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C. R. Acad. Sci. Paris, Ser. I 344 (2007) 247–252
COMPTES RENDUS MATHEMATIQUE
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Topology/Geometry

String topology for loop stacks

Kai Behrend [a], Grégory Ginot [b], Behrang Noohi [c], Ping Xu [d,1]

[a] *Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, B.C., Canada V6T 1Z2*
[b] *École normale supérieure de Cachan et université Paris 13, CMLA, 61, avenue du Président Wilson, 94230 Cachan cedex, France*
[c] *Max Planck Institut für Mathematik, Vivastsgasse 7, 53111 Bonn, Germany*
[d] *Pennsylvania State University, 210 McAllister Building, University Park, PA 16802, USA*

Received 9 August 2006; accepted 11 October 2006

Presented by Charles-Michel Marle

Abstract

We prove that the homology groups of the free loop stack of an oriented stack are equipped with a canonical loop product and coproduct, which makes it into a Frobenius algebra. Moreover, the shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ admits a BV algebra structure. ***To cite this article: K. Behrend et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Topologie des cordes pour les lacets libres d'un champ. On munit les groupes d'homologie du champ des lacets libres d'un champ orienté d'un produit et d'un coproduit induisant une structure d'algèbre de Frobenius. De plus, l'homologie en degrés décalés $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ est une algèbre BV. ***Pour citer cet article : K. Behrend et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Dans cette Note on généralise le produit de Chas et Sullivan [3], défini sur les groupes d'homologie de l'espace des lacets libres d'une variété orientée, aux champs. On généralise également deux autres constructions de la topologie des cordes : l'opérateur BV [3] et le coproduit [4]. Pour ce faire, la bonne notion de champ est celle de champ différentiel muni d'une diagonale normalement non-singulière orientée (cf. [1]). Nous dirons d'un tel champ qu'il est *orienté*. Rappelons qu'un champ est dit topologique s'il est représentable par un groupoide topologique et que les champs différentiels sont ceux représentables par un groupoide de Lie. A un champ topologique $\mathfrak{X}$, on peut associer fonctoriellement un champ $\mathrm{L}\mathfrak{X} = \mathrm{hom}(S^1, \mathfrak{X})$ qui est topologique. Ici, hom désigne le champ des morphismes de champ [9]. De fait, pour toute présentation Γ de $\mathfrak{X}$, on donne une construction naturelle d'un groupoide $\mathrm{L}\Gamma$ représen-

E-mail addresses: behrend@maths.ubc.ca (K. Behrend), ginot@cmla.ens-cachan.fr (G. Ginot), bnoohi@mpim-bonn.mpg.de (B. Noohi), ping@math.psu.edu (P. Xu).
1 Research partially supported by NSF grants DMS-0306665 and DMS-0605725 & NSA grant H98230-06-1-0047.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.10.006

tant $\mathrm{L}\mathfrak{X}$. Cette construction s'obtient comme la limite, sur tous les sous-ensembles finis $P = \{0 < p_1 < \cdots < p_{n-1}\}$ de $S^1 = [0,1]/\{0 \sim 1\}$ ($n \geqslant 1$), du groupoide des morphismes de groupoides stricts $S^P \to M\Gamma$. Le groupoide $M\Gamma$ est le groupoide des carrés commutatifs dans la catégorie Γ et S^P est le groupoide $S_0^P \times_{S^1} S_0^P \rightrightarrows S_0^P$ où, en notant $p_n = 1 = 0$, S_0^P est la réunion disjointe $\coprod_{i=1}^n [p_{i-1}, p_i]$.

De manière similaire à [1,2], pour tout champ différentiel orienté $\mathfrak{X}$ de dimension d, on construit un produit

$$\star : H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X})$$

et un coproduit

$$\delta : H_\bullet(\mathrm{L}\mathfrak{X}) \longrightarrow \bigoplus_{i+j=\bullet-d} H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X}).$$

Les propriétés de fonctorialité de $\mathrm{L}\mathfrak{X}$ lui confèrent une action de S^1. On obtient alors un opérateur $D : H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$ qui est l'application composée :

$$H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{\times\omega} H_{\bullet+1}(\mathrm{L}\mathfrak{X} \times S^1) \longrightarrow H_{\bullet+1}(\mathrm{L}\mathfrak{X}),$$

où $\omega \in H_1(S^1)$ est la classe fondamentale de S^1. La dernière flèche est induite par l'action de S^1 sur $\mathrm{L}\mathfrak{X}$. Les propriétés de fonctorialité et de naturalité des morphismes de Gysin (cf. [1] Proposition 2.2) autorisent l'utilisation des méthodes opéradiques de Cohen, Jones et Voronov [5,6] pour montrer que D est un opérateur BV. Les résultats principaux de cette note sont résumés dans le théorème suivant :

