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Partial Differential Equations

Some isoperimetric problems for the principal eigenvalues of second-order elliptic operators in divergence form

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Abstract

We prove various optimization results for the principal eigenvalues of general second-order elliptic operators in divergence form with Dirichlet boundary condition in C^2 bounded nonempty domains of $\mathbb{R}^n$. In particular, we obtain a 'Faber–Krahn' type inequality for these operators. The proofs use a new rearrangement technique. ***To cite this article: F. Hamel et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Quelques problèmes isopérimétriques pour la première valeur propre d'opérateurs elliptiques d'ordre deux sous forme divergence. On montre divers résultats d'optimisation pour la première valeur propre d'opérateurs elliptiques généraux du second ordre sous forme divergence avec condition au bord de Dirichlet dans des domaines bornés non vides de classe C^2 de $\mathbb{R}^n$. En particulier, on obtient une inégalité de type « Faber–Krahn » pour ces opérateurs. Les preuves utilisent une nouvelle méthode de réarrangement. ***Pour citer cet article : F. Hamel et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit $n \geqslant 1$ et $\mathcal{C}$ l'ensemble des domaines bornés non vides de classe C^2 de $\mathbb{R}^n$. Si $E \subset \mathbb{R}^n$ est mesurable, on désigne par $|E|$ la mesure de Lebesgue de E. Pour tout $x \in \mathbb{R}^n \setminus \{0\}$, soit

$$e_r(x) = \frac{x}{|x|},$$

où $|x|$ est la norme euclidienne de x.

Si $\Omega \in \mathcal{C}$, on note Ω^* la boule euclidienne de centre 0 dans $\mathbb{R}^n$ telle que $|\Omega^*| = |\Omega|$. On définit également

$$L^\infty_+(\Omega) = \left\{ f \in L^\infty(\Omega);\ \exists \gamma > 0 \text{ tel que } f(x) \geqslant \gamma \text{ p.p. } x \in \Omega \right\}.$$

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Pour tout champ $v:\Omega\to\mathbb{R}^n$ mesurable, on dit que $v\in L^\infty(\Omega,\mathbb{R}^n)$ si $|v|\in L^\infty(\Omega)$ (où $|v|$ désigne la norme euclidienne de v), et on pose $\|v\|_{L^\infty(\Omega,\mathbb{R}^n)}=\||v|\|_{L^\infty(\Omega)}$.

Dans [4,5], nous avons établi une inégalité de Faber–Krahn pour les opérateurs $-\Delta+v\cdot\nabla$ (Théorème 0.1), où $v\in L^\infty(\Omega,\mathbb{R}^n)$. Nous étendons ici cette inégalité au cas des opérateurs elliptiques du second ordre sous forme divergence.

Précisons nos notations. Soit $\mathcal{S}_n(\mathbb{R})$ l'ensemble des matrices symétriques de taille $n\times n$ à valeurs réelles. Si $\Omega\in\mathcal{C}$, on considère un opérateur elliptique du second ordre $L=-\operatorname{div}(A\nabla)+v\cdot\nabla+V$ dans Ω avec condition au bord de Dirichlet. On supposera toujours que $A\in W^{1,\infty}(\Omega,\mathcal{S}_n(\mathbb{R}))$ (de sorte qu'on peut considérer que $A\in C(\overline{\Omega})$, quitte à modifier A sur un ensemble de mesure nulle) et que A est uniformément elliptique sur Ω, ce qui signifie qu'il existe $\gamma>0$ tel que $A(x)\geqslant\gamma\,\mathrm{Id}$ pour tout $x\in\overline{\Omega}$. De manière générale, si $\Lambda\in L^\infty_+(\Omega)$, on dit que $A\geqslant\Lambda\,\mathrm{Id}$ presque partout dans Ω si pour presque tout $x\in\Omega$ et tout $\xi\in\mathbb{R}^n$,

$$A(x)\xi\cdot\xi\geqslant\Lambda(x)|\xi|^2.$$

On suppose aussi que $v\in L^\infty(\Omega,\mathbb{R}^n)$ et $V\in L^\infty(\Omega)$. On désigne par $\lambda_1(\Omega,A,v,V)$ la première valeur propre de L (voir [2]). Nous établissons une inégalité de type Faber–Krahn pour l'opérateur L:

Théorème 0.1. *Soit $\Omega\in\mathcal{C}$ qui n'est pas une boule, $\overline{M}_A\geqslant\underline{m}_\Lambda>0$, $\tau_1\geqslant0$ et $\tau_2\geqslant0$. On se donne $A\in W^{1,\infty}(\Omega,\mathcal{S}_n(\mathbb{R}))$, $\Lambda\in L^\infty_+(\Omega)$, $v\in L^\infty(\Omega,\mathbb{R}^n)$ et $V\in L^\infty(\Omega)$ vérifiant*

$$\|A\|_{W^{1,\infty}(\Omega,\mathcal{S}_n(\mathbb{R}))}\leqslant\overline{M}_A,\quad \operatorname*{ess\,inf}_{\Omega}\Lambda\geqslant\underline{m}_\Lambda,\quad A\geqslant\Lambda\,\mathrm{Id}\quad \textit{p.p. dans } \Omega,$$
$$\|v\|_{L^\infty(\Omega,\mathbb{R}^n)}\leqslant\tau_1\quad \textit{et}\quad \|V\|_{L^\infty(\Omega)}\leqslant\tau_2.$$

