Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 344 (2007) 195–198
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Probability Theory/Statistics

A new family of symmetric bivariate copulas ☆

Fabrizio Durante

Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, A-4040 Linz, Austria

Received 11 February 2006; accepted after revision 1 December 2006

Available online 9 January 2007

Presented by Paul Deheuvels

Abstract

A new class of copulas, depending on an univariate function, is introduced and its properties (dependence, ordering, symmetry) are studied. ***To cite this article: F. Durante, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***
© 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé

Une nouvelle famille de copules symétriques bivariées. On introduit une nouvelle famille de copules, qui dépendent d'une fonction unidimensionnelle, et l'on étudie ses propriétés (dépendance, ordre, symétrie). ***Pour citer cet article : F. Durante, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***
© 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Copulas have proved to be very useful in the construction of multivariate distribution functions (d.f., for short) with a variety of dependence structures. In fact, in view of *Sklar's Theorem* [7], the construction of a multivariate d.f. can be reduced to the choice of a copula and of two univariate d.f.'s and, moreover, it is precisely the copula that captures the dependence properties. Several families of copulas are collected, for example, in [4], and many others have been introduced recently (see, e.g., [1,2,6]). Here, we present a family of (bivariate) copulas, which includes many other families, such as the well-known Cuadras–Augé [3], we study its properties in details and provide several examples.

2. Characterization of the new class

We recall that a (*bivariate*) *copula* is a function $C:[0,1]^2 \to [0,1]$ such that: (a) $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for every t in $[0,1]$; (b) C is 2-*increasing*, viz., for every $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, $V_C([x_1,x_2] \times [y_1,y_2]) := C(x_1,y_1) + C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) \geqslant 0$. In particular, every copula is increasing in each place and 1-Lipschitz.

☆ A first version of this paper was written when the author was Ph.D. student at the Department of Mathematics of the University of Lecce (Italy). This work was partially supported by the Italian M.I.U.R. through the project "Metodi stocastici in finanza matematica".

E-mail address: fabrizio.durante@jku.at (F. Durante).

1631-073X/$ – see front matter © 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2006.12.004

The importance of copulas in statistics stems from *Sklar's Theorem*: given a pair of continuous random variables (X, Y) with joint d.f. H and margins F and G, then there exists a unique copula C such that $H(x, y) = C(F(x), G(y))$ for every x and y in $\overline{\mathbb{R}}$. Important examples of copulas are: $\Pi(x, y) = xy$, which is associated to a pair of independent random variables, $M(x, y) = \min(x, y)$ and $W(x, y) = \max(x + y - 1, 0)$. Each copula C satisfies $W \prec C \prec M$, where $\prec$ denote the *concordance order*, viz. $A \prec B$ if, and only if, $A(x, y) \leqslant B(x, y)$ for all x, y in $[0, 1]$.

For every mapping $f : [0, 1] \to [0, 1]$, we consider the function C_f given, for every $x, y \in [0, 1]$, by

$$C_f(x, y) := \min(x, y) f\big(\max(x, y)\big). \tag{1}$$

It is obvious that every function C_f is symmetric (i.e., $C(x, y) = C(y, x)$) and the copulas Π and M are of this type: it suffices to take, respectively, $f(t) = t$ and $f(t) = 1$ for all $t \in [0, 1]$. Our aim is to study under which conditions on f, C_f is a copula. Notice that, in view of the properties of a copula, it is quite natural to require that f is increasing and continuous.

Theorem 2.1. *Let $f : [0, 1] \to [0, 1]$ be a continuous function, differentiable except at finitely many points. Let C_f be the function defined by* (1)*. Then C_f is a copula if, and only if,*

(i) $f(1) = 1$;
(ii) *f is increasing*;
(iii) *the function $t \mapsto f(t)/t$ is decreasing on* $(0, 1]$.

To prove this result, we use the following lemma, which can be proved with simple considerations of real analysis:

Lemma 2.2. *Let $f : [0, 1] \to [0, 1]$ be a continuous and increasing function, differentiable except at finitely many points. The following statements are equivalent*:

- *for every $s, t \in (0, 1]$, with $s \leqslant t$, $sf(s) + tf(t) \geqslant 2sf(t)$*;
- *the function $t \mapsto f(t)/t$ is decreasing on* $(0, 1]$.

