Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 344 (2007) 219–224
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Mathematical Analysis

p-adic repellers in $\mathbb{Q}_p$ are subshifts of finite type

Aihua Fan [a,b], Lingmin Liao [a,b], Yue Fei Wang [c], Dan Zhou [b]

[a] *Department of Mathematics, Wuhan University, 430072 Wuhan, China*
[b] *LAMFA, UMR 6140 CNRS, université de Picardie, 33, rue Saint Leu, 80039 Amiens, France*
[c] *Institute of Mathematics, AMSS, CAS, 55 East Road Zhongguancun, 100080 Beijing, China*

Received 21 November 2006; accepted 12 December 2006

Available online 24 January 2007

Presented by Jean-Pierre Kahane

Abstract

We prove that any p-adic transitive weak repeller is isometrically conjugate to a subshift of finite type where a suitable metric is defined. ***To cite this article: A. Fan et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Les répulseurs p-adiques dans $\mathbb{Q}_p$ sont des sous-shifts de type fini. Nous prouvons que tout répulseur faible transitif p-adique est isométriquement conjugué à un sous-shift de type fini où une métrique convenable est définie. ***Pour citer cet article : A. Fan et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit $p \geqslant 2$ un nombre premier et $\mathbb{Q}_p$ le corps des nombres p-adiques. Soit $f : X \to \mathbb{Q}_p$ une application définie sur un compact ouvert X de $\mathbb{Q}_p$ à valeurs dans $\mathbb{Q}_p$. Nous supposons que : (i) $f^{-1}(X) \subset X$; (ii) $X = \bigsqcup_{i\in I} B_{p^{-\tau}}(c_i)$ (une réunion finie de boules disjointes de centres c_i et rayon $p^{-\tau}$, $\tau \in \mathbb{Z}$) et pour tout $i \in I$ il existe un entier $\tau_i \in \mathbb{Z}$ tel que

$$\big|f(x) - f(y)\big|_p = p^{\tau_i}|x-y|_p \quad (\forall x, y \in B_{p^{-\tau}}(c_i)). \tag{1}$$

Pour une telle application f, définissons son ensemble de Julia par

$$J_f = \bigcap_{n=0}^{\infty} f^{-n}(X). \tag{2}$$

Il est clair que $f^{-1}(J_f) = J_f$ et puis $f(J_f) \subset J_f$. Nous nous proposons d'étudier le système dynamique (J_f, f). Le triplet (X, J_f, f) s'appelle *répulseur faible* (p-adique) si tous les τ_i dans (1) sont positifs, dont au moins un est strictement positif. On l'appelle *répulseur* (p-adique) si tous les τ_i dans (1) sont strictement positifs.

E-mail addresses: ai-hua.fan@u-picardie.fr (A. Fan), lingmin.liao@u-picardie.fr (L. Liao), wangyf@amss.ac.cn (Y.F. Wang), maureen28@sina.com (D. Zhou).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.12.007

Supposons que tous les τ_i sont positifs. Pour tout $i \in I$, posons

$$I_i := \{ j \in I:\ B_j \cap f(B_i) \neq \emptyset \} = \{ j \in I:\ B_j \subset f(B_i) \}.$$

Puis definissons la matrice $A = (A_{i,j})_{I \times I}$, dite matrice d'incidence, par

$$A_{ij} = 1 \quad \text{si } j \in I_i; \qquad A_{ij} = 0 \quad \text{sinon.}$$

Si A est irréductible, on dit que (X, J_f, f) est *transitif*.

Soit Σ_A l'espace de sous-shift défini par A et soit σ le décalage à gauche sur Σ_A. Nous équipons Σ_A de la métrique d_f définie comme suit. Pour $x = (x_0, x_1, \ldots, x_n, \ldots) \in \Sigma_A$ et $y = (y_0, y_1, \ldots, y_n, \ldots) \in \Sigma_A$, définissons

$$d_f(x, y) = p^{-\tau_{x_0} - \tau_{x_1} - \cdots - \tau_{x_{n-1}} - \kappa(x_n, y_n)}$$

où $n = n(x, y) = \min\{i \geqslant 0:\ x_i \neq y_i\}$ et $p^{-\kappa(x_n, y_n)} = |c_{x_n} - c_{y_n}|_p$. Cette métrique d_f definit la même topologie que la métrique classique definie par $d(x, y) = p^{-n(x,y)}$.