Alors il existe une constant $\eta=\eta(\Omega,n,\overline{M}_A,\underline{m}_\Lambda,\tau_1)>0$ ne dépendant que de $\Omega,n,\overline{M}_A,\underline{m}_\Lambda$ et τ_1, et une fonction radiale $\Lambda^*>0$ de classe $C^\infty(\overline{\Omega^*})$ tels que

$$\operatorname*{ess\,inf}_{\Omega}\Lambda\leqslant\min_{\overline{\Omega^*}}\Lambda^*\leqslant\max_{\overline{\Omega^*}}\Lambda^*\leqslant\operatorname*{ess\,sup}_{\Omega}\Lambda,\qquad \left\|(\Lambda^*)^{-1}\right\|_{L^1(\Omega^*)}=\left\|\Lambda^{-1}\right\|_{L^1(\Omega)}$$

et

$$\lambda_1\big(\Omega^*,\Lambda^*\,\mathrm{Id},\tau_1e_r,-\tau_2\big)\leqslant\lambda_1(\Omega,A,v,V)-\eta. \tag{1}$$

Lorsque $L=-\Delta+v\cdot\nabla$, l'inégalité (1) redonne bien celle obtenue dans le Théorème 0.1 de [4]. On notera toutefois que, dans le Théorème 0.1 de la présente note, contrairement à la conclusion du Théorème 0.1 de [4], on ne donne pas une description complète des cas d'égalité dans (1) quand Ω est une boule. Nous montrons aussi des résultats d'optimisation pour la première valeur propre de l'opérateur L lorsque ses coefficients varient en satisfaisant certaines contraintes: bornes sur les normes L^p ou les fonctions de distribution des coefficients de L, contraintes sur le déterminant et la trace de A (ou une autre fonction symétrique des valeurs propres de A). Les preuves utilisent une nouvelle méthode de réarrangement des fonctions, qui étend celle décrite dans [4,5]. Nous renvoyons à la version en anglais ci-dessous pour des énoncés complets, et à [6] pour leurs preuves et celles d'autres résultats voisins.

1. Statement of the results

Let $n\geqslant1$. If $E\subset\mathbb{R}^n$ is any measurable subset, $|E|$ denotes the Lebesgue measure of E. Denote by $\mathcal{C}$ the class of C^2 nonempty bounded domains of $\mathbb{R}^n$ and, for all $\Omega\in\mathcal{C}$, let Ω^* be the Euclidean ball centered at 0 and satisfying $|\Omega^*|=|\Omega|$. If $v:\Omega\to\mathbb{R}^n$ is measurable and $1\leqslant p\leqslant+\infty$, say that $v\in L^p(\Omega,\mathbb{R}^n)$ if $|v|$ (where $|\cdot|$ stands for the Euclidean norm) belongs to $L^p(\Omega)$, and write $\|v\|_{L^p(\Omega,\mathbb{R}^n)}$ instead of $\||v|\|_{L^p(\Omega)}$. The celebrated Rayleigh–Faber–Krahn inequality asserts that, among all smooth nonempty bounded domains with fixed Lebesgue measure, the only ones which minimize the principal eigenvalue of the Laplacian under Dirichlet boundary condition are balls ([3,7–9]). In [4,5], we generalized this result to the case of the Laplace operator with a drift term.

In the present Note, we extend this 'Faber–Krahn' type inequality to general second-order elliptic operators in divergence form, and prove other optimization results for the principal eigenvalue of these operators. As in [4,5], the proofs rely on a rearrangement technique, interesting in its own right, and different from Schwarz and Steiner symmetrizations.

Denote by $\mathcal{S}_n(\mathbb{R})$ the set of $n \times n$ real-valued symmetric matrices. Let $\Omega \in \mathcal{C}$. We are interested in second-order elliptic operators of the form $L = -\operatorname{div}(A\nabla) + v \cdot \nabla + V$ on Ω with Dirichlet boundary condition. We always assume that $A \in W^{1,\infty}(\Omega, \mathcal{S}_n(\mathbb{R}))$ (so that, up to modification on a zero measure set, A can be assumed to be continuous on $\overline{\Omega}$), and that A is uniformly elliptic in Ω, which means that there exists $\gamma > 0$ such that $A(x) \geqslant \gamma \operatorname{Id}$ for all $x \in \overline{\Omega}$. More generally, we may consider the case when A is bounded from below by some positive function and, for

$$\Lambda \in L^\infty_+(\Omega) = \Big\{ f \in L^\infty(\Omega);\ \operatorname*{ess\,inf}_{\Omega} f > 0 \Big\},$$

we say that $A \geqslant \Lambda \operatorname{Id}$ in Ω if, for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$,

$$A(x)\xi \cdot \xi \geqslant \Lambda(x)|\xi|^2.$$

We also assume that $v \in L^\infty(\Omega, \mathbb{R}^n)$ and $V \in L^\infty(\Omega)$. Under these assumptions, the operator L has a principal eigenvalue (see [2]), which will be denoted by $\lambda_1(\Omega, A, v, V)$.