Proof. [Theorem 2.1] It is immediate that C_f satisfies the boundary conditions (a) for a copula if, and only if, $f(1) = 1$. We now prove that C_f is 2-increasing if, and only if, (ii) and (iii) hold. Let x, x', y, y' be in $[0, 1]$ with $x \leqslant x'$ and $y \leqslant y'$. First, we suppose that the rectangle $R := [x, x'] \times [y, y']$ is a subset of $\{(x, y) \in [0, 1]^2\colon\ x \geqslant y\}$. Then $V_C([x, x'] \times [y, y']) = (y' - y)(f(x') - f(x)) \geqslant 0$ if, and only if, f is increasing. Analogously, the 2-increasing property is equivalent to (ii) if R is contained in $\{(x, y) \in [0, 1]^2\colon\ x \leqslant y\}$. If, instead, the diagonal of R lies on the diagonal $\{(x, y) \in [0, 1]^2\colon\ y = x\}$ of the unit square, in view of Lemma 2.2, $V_C([x, x'] \times [x, x']) = xf(x) + x'f(x') - 2xf(x') \geqslant 0$ if, and only if, (iii) holds. Now, the assertion follows directly from the fact that every rectangle $R = [x, x'] \times [y, y']$ can be decomposed into the finite union of rectangles R_i of the described type and $V_C(R) = \sum V_C(R_i)$. $\square$

We call a *generator* a function f that satisfies the assumptions of Theorem 2.1. Such a f has an interesting probabilistic interpretation: if U and V be random variables uniformly distributed on $[0, 1]$ with copula C_f of type (1), then $f(t) = P(\max(U, V) \leqslant t \mid U \leqslant t)$.

Notice that if f is a generator, then f is star-shaped, i.e., $f(\alpha x) \geqslant \alpha f(x)$ for all $\alpha \in [0, 1]$. Moreover, every concave function is a generator.

Some examples of parametric families of copulas $\{C_\alpha\}$, generated by $\{f_\alpha\}$ via (1), are now given:

Example 1 *(Fréchet copulas).* Given $f_\alpha(t) := \alpha t + (1 - \alpha)$ $(\alpha \in [0, 1])$, the corresponding copula is $C_\alpha = \alpha\Pi + (1 - \alpha)M$, which is a convex sum of Π and M and, therefore, is a member of the *Fréchet family* of copulas (see [4]). Notice that $C_0 = M$ and $C_1 = \Pi$.

Example 2 *(Cuadras–Augé copulas).* Given $f_\alpha(t) := t^\alpha$ $(\alpha \in [0, 1])$, C_α is defined by

$$C_\alpha(x, y) = \min(x, y)\big(\max(x, y)\big)^\alpha = \begin{cases} xy^\alpha, & \text{if } x \leqslant y; \\ x^\alpha y, & \text{if } x > y. \end{cases}$$

Then C_α describes the *Cuadras–Augé* family of copulas (see [3]). Notice that $C_0 = M$ and $C_1 = \Pi$.

Example 3 *(Ordinal sum).* Given $f_\alpha(t) := \min(\alpha t, 1)$ $(\alpha \geqslant 1)$, C_α is defined by

$$C_\alpha(x, y) = \begin{cases} \alpha xy, & \text{if } (x, y) \in [0, 1/\alpha]^2; \\ x \wedge y, & \text{otherwise}; \end{cases}$$

viz. C_α is the *ordinal sum* $(\langle 0, 1/\alpha, \Pi \rangle)$ (see [4]). Notice that $C_1 = \Pi$ and $C_\infty = M$. It can be also proved that a copula of type (1) is associative (regarded as a binary operation on [0, 1]) if, and only if, it is an ordinal sum $(\langle 0, 1/\alpha, \Pi \rangle)$ for a suitable α in [0, 1].

Notice that every copula of type (1), except for Π, has a singular component on the main diagonal of the unit square.

3. Properties of the new class

In this section we give the most important properties of dependence and association for a copula C_f. For the definitions of these concepts, we refer to [4].