Théorème 0.1. *Soit $\mathfrak{X}$ un champ orienté de dimension d.*

(i) *L'homologie en degrés décalés $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$, munie du produit $\star : \mathbb{H}_*(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_*(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_*(\mathrm{L}\mathfrak{X})$ et de l'opérateur $D : \mathbb{H}_*(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_{*+1}(\mathrm{L}\mathfrak{X})$, est une algèbre BV.*
(ii) *De plus l'homologie $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ est une algèbre de Frobenius non nécessairement (co)unitaire.*

L'homologie du champ d'inertie $\Lambda\mathfrak{X}$ associé à $\mathfrak{X}$ est aussi munie d'une structure d'algèbre de Frobenius [2]. Par ailleurs, il existe un morphisme naturel de champs $\Phi : \Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$.

Théorème 0.2. *L'application induite $\Phi_* : H_\bullet(\Lambda\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X})$ est un morphsime d'algèbres de Frobenius.*

1. Introduction

In this Note, we generalize the loop product [3] and coproduct [4], as well as the BV-operator [3], for loop homology of manifolds to stacks. The relevant notion is oriented differential stacks [1], which we simply call oriented stacks in the Note. In fact, if $\mathfrak{X}$ is a topological stack, there is a functorial construction of the free loop stack $\mathrm{L}\mathfrak{X}$ which is a topological stack. Indeed for any presentation Γ of the stack $\mathfrak{X}$, we present a natural construction of a topological groupoid, which represents the free loop stack $\mathrm{L}\mathfrak{X}$. Similar to the constructions in [1,2], for any oriented differential stack of dimension d, we construct a product

$$\star : H_\bullet(\mathrm{L}\mathfrak{X}) \otimes H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet-d}(\mathrm{L}\mathfrak{X}),$$

and a coproduct

$$\delta : H_\bullet(\mathrm{L}\mathfrak{X}) \longrightarrow \bigoplus_{i+j=\bullet-d} H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X})$$

which makes $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ into a Frobenius algebra. Furthermore, we give a natural map $\Phi : \Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$, where $\Lambda\mathfrak{X}$ is the inertia stack of $\mathfrak{X}$, which induces a nontrivial morphism of Frobenius algebras in homology.

Due to its functorial property, $\mathrm{L}\mathfrak{X}$ admits a natural S^1-action which induces a square zero operator $D : H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$. We prove that the shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ together with the string product $\star : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ and the operator $D : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_{\bullet+1}(\mathrm{L}\mathfrak{X})$ becomes a BV-algebra.

2. Free loop stack

We use the conventions and notations from [1,2]. Let $\mathfrak{X}$ be a topological stack and A a compactly generated topological space. We define the stack $\mathrm{Map}(A, \mathfrak{X})$, called the **mapping stack** from A to $\mathfrak{X}$, by the rule

$$T \in \mathbf{Top} \mapsto \hom(T \times A, \mathfrak{X}).$$

Here, the right-hand side stands for the groupoid of stack morphisms from $T \times A$ to $\mathfrak{X}$. The mapping stack $\mathrm{Map}(A, \mathfrak{X})$ is functorial in A and $\mathfrak{X}$.

Lemma 2.1. *If A is compact, then* $\mathrm{Map}(A, \mathfrak{X})$ *is a topological stack.*

In the case where $\mathfrak{X}$ is a manifold, $\mathrm{Map}(A, \mathfrak{X})$ represents the usual mapping space with the compact-open topology.

When $A = S^1$ is the unit circle, we denote $\mathrm{Map}(S^1, \mathfrak{X})$ by $\mathrm{L}\mathfrak{X}$ and call it the **free loop stack** of $\mathfrak{X}$. By functoriality of mapping stacks, for every $t \in S^1$ we have the corresponding evaluation map $\mathrm{ev}_t : \mathrm{L}\mathfrak{X} \to \mathfrak{X}$.