Théorème 0.1. *Soit (X, J_f, f) un répulseur faible transitif p-adique avec A comme matrice d'incidence. La dynamique $(J_f, f, |\cdot|_p)$ est isométriquement conjuguée à la dynamique (Σ_A, σ, d_f).*

Remarque 0.2. Si la matrice d'incidence A n'est pas irréductible, l'ensemble d'indices I est décomposé en classes d'indices $I^{(1)}, I^{(2)}, \ldots, I^{(k)}$. On peut arranger $I^{(1)}, I^{(2)}, \ldots, I^{(k)}$ de sorte que A se présente comme une matrice triangulaire inférieure avec des sous-matrices irréductibles $A_1, A_2, \ldots, A_k$ sur sa diagonale (voir [1]). Alors nous pouvons appliquer le Théorème 0.1 à $f : X^{(t)} \to \mathbb{Q}$ pour tout $1 \leqslant t \leqslant k$, où $X^{(t)} = \bigcup_{i \in I^{(t)}} B_{p^{-\tau}}(c_i)$.

Remarque 0.3. Le Théorème 0.1 est valable non seulement pour les dynamiques p-adiques, mais aussi pour toute dynamique ultramétrique satisfaisant aux conditions (i) et (ii).

Le résultat suivant nous permet d'appliquer le Théorème 0.1 aux dynamiques polynomiales :

Théorème 0.4. *Soit $f \in \mathbb{Q}_p[x]$ un polynôme tel que $f'(x) \neq 0$ pour tout $x \in \mathbb{Q}_p$. Il existe un compact ouvert $X \subset \mathbb{Q}_p$ et un entier τ tels que les conditions* (i) *et* (ii) *soient satisfaites. De plus, pour tout $x \notin J_f$ on a* $\lim_{n\to\infty} |f^n(x)|_p = \infty$.

1. Statement of results

Let $p \geqslant 2$ be a prime number and $\mathbb{Q}_p$ be the field of p-adic numbers. Let $f : X \to \mathbb{Q}_p$ be a map from a compact open set X of $\mathbb{Q}_p$ into $\mathbb{Q}_p$. We assume that (i) $f^{-1}(X) \subset X$; (ii) $X = \bigsqcup_{i \in I} B_{p^{-\tau}}(c_i)$ can be written as a finite disjoint union of balls of centers c_i and of the same radius $p^{-\tau}$ (with some $\tau \in \mathbb{Z}$) such that for each $i \in I$ there is an integer $\tau_i \in \mathbb{Z}$ such that

$$\left| f(x) - f(y) \right|_p = p^{\tau_i} |x - y|_p \quad (\forall x, y \in B_{p^{-\tau}}(c_i)). \tag{3}$$

For such a map f, define its Julia set by

$$J_f = \bigcap_{n=0}^{\infty} f^{-n}(X). \tag{4}$$

It is clear that $f^{-1}(J_f) = J_f$ and then $f(J_f) \subset J_f$. We will study the dynamical system (J_f, f).

The triple (X, J_f, f) is called a *p-adic weak repeller* if all τ_i in (3) are nonnegative, but at least one is positive. We call it a *p-adic repeller* if all τ_i in (3) are positive. For later convenience, we will write $\|f\| = p^{\tau_i}$ for any map having the property (3), which could be called the expanding ratio (resp. contractive ratio) of f on the ball $B_{p^{-\tau}}(c_i)$ when $\tau_i \geqslant 0$ (resp. $\tau_i \leqslant 0$).

In this Note we consider the expanding case, i.e. all τ_i are nonnegative. For any $i \in I$, let

$$I_i := \{ j \in I:\ B_j \cap f(B_i) \neq \emptyset \} = \{ j \in I:\ B_j \subset f(B_i) \}$$

(the second equality holds because of the expansiveness and of the ultrametric property). Then define a matrix $A = (A_{i,j})_{I \times I}$, called incidence matrix, by

$$A_{ij} = 1 \quad \text{if } j \in I_i; \qquad A_{ij} = 0 \quad \text{otherwise.}$$

If A is irreducible, we say that (X, J_f, f) is *transitive*. That A is irreducible means, for any pair $(i, j) \in I \times I$ there is positive integer m such that $A^m_{ij} > 0$.