Proposition 3.1. *Let C_f and C_g be two copulas of type* (1). *Then $C_f \prec C_g$ if, and only if, $f(t) \leqslant g(t)$ for all $t \in [0, 1]$.*

Thus, the concordance order between C_f and C_g can be reduced into the pointwise order between the respective generators. Because of condition (iii) of Theorem 2.1, for every generator f we have $t \leqslant f(t) \leqslant 1$ on [0, 1], and, hence, for every copula C_f, $\Pi \prec C_f \prec M$. In other words, every C_f is *positively quadrant dependent*, and, therefore, it is suitable to describe the positive dependence of a random pair (X, Y). However, it is very simple to introduce, in the same way, a copula to describe a negative dependence. It suffices to consider the copula C_f^* given by

$$C_f^*(x, y) := x - C(x, 1 - y) = \begin{cases} x(1 - f(1 - y)), & \text{if } x + y \leqslant 1; \\ x - (1 - y) f(x), & \text{otherwise.} \end{cases}$$

In this case, $f_1(t) = t$ generates Π and $f_2(t) = 1$ generates W. It is obvious that the properties of a copula C_f^* can be obtained in a simple way from the corresponding properties of a copula C_f.

If f and g are two generators, then $h := \min(f, g)$ is also a generator. Moreover, the pointwise minimum between C_f and C_g is equal to C_h. Therefore, the class of copulas of type (1) is stable with respect to the minimum (see [5] for a discussion on the lattice structure of the set of copulas).

Example 4. Consider the family $\{f_\alpha\}$ $(\alpha \geqslant 1)$, given by $f_\alpha(t) := 1 - (1 - t)^\alpha$. It is easily proved by differentiation that every f_α is increasing with $f_\alpha(t)/t$ decreasing on (0, 1]. Therefore, this family generates a family of copulas C_α, that is positively ordered, viz. $C_\alpha \leqslant C_\beta$ if $\alpha \leqslant \beta$. Another positively ordered family is that generated by $g_\alpha(t) := (1 + \alpha)t/(\alpha t + 1)$ for every $\alpha \geqslant 0$.

Proposition 3.2. *Let (X, Y) be a continuous random pair with copula C_f of type* (1). *Then*

(a) *Y is left tail decreasing in X*;
(b) *Y is stochastically increasing in X if, and only if, f' is decreasing a.e. on* [0, 1];
(c) *X and Y are left corner set decreasing.*

Proposition 3.3. *Let C_f be a copula of type* (1). *Then, the lower tail dependence of C_f is $f(0^+)$ and the upper tail dependence of C_f is $1 - f'(1^-)$.*

Proposition 3.4. *The values of Kendall's tau, Spearman's rho, Gini's gamma and Blomqvist's beta of a copula C_f of type* (1) *are, respectively, given by*

$$\tau_{C_f} = 4\int_0^1 x f^2(x)\,\mathrm{d}x - 1, \qquad \rho_{C_f} = 12\int_0^1 x^2 f(x)\,\mathrm{d}x - 3,$$

$$\gamma_{C_f} = 4\left(\int_0^{1/2} x\bigl[f(x) + f(1-x)\bigr]\,\mathrm{d}x + \int_{1/2}^1 f(x)\,\mathrm{d}x\right) - 2, \qquad \beta_{C_f} = 2f\left(\frac{1}{2}\right) - 1.$$

Proposition 3.5. *Let (X, Y) be random variables with copula C_f of type* (1)*.*

(a) *If X and Y are identically distributed, then X and Y are exchangeable.*
(b) *If X and Y are symmetric about a and b, respectively $(a, b \in \mathbb{R})$, then (X, Y) is radially symmetric about (a, b) if, and only if, $C_f = \alpha\Pi + (1-\alpha)M$ for some $\alpha \in [0, 1]$.*
(c) *If X and Y are symmetric about a and b, respectively $(a, b \in \mathbb{R})$, then (X, Y) is jointly symmetric about (a, b) if, and only if, $C_f = \Pi$.*

References

[1] A. Alfonsi, D. Brigo, New families of copulas based on periodic functions, Comm. Statist. Theory Methods 34 (2005) 1437–1447.
[2] C. Amblard, S. Girard, Une famille semi-paramétriques de copules symétriques bivariées, C. R. Acad. Sci. Paris, Ser. I 333 (2001) 129–132.
[3] C.M. Cuadras, J. Augé, A continuous general multivariate distribution and its properties, Comm. Statist. A Theory Methods 10 (1981) 339–353.
[4] R.B. Nelsen, An Introduction to Copulas, Springer, New York, 1999.
[5] R.B. Nelsen, M. Úbeda Flores, The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas, C. R. Acad. Sci. Paris, Ser. I 341 (2005) 583–586.
[6] J.A. Rodríguez Lallena, M. Úbeda Flores, A new class of bivariate copulas, Statist. Probab. Lett. 66 (2004) 315–325.
[7] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris 8 (1959) 229–231.