Let us now describe, for any presentation $\Gamma : \Gamma_1 \rightrightarrows \Gamma_0$ of a topological stack $\mathfrak{X}$, a natural useful construction of a groupoid which represents the free loop stack $\mathrm{L}\mathfrak{X}$. We use this presentation at the end of Section 5. Note that our construction is somehow similar to the construction of the fundamental groupoid of a groupoid [8]. Let $P \subset S^1$ be a finite subset of S^1 which contains the base point $0 \sim 1 \in S^1$. The points of P are labeled according to increasing angle as $P_0, P_1, \ldots, P_n$ in such a way that $P_0 = P_n$ is the base point of S^1. Write I_i for the closed interval $[P_{i-1}, P_i]$. Let S_0^P be the disjoint union $S_0^P = \coprod_{i=1}^n I_i$. There is a canonical map $S_0^P \to S^1$. Let S_1^P be the fiber product $S_1^P = S_0^P \times_{S^1} S_0^P$. There is an obvious topological groupoid structure $S_1^P \rightrightarrows S_0^P$. The compact-open topology induces a topological groupoid structure on $\mathrm{L}^P\Gamma : \mathrm{L}_1^P\Gamma \rightrightarrows \mathrm{L}_0^P\Gamma$, where $\mathrm{L}_0^P\Gamma$ is the set of continuous strict groupoid morphisms $[S_1^P \rightrightarrows S_0^P] \to [\Gamma_1 \rightrightarrows \Gamma_0]$ and $\mathrm{L}_1^P\Gamma$ is the set of strict continuous groupoid morphisms $[S_1^P \rightrightarrows S_0^P] \to [M_1\Gamma \rightrightarrows M_0\Gamma]$. Here $M\Gamma = [M_1\Gamma \rightrightarrows M_0\Gamma]$ is the morphism groupoid of Γ. Recall that the groupoid $M\Gamma$ is the groupoid where $M_1\Gamma$ is the set of commutative squares

$$\begin{array}{ccc} t(h) & \xleftarrow{\;g\;} & t(k) \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle k} \\ s(h) & \xleftarrow{\;h^{-1}gk\;} & s(k) \end{array} \tag{1}$$

in the underlying category of Γ. The source and target maps are the horizontal arrows as in square (1) and the groupoid multiplication is by superposition of squares. Thus we have $M_0\Gamma \cong \Gamma_1$ and $M_1\Gamma \cong \Gamma_3 = \Gamma_1 \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} \Gamma_1$.

Remark 1. Note that this topological groupoid structure is analogous to the one used in Section 2 of [2].

Lemma 2.2. *Let Γ be a groupoid representing a topological stack $\mathfrak{X}$. The limit*

$$\mathrm{L}\Gamma = \varinjlim_{P \subset S^1} \mathrm{L}^P\Gamma$$

represents the free loop stack $\mathrm{L}\mathfrak{X}$.

It is easy to represent evaluation map and functorial properties of the free loop stack at the groupoid level with this model.

3. Loop product

In this section we consider *oriented* stacks. Recall that a differential stack $\mathfrak{X}$ is called oriented if the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is an oriented normally nonsingular morphism [1]. For instance, oriented manifolds and oriented orbifolds are oriented stacks. More generally, the quotient stack of a compact Lie group acting by orientation preserving automorphisms on an oriented manifold is an oriented stack.

Consider the Cartesian diagram

$$\begin{array}{ccc} \mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X} & \longrightarrow & \mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle(\mathrm{ev}_0,\mathrm{ev}_0)} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X}\times\mathfrak{X} \end{array}$$

The fact that $\mathfrak{X}$ is topological implies ([9], Proposition 16.1) that there is a natural equivalence of stacks

$$\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\cong \mathrm{Map}(8,\mathfrak{X}).$$

The map $S^1\to S^1\vee S^1$ that pinches $\frac{1}{2}$ to 0, induces a natural map $m:\mathrm{Map}(8,\mathfrak{X})\to \mathrm{L}\mathfrak{X}$, called the *Pontrjagin multiplication*. Putting these together, we have the following augmented Cartesian square:

$$\begin{array}{ccccc} \mathrm{L}\mathfrak{X} & \xleftarrow{m} & \mathrm{Map}(8,\mathfrak{X}) & \longrightarrow & \mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X} \\ & & \downarrow & & \downarrow{\scriptstyle e=(\mathrm{ev}_0,\mathrm{ev}_0)} \\ & & \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X}\times\mathfrak{X} \end{array} \tag{2}$$