Given I and the irreducible incidence matrix A as above. Let Σ_A be the corresponding subshift space and let σ be the shift transformation on Σ_A. We equip Σ_A with a metric d_f depending on the dynamics which is defined as follows. First for $i, j \in I,\ i \neq j$, let $\kappa(i, j)$ be the integer such that $|c_i - c_j|_p = p^{-\kappa(i,j)}$. It clear that $\kappa(i, j) < \tau$. By the ultra-metric inequality, we have

$$|x - y|_p = |c_i - c_j|_p \quad (i \neq j,\ \forall x \in B_{p^{-\tau}}(c_i),\ \forall y \in B_{p^{-\tau}}(c_j)). \tag{5}$$

For $x = (x_0, x_1, \ldots, x_n, \ldots) \in \Sigma_A$ and $y = (y_0, y_1, \ldots, y_n, \ldots)$ in Σ_A, define

$$d_f(x, y) = p^{-\tau_{x_0}-\tau_{x_1}-\cdots-\tau_{x_{n-1}}-\kappa(x_n,y_n)} \quad (\text{if } n \neq 0), \qquad d_f(x, y) = p^{-\kappa(x_0,y_0)} \quad (\text{if } n = 0) \tag{6}$$

where $n = n(x, y) = \min\{i \geqslant 0\colon x_i \neq y_i\}$. It is clear that d_f defines the same topology as the classical metric which is defined by $d(x, y) = p^{-n(x,y)}$.

Theorem 1.1. *Let (X, J_f, f) be a transitive p-adic weak repeller with incidence matrix A. Then the dynamics $(J_f, f, |\cdot|_p)$ is isometrically conjugate to the shift dynamics (Σ_A, σ, d_f).*

Remark 1.2. If the incidence matrix A is not irreducible, the index set I is partitioned into classes of indexes $I^{(1)}, I^{(2)}, \ldots, I^{(k)}$. We can arrange $I^{(1)}, I^{(2)}, \ldots, I^{(k)}$ in an order so that A is written as lower triangular matrix with irreducible sub-matrixes $A_1, A_2, \ldots, A_k$ on its diagonal (see [1]). We can then apply Theorem 1.1 to $f : X^{(t)} \to \mathbb{Q}$ for each $1 \leqslant t \leqslant k$, where $X^{(t)} = \bigcup_{i \in I^{(t)}} B_{p^{-\tau}}(c_i)$.

Remark 1.3. Theorem 1.1 holds not only for p-adic dynamics but also for any ultrametric dynamics which satisfy the two conditions (i) and (ii) listed at the beginning of the note and the property stated in Lemma 2.1 below.

The following result allows us to apply Theorem 1.1 to polynomial dynamics:

Theorem 1.4. *Let $f \in \mathbb{Q}_p[x]$ be a polynomial with $f'(x) \neq 0$ for all $x \in \mathbb{Q}_p$. There exists a compact-open set $X \subset \mathbb{Q}_p$ and an integer τ such that conditions* (i) *and* (ii) *are satisfied. Moreover, for $x \notin J_f$, $\lim_{n\to\infty} |f^n(x)|_p = \infty$.*

Here is a class of polynomials to which the above results can apply. Let $f(x) = p^{-m}P(x) \in \mathbb{Q}_p[x]$ with $P \in \mathbb{Z}_p[x]$ and $m > 0$ (otherwise $f \in \mathbb{Z}_p[x]$). Assume first that

(a) $|P(x)|_p \geqslant |x|_p\ \forall x \notin \mathbb{Z}_p$.

Let

$$X = \bigcup_{i \in I_0} B_{p^{-m}}(i), \quad \text{where } I_0 = \{0 \leqslant i < p^m\colon\ P(i) \equiv 0 \bmod p^m\}.$$

Assume further that

(b) $f'(x) \neq 0$ for all $x \in X$.

