration by parts on each cut $\mathcal{C}_\alpha$, that $[\![\hat{\mathbf{u}}]\!]_{\mathcal{C}_\alpha}$ is actually an infinitesimal rigid displacement; hence $\hat{\mathbf{u}}$ belongs to $\mathcal{V}$. It follows that $\mathbf{Z}_{|\Omega^*} = \mathbf{0}$ and so $\mathbf{Z} = \mathbf{0}$. □

Since the matrix fields $(\widetilde{\nabla_s \mathbf{u}_i^\alpha})_{\alpha,i}$ and $(\widetilde{\nabla_s \mathbf{r}_i^\alpha})_{\alpha,i}$ are linearly independent in $\mathbb{L}_s^2(\Omega)$, we also have:

Corollary 2.1. *Assume that all the cuts $\mathcal{C}_\alpha$ are planar. Then the space $\mathbb{K}$ is of dimension $6N$.*

Note that these results could be integrated in the definition of the de Rham complex for symmetric matrices, as in [4] for elasticity and [9] for magnetostatics.

Corollary 2.2. *Assume that all the cuts $\mathcal{C}_\alpha$ are planar. Then $\Sigma_{ad}(\Omega) = \mathbb{K} \overset{\perp}{\oplus} \mathbb{X}$ with*

$$\mathbb{X} = \big\{\mathbf{S} \in \Sigma_{ad}(\Omega)\colon \langle \mathbf{Sn}, \mathbf{e}_i\rangle_{\mathcal{C}_\alpha} = 0,\ \langle \mathbf{Sn}, \mathbf{P}_i\rangle_{\mathcal{C}_\alpha} = 0,\ \alpha = 1, \ldots, N,\ i = 1, 2, 3\big\}.$$

Moreover, the definition of the space $\mathbb{X}$ is independent of the way the cuts are defined.

Using the above results, we generalize the second statement of Beltrami's completeness theorem given in [8] (Theorem 2.2(ii)), which is correct only in a simply-connected domain, to the case of a not simply-connected domain:

Theorem 2.3. *Assume that all the cuts $\mathcal{C}_\alpha$ are planar. Then* $\mathbf{CURL\,CURL}(\mathbb{H}_{0,s}^2(\Omega)) = \mathbb{X}$.

Finally, using the first statement of Beltrami's completeness theorem ([8] Theorem 2.2(i)), one can prove the following result:

Theorem 2.4. *We have* $\mathbb{L}_s^2(\Omega) = \mathbf{CURL}\,\mathbf{CURL}(\mathbb{H}_s^2(\Omega)) \overset{\perp}{\oplus} \mathbb{Y}$, *with*

$$\mathbb{Y} = \left\{\mathbf{S} \in \mathbb{L}_s^2(\Omega) \colon\ \mathbf{S} = \nabla_s \mathbf{u},\ \mathbf{u} \in \mathbf{H}^1(\Omega),\ \mathbf{u}_{|\Gamma_q} = \mathbf{a}_q + \mathbf{b}_q \wedge \mathbf{id}_\Omega,\ q = 0, \dots, Q\right\}.$$

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