By Proposition 2.2 in [1], since Δ is an oriented normally nonsingular morphism of codimension d, we have a Gysin map $G^e_\Delta: H_\bullet(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X})\to H_{\bullet-d}(\mathrm{Map}(8,\mathfrak{X}))$. We define the *loop product* to be the following composition

$$H_\bullet(\mathrm{L}\mathfrak{X})\otimes H_\bullet(\mathrm{L}\mathfrak{X})\cong H_\bullet(\mathrm{L}\mathfrak{X}\times\mathrm{L}\mathfrak{X})\xrightarrow{G^e_\Delta} H_{\bullet-d}\big(\mathrm{Map}(8,\mathfrak{X})\big)\xrightarrow{m_*} H_{\bullet-d}(\mathrm{L}\mathfrak{X}).$$

Theorem 3.1. *Let $\mathfrak{X}$ be an oriented stack of dimension d. The loop product induces a structure of associative and graded commutative algebra for the shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}):=H_{\bullet+d}(\mathrm{L}\mathfrak{X})$.*

Similar to the string product of inertia stacks, there is also a "twisted" version of loop product. Let α be a class in $H^r(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X})$. The *twisted loop product* $\star_\alpha: H_\bullet(\mathrm{L}\mathfrak{X})\otimes H_\bullet(\mathrm{L}\mathfrak{X})\to H_{\bullet-d-r}(\mathrm{L}\mathfrak{X})$ is defined, for all $x,y\in H_\bullet(\mathrm{L}\mathfrak{X})$, by

$$x\star_\alpha y=m_*\big(G^e_\Delta(x\times y)\cap\alpha\big).$$

Recall some notations from [1]: let $p_{12},p_{23}:\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\to\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}$ be, respectively, the projections on the first two and the last two factors. Also $(m\times 1)$ and $(1\times m)$: $\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\to\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}$ denote the Pontrjagin multiplication of the two first factors and two last factors respectively.

Theorem 3.2. *Let α be a class in $H^r(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X})$. If α satisfies the 2-cocycle condition*

$$p_{12}^*(x)\cup(m\times 1)^*(\alpha)=p_{23}^*(\alpha)\cup(1\times m)^*(\alpha)$$

in $H^\bullet(\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X})$, then $\star_e: H_i(\mathrm{L}\mathfrak{X})\otimes H_j(\mathrm{L}\mathfrak{X})\to H_{i+j-d-r}(\mathrm{L}\mathfrak{X})$ is associative.

4. BV-structure

In this section, we assume that singular homology is taken with coefficients in a field of characteristic different from 2. In this case, the homology of the free loop space of a manifold is known to be a BV-algebra [3]. The operadic approach of Cohen, Jones and Voronov [5,6] for constructing the BV-structure relies on the existence of evaluation maps, existence of Gysin maps and their functoriality and naturality properties. Thanks to Proposition 2.2 in [1], one can adapt their approach to stacks.

By the functorial properties of Lemma 2.1, the free loop stack $\mathrm{L}\mathfrak{X}$ inherits an S^1-action. Introduce an operator $D: H_\bullet(\mathrm{L}\mathfrak{X})\to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$ by the composition

$$H_\bullet(\mathrm{L}\mathfrak{X})\xrightarrow{\times\omega} H_{\bullet+1}\big(\mathrm{L}\mathfrak{X}\times S^1\big)\longrightarrow H_{\bullet+1}(\mathrm{L}\mathfrak{X}),$$

where $\omega\in H_1(S^1)$ is the fundamental class and the last arrow is induced by the action. It is not hard to check that $D^2=0$.

Theorem 4.1. *Let $\mathfrak{X}$ be an oriented stack of dimension d. The shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ admits a BV-algebra structure given by the loop product $\star : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ and the operator $D : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_{\bullet+1}(\mathrm{L}\mathfrak{X})$.*

Example 1. When $\mathfrak{X}$ is an oriented manifold M, then $\mathrm{L}\mathfrak{X}$ is the free loop space of M. Then the BV-structure on $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ coincides with the one of Chas and Sullivan [3,5].

If $\mathfrak{X}$ is a global quotient orbifold which is oriented, then the BV-structure on $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ coincides with the one introduced (in characteristic zero) in [7].