In order to apply Theorem 1.1 or Theorem 1.4 to this polynomial $p^{-m}P(x)$, in general, we have to find a finer partition of X such that (3) holds on each component of this new partition.

2. Proofs

Proof of Theorem 1.1. The proof consists of the following lemmas:

Lemma 2.1. *For each $i \in I$, the restricted map $f : B_{p^{-\tau}}(c_i) \to B_{p^{-\tau+\tau_i}}(f(c_i))$ is a bijection.*

Proof. The injectivity and the inclusion $f(B_{p^{-\tau}}(c_i)) \subseteq B_{p^{-\tau+\tau_i}}(f(c_i))$ are direct consequences of the hypothesis (3). Since $f(B_{p^{-\tau}}(c_i))$ is the continuous image of a compact set and then is closed, for the surjectivity, it suffices to prove that $f(B_{p^{-\tau}}(c_i))$ is dense in $B_{p^{-\tau+\tau_i}}(f(c_i))$. For an arbitrary integer $n \geqslant 1$, consider the p^n points $c_i + kp^\tau$ $(0 \leqslant k < p^n)$ in the ball $B_{p^{-\tau}}(c_i)$. Any two such different points $c_i + k'p^\tau$ and $c_i + k''p^\tau$ has a distance strictly larger than $p^{-n-\tau}$. So, by the hypothesis (3),

$$\left|f(c_i + k'p^\tau) - f(c_i + k''p^\tau)\right|_p > p^{-n-\tau+\tau_i}.$$

Then the p^n image points $f(c_i + kp^\tau)$ belong to p^n different balls of radius $p^{-n-\tau+\tau_i}$. Thus each ball of radius $p^{-n-\tau+\tau_i}$ contained in $B_{p^{-\tau+\tau_i}}(f(c_i))$ contains an image point. Since n is arbitrarily large, we get the density. □

Let $I_{\exp} \subset I$ (resp. I_{iso}) be the subset of indexes $i \in I$ such that $\tau_i > 0$ (resp. $\tau_i = 0$). A ball $B_{p^{-\tau}}(c_i)$ is said to be expanding (resp. isometric) if $i \in I_{\exp}$ (resp. $i \in I_{\text{iso}}$).

Lemma 2.2. *For each isometric ball $B_{p^{-\tau}}(c_i)$, there exists an integer $n \geqslant 1$ such that $f^n(B_{p^{-\tau}}(c_i))$ is an expanding ball.*

Proof. If $B_{p^{-\tau}}(c_i)$ is an isometric ball, by Lemma 2.1, the image $f(B_{p^{-\tau}}(c_i))$ is still a ball of the same size as $B_{p^{-\tau}}(c_i)$. Suppose that the conclusion of the lemma is not true. Then all balls $f^k(B_{p^{-\tau}}(c_i))$ $(k = 0, 1, \ldots)$ are isometric balls. Since there is a finite number of isometric balls, there are $0 \leqslant k' < k''$ such that $f^{k'}(B_{p^{-\tau}}(c_i)) = f^{k''}(B_{p^{-\tau}}(c_i))$. This contradicts the irreducibility. □

Lemma 2.3. *For each $x \in J_f$ there is a unique sequence $(j_n)_{n \geqslant 0} \in \Sigma_A$ such that*

$$x \in B_{p^{-\tau}}(c_{j_0}), \quad f(x) \in B_{p^{-\tau}}(c_{j_1}), \ldots, f^n(x) \in B_{p^{-\tau}}(c_{j_n}), \ldots.$$

Proof. This is just because $f(J_f) \subset J_f \subset X$ and $\{B_{p^{-\tau}}(c_i)\}_{i \in I}$ is a partition of X. □

Denote by $h : J_f \to \Sigma_A$ the map $x \mapsto (j_n)_{n \geqslant 0}$ and call $(j_n)_{n \geqslant 0}$ the code sequence of x.

Lemma 2.4. *For each sequence $(j_n)_{n \geqslant 0} \in \Sigma_A$, there are an infinite number of j_n's belonging to $I_{\exp}$.*

Proof. When $A_{i,j} = 1$, we say j is an issue of i. That $i \in I_{\text{iso}}$ means i has only one issue. Using the same argument as in the proof of Lemma 2.2 but in a different presentation, we can prove that there is no sequence $(j_n)_{n \geqslant 0}$ in Σ_A which ends with j_n's having only one issue. □

If j is an issue of i, let $T_{ij} : B_{p^{-\tau}}(c_j) \to B_{p^{-\tau}}(c_i)$ be the inverse map restricted on $B_{p^{-\tau}}(c_j)$ of $f : B_{p^{-\tau}}(c_i) \to f(B_{p^{-\tau}}(c_i))$. Remark that if $i \in I_{\text{iso}}$, then T_{ij} is an isometry and if $i \in I_{\exp}$, then T_{ij} is a contraction with $p^{-\tau_i}$ as its contraction ratio.

Lemma 2.5. *For each sequence $(j_n)_{n \geqslant 0} \in \Sigma_A$ and for any choice (b_n) with $b_n \in B_{p^{-\tau}}(c_{j_n})$ $(n = 0, 1, 2, \ldots)$, the following limit*

$$x = \lim_{n \to \infty} T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}(b_n) \tag{7}$$

exists and is independent of the choice $(b_n)_{n \geqslant 0}$. The point x belongs to J_f and it has (j_n) as its code sequence.

Proof. Let $x_n = T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}(b_n)$. We have

$$|x_{n+m} - x_n|_p \leqslant \|T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}\| \operatorname{diam}(X)$$

where $\operatorname{diam}(X)$ denotes the diameter of X. By Lemma 2.4, $\|T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}\|$ tends to zero. Thus we have proved the existence of the limit by the Cauchy criterion. Let (b'_n) is another choice. Let x'_n be the corresponding x_n. We have

$$|x'_n - x_n|_p \leqslant \|T_{j_0 j_1} T_{j_1 j_2} \cdots T_{j_{n-1} j_n}\| \operatorname{diam}(X)$$

which proves the independence. Recall that $f \circ T_{ij}(x) = x$. Then the last assertions in the lemmas are obviously true. □

Denote by $h^*: \Sigma_A \to J_f$ the map $(j_n) \mapsto x$ where x is well determined by (7) in Lemma 2.5.

Lemma 2.6. *The map $h: (J_f, |\cdot|_p) \to (\Sigma_A, d_f)$ is an isometric homeomorphism.*

Proof. Lemma 2.5 and the definition of code show $h \circ h^* = \mathrm{Id}_{\Sigma_A}$ and $h^* \circ h = \mathrm{Id}_{J_f}$. So, h is a bijection. It remains to show that h is isometric. For $\bar{x} = (j_0 j_1 \cdots j_n \cdots) \in \Sigma_A$ and $\bar{y} = (j'_0 j'_1 \cdots j'_n \cdots) \in \Sigma_A$, let $x = h^*(\bar{x})$ and $y = h^*(\bar{y})$. Let $n = \min\{i \geqslant 0\colon\ j_i \neq j'_i\}$. If $n = 0$, it is clear that $d_f(\bar{x}, \bar{y}) = p^{-\kappa(j_0, j'_0)} = |x - y|_p$. If $n \neq 0$,

$$\begin{aligned} |x-y|_p &= \left|h^*(\bar{x}) - h^*(\bar{y})\right|_p = \left|\lim_{n\to\infty} T_{j_0 j_1} \cdots T_{j_{n-1} j_n}(b_n) - \lim_{n\to\infty} T_{j'_0 j'_1} \cdots T_{j'_{n-1} j'_n}(b'_n)\right|_p \\ &= \|T_{j_0 j_1} \cdots T_{j_{n-2} j_{n-1}}\| \cdot \left|T_{j_{n-1} j_n}(x') - T_{j_{n-1} j'_n}(y')\right|_p, \end{aligned}$$

where $x' = \lim_{k\to\infty} T_{j_n j_{n+1}} \cdots T_{j_{n+k-1} j_{n+k}}(b_{n+k})$ and $y' = \lim_{k\to\infty} T_{j'_n j'_{n+1}} \cdots T_{j'_{n+k-1} j'_{n+k}}(b'_{n+k})$. Since

$$\|T_{j_0 j_1} \cdots T_{j_{n-2} j_{n-1}}\| = p^{-\tau_{j_0} - \cdots - \tau_{j_{n-2}}},$$

we have only to show that

$$\left|T_{j_{n-1} j_n}(x') - T_{j_{n-1} j'_n}(y')\right|_p = p^{-\tau_{j_{n-1}} - \kappa(j_n, j'_n)}.$$