5. Frobenius structure and inertia stack

It is known [4] that there is also a coproduct on the homology of a free loop manifold which induces a Frobenius algebra structure. Also, in [2] it is shown that the homology of the inertia stack of an oriented stack $\mathfrak{X}$ admits a Frobenius algebra structure. Thus it is reasonable to expect that such a structure exist on $H_\bullet(\mathrm{L}\mathfrak{X})$ as well. We show that this is indeed the case. Let $\mathrm{ev}_0, \mathrm{ev}_{1/2} : \mathrm{L}\mathfrak{X} \to \mathfrak{X}$ be the evaluation maps defined in Section 2.

Lemma 5.1. *The stack $\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ fits into a Cartesian square*

$$\begin{array}{ccc} \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} & \longrightarrow & \mathrm{L}\mathfrak{X} \\ \downarrow & & \downarrow {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_{1/2})} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array} \tag{3}$$

where $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is the diagonal.

According to Proposition 2.2 of [1], if $\mathfrak{X}$ is an oriented differential stack of dimension d, the Cartesian square (3) yields a Gysin map

$$G_\Delta^{(\mathrm{ev}_0, \mathrm{ev}_{1/2})} : H_\bullet(\mathrm{L}\mathfrak{X}) \longrightarrow H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}).$$

By diagram (2), there is a map $\mathrm{Map}(8, \mathfrak{X}) \cong \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} \xrightarrow{\rho} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}$. Thus we obtain a degree d map

$$\delta : H_\bullet(\mathrm{L}\mathfrak{X}) \xrightarrow{G_\Delta^{(\mathrm{ev}_0, \mathrm{ev}_{1/2})}} H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \xrightarrow{\rho_*} H_{\bullet-d}(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \cong \bigoplus_{i+j=\bullet-d} H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X}).$$

Theorem 5.2. *Let $\mathfrak{X}$ be an oriented stack of dimension d. Then $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ is a Frobenius algebra, where both operations $\star$ and δ are of degree d.*

We now introduce a morphism of stacks $\Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$. Let Γ be a groupoid representing $\mathfrak{X}$ and $\Lambda\Gamma$ its inertia groupoid representing $\Lambda\mathfrak{X}$. We refer to [1,2] for details. Following the notations as in Lemma 2.2, we take $P = \{1\} \subset S^1$ as a trivial subset of S^1. Any element $(g,h) \in S\Gamma \rtimes \Gamma$ (i.e. $g \in \Gamma_1$ with $s(g) = t(g)$) determines a commutative diagram $D(g,h)$ in the category Γ

$$[\mathrm{D(g,h)}]: \quad \begin{array}{ccc} t(h) & \xleftarrow{g} & t(h) \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h} \\ s(h) & \xleftarrow{h^{-1}gh} & s(h), \end{array}$$

thus an element of $M_1\Gamma$. In particular it induces a (constant) groupoid morphism $[S_1^P \rightrightarrows S_0^P] \to [M_1\Gamma \rightrightarrows M_0\Gamma]$. The map $(g,h) \mapsto D(g,h)$ is easily seen to be a groupoid morphism. We denote by $\Phi : \Lambda\Gamma \to \mathrm{L}\Gamma$ its composition with the inclusion $\mathrm{L}^P\Gamma \to \mathrm{L}\Gamma$.

Lemma 5.3. *The map $\Phi : \Lambda\Gamma \to \mathrm{L}\Gamma$ induces a map of stacks $\Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$.*

Thus there is an induced map $\Phi_\star : H_\bullet(\Lambda\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X})$.

Theorem 5.4. *The map* $\Phi_\star : H_\bullet(\Lambda\mathfrak{X}) \to H_\bullet(\mathrm{L}\mathfrak{X})$ *is a morphism of Frobenius algebras.*

If $\mathfrak{X} = [*/G]$ with G being a compact Lie group, then $\mathrm{L}[*/G]$ is homotopy equivalent to $\Lambda[*/G]$ and the map $\Phi : H_\bullet(\Lambda[*/G]) \to H_\bullet(\mathrm{L}[*/G])$ is an isomorphism of Frobenius algebras. This Frobenius structure is studied (with real coefficients) in [2]. In this case, the BV-operator is trivial.

Acknowledgements

We would like to thank several institutions for their hospitality while work on this project was being done: Erwin Schrödinger International Institute for Mathematical Physics (Ginot, Noohi and Xu), Université Pierre et Marie Curie (Xu), Zhejiang University (Xu) and École normale supérieure de Cachan (Noohi).

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