In fact, since $T_{j_{n-1} j_n}(x')$ and $T_{j_{n-1} j'_n}(y')$ are both in the same ball $B_{p^{-\tau}}(j_{n-1})$, the expanding property (3) shows

$$|x' - y'|_p = \left|f\big(T_{j_{n-1} j_n}(x')\big) - f\big(T_{j_{n-1} j'_n}(y')\big)\right|_p = p^{\tau_{j_{n-1}}} \left|T_{j_{n-1} j_n}(x') - T_{j_{n-1} j'_n}(y')\right|_p.$$

Then we conclude by the definition of κ (see (5) and the fact that $x' \in B_{p^{-\tau}}(j_n)$ and $y' \in B_{p^{-\tau}}(j'_n)$ $(j_n \neq j'_n)$, which give $|x' - y'|_p = p^{-\kappa(j_n, j'_n)}$. □

Lemma 2.7. *We have $h \circ f = \sigma \circ h$.*

Proof. Recall that $h(x)$ is the code sequence of x and that $f \circ T_{ij}(x) = x$. Using these facts and the expression (7) of x, we have

$$h \circ f(x) = h \circ f\Big(\lim_n T_{j_0 j_1} \cdots T_{j_{n-1} j_n}(b_n)\Big) = h\Big(\lim_n T_{j_1 j_2} \cdots T_{j_{n-1} j_n}(b_n)\Big) = (j_1 \cdots j_n \cdots) = \sigma \circ h(x). \qquad \square$$

Proof of Theorem 1.4. Assume $f(x) = \sum_{j=0}^n a_j x^j$ with $a_n \neq 0$. There exists an integer ℓ such that if $|x|_p \geqslant p^\ell$, we have $|a_{n-k}|_p / |x|_p^k < |a_n|_p$ $(1 \leqslant k \leqslant n)$ and $|x|_p^{n-1} |a_n|_p \geqslant p$, so that

$$\left|f(x)\right|_p = |x|_p^n \,|\, a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \,|_p = |x|_p^n |a_n|_p \geqslant p|x|_p. \tag{8}$$

Let $X = \{x \in \mathbb{Q}_p\colon\ |x|_p < p^\ell\}$. By (8), we have $f(\mathbb{Q}_p \setminus X) \subset \mathbb{Q}_p \setminus X$. Hence, $f^{-1}(X) \subset X$. Also by (8), we get $\lim_{n\to\infty} |f^n(x)|_p = \infty$ for $x \notin X$. Furthermore, we have $\lim_{n\to\infty} |f^n(x)|_p = \infty$ for $x \notin J_f := \bigcap_{n=0}^\infty f^{-n}(X)$. In fact, for $x \notin J_f$, there exists $n_0 \geqslant 0$ such that $f^{n_0}(x) \notin X$, thus $\lim_{n\to\infty} |f^{n+n_0}(x)|_p = \infty$.

The strict differentiability of $f \in \mathbb{Q}_p[x]$ (see [2], p. 78) and the fact $f'(x) \neq 0$ for all $x \in \mathbb{Q}_p$ imply that for $a \in X$, there exists an integer $\tau(a)$ such that $|f(x) - f(y)|_p = |f'(a)|_p |x - y|_p$ for all $x, y \in B_{p^{-\tau(a)}}(a)$. Then the compactness of X implies that there is an integer τ such that condition (ii) is satisfied. □

3. Examples

Example 1. Let $a \in \mathbb{Z}_p$ and $m \geqslant 1$ be an integer. Consider the transformation $f_m: \mathbb{Q}_p \to \mathbb{Q}_p$ given by

$$f_{m,a}(x) = \frac{x^p - ax}{p^m}.$$

In the case where $m = 1$ and $a \equiv 1 (\text{mod } p)$, the map $f_{m,a} : \mathbb{Z}_p \to \mathbb{Z}_p$ was shown, by C.F. Woodcock and N.P. Smart [3], to be topologically conjugate to the full shift on the symbolic system with p symbols.

It is easy to see that $|f'_{m,a}(x)|_p = p^m$. We have even $|f_{m,a}(x) - f_{m,a}(y)|_p = p^m|x - y|_p$ for all $x, y \in \mathbb{Z}_p$, $|x - y|_p < 1$ and $|x^p - ax|_p \geqslant |x|_p$ for $x \notin \mathbb{Z}_p$. Let $I_{m,a} = \{0 \leqslant k < p^m\colon\ k^p - ak \equiv 0 \pmod{p^m}\}$ and let $X_{m,a} = \bigsqcup_{k \in I_{m,a}} (k + p^m\mathbb{Z}_p)$. If $a \equiv 1 \pmod p$, $x^p - ax = 0$ has p solutions on $\mathbb{Z}_p$, by Hensel lemma. Then by Theorem 1.1, on the Julia set $J_{m,a}$, $f_{m,a}$ is conjugate to the full shift on the symbolic space of p symbols. If $a \not\equiv 1 \pmod p$, then $I_{m,a} = \{0\}$ and $J_{m,a} = \{0\}$ is the singleton consisting of the repeller fixed point 0. In both cases, for every $x \notin J_{m,a}$ we have $\lim_{n\to\infty} |f^n_{m,a}(x)|_p = \infty$.

Example 2. Let $c = c_0/p^\tau \in \mathbb{Q}_p$ with $|c_0|_p = 1$ and $\tau \geqslant 1$. Consider the p-adic logistic map $f_c : \mathbb{Q}_p \to \mathbb{Q}_p$ defined by

$$f_c(x) = cx(x-1) = \frac{c_0 x(x-1)}{p^\tau}.$$

Let $I_c = \{0 \leqslant k < p^\tau\colon\ k(k-1) = 0 (\text{mod } p^\tau)\}$. It is clear that $I_c = \{0, 1\}$. Notice that

$$\left|f_c(x) - f_c(y)\right|_p = p^\tau |x - y|_p \left|1 - (x + y)\right|_p \quad (\forall x, y).$$

So, we get $|f_c(x) - f_c(y)|_p = p^\tau|x - y|_p$ whenever $x, y \in B_{p^{-\tau}}(0)$ or $x, y \in B_{p^{-\tau}}(1)$. Let J_c be the Julia set of f_c, by Theorem 1.1, (J_c, f_c) is always conjugate to the full shift on the symbolic space of two symbols.

Example 3. Consider the polynomial $f : \mathbb{Q}_2 \to \mathbb{Q}_2$ defined by

$$f(x) = \frac{x(x-1)(x+1)}{2}.$$

It is easy to see $|f'(x)|_p = 2$, $|f''(x)/2|_p \leqslant 1$, $|f'''(x)/6|_p = 2$ on $2\mathbb{Z}_2$ and $|f'(x)|_p = 1$, $|f''(x)/2|_p = 2$, $|f'''(x)/6|_p = 2$ on $1 + 2\mathbb{Z}_2$. Then by

$$f(x) - f(y) = (x - y)f'(y) + \frac{(x-y)^2}{2} f''(y) + \frac{(x-y)^3}{6} f'''(y),$$

we have $|f(x) - f(y)|_p = |f'(y)|_p|x - y|_p$ for $x, y \in \mathbb{Z}_2$, $|x - y|_p \leqslant 1/4$. Thus we can take $X = \bigsqcup_{k=0}^3 (k + 4\mathbb{Z}_2)$. Then the conditions (i) and (ii) are satisfied and incidence matrix, which is irreducible, is equal to

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In this case we have $\tau_0 = \tau_2 = 2$ and $\tau_1 = \tau_3 = 1$. The topological entropy of (J_f, f) is equal to $\log 1.6956\ldots$ where $1.6956\ldots$ is the maximal eigenvalue of A.

References

[1] D. Lind, B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, 1995.

[2] W.H. Schikhof, Ultrametric Calculus, Cambridge University Press, 1984.

[3] C.F. Woodcock, N.P. Smart, p-adic chaos and random number generation, Experiment Math. 7 (4) (1998) 333–342.