We are interested in some optimization problems for $\lambda_1(\Omega, A, v, V)$ when Ω, A, v and V vary, satisfying various kinds of constraints. To cope with these problems, we associate to L an operator L^* on Ω^* with radial coefficients satisfying the same constraints as those of L, in such a way that the principal eigenvalue of L^* is not too much larger than $\lambda_1(\Omega, A, v, V)$. To state our results, we need two more notations: if $\Omega \in \mathcal{C}$, for all measurable $u : \Omega \to \mathbb{R}$, the distribution function of u is given by

$$\mu_u(t) = \big|\{x \in \Omega;\ u(x) > t\}\big|$$

for all $t \in \mathbb{R}$. For all $x \in \mathbb{R}^n \setminus \{0\}$, denote $e_r(x) = x/|x|$.

Here is our first result:

Theorem 1.1. *Let $\Omega \in \mathcal{C}$, $A \in W^{1,\infty}(\Omega, \mathcal{S}_n(\mathbb{R}))$, $\Lambda \in L^\infty_+(\Omega)$, $v \in L^\infty(\Omega, \mathbb{R}^n)$ and $V \in C(\overline{\Omega})$. Assume that $A \geqslant \Lambda \operatorname{Id}$ a.e. in Ω, and that $\lambda_1(\Omega, A, v, V) \geqslant 0$. Then, for all $\varepsilon > 0$, there exist three radially symmetric $C^\infty(\overline{\Omega^*})$ fields $\Lambda^* > 0$, $\omega^* \geqslant 0$ and $\overline{V}^* \leqslant 0$ such that, for $v^* = \omega^* e_r$ in $\overline{\Omega^*}\backslash\{0\}$,*

$$\begin{cases} \operatorname*{ess\,inf}_{\Omega} \Lambda \leqslant \min_{\overline{\Omega^*}} \Lambda^* \leqslant \max_{\overline{\Omega^*}} \Lambda^* \leqslant \operatorname*{ess\,sup}_{\Omega} \Lambda, & \big\|(\Lambda^*)^{-1}\big\|_{L^1(\Omega^*)} = \big\|\Lambda^{-1}\big\|_{L^1(\Omega)}, \\ \|v^*\|_{L^\infty(\Omega^*,\mathbb{R}^n)} \leqslant \|v\|_{L^\infty(\Omega,\mathbb{R}^n)}, & \big\||v^*|^2(\Lambda^*)^{-1}\big\|_{L^1(\Omega^*)} = \big\||v|^2\Lambda^{-1}\big\|_{L^1(\Omega)}, \\ \mu_{|\overline{V}^*|} = \mu_{(\overline{V}^*)^-} \leqslant \mu_{V^-}, \end{cases} \tag{2}$$

and

$$\lambda_1\big(\Omega^*, \Lambda^* \operatorname{Id}, v^*, \overline{V}^*\big) \leqslant \lambda_1(\Omega, A, v, V) + \varepsilon.$$

There also exists a nonpositive radially symmetric $L^\infty(\Omega^*)$ field V^* such that $\mu_{V^*} = \mu_{-V^-}$, $V^* \leqslant \overline{V}^* \leqslant 0$ and $\lambda_1(\Omega^*, \Lambda^* \operatorname{Id}, v^*, V^*) \leqslant \lambda_1(\Omega^*, \Lambda^* \operatorname{Id}, v^*, \overline{V}^*) \leqslant \lambda_1(\Omega, A, v, V) + \varepsilon$.

If one further assumes that Λ is equal to a constant $\gamma > 0$ in Ω, then there exist two radially symmetric bounded functions $\omega_0^* \geqslant 0$ and $V_0^* \leqslant 0$ in Ω^* such that, for $v_0^* = \omega_0^* e_r$,

$$\begin{cases} \|v_0^*\|_{L^\infty(\Omega^*,\mathbb{R}^n)} \leqslant \|v\|_{L^\infty(\Omega,\mathbb{R}^n)}, & \|v_0^*\|_{L^2(\Omega^*,\mathbb{R}^n)} \leqslant \|v\|_{L^2(\Omega,\mathbb{R}^n)}, \\ -\max_{\overline{\Omega}} V^- \leqslant V_0^* \leqslant 0 \quad \text{a.e. in } \Omega^*, & \big\|V_0^*\big\|_{L^p(\Omega^*)} \leqslant \|V^-\|_{L^p(\Omega)} \quad \text{for all } 1 \leqslant p \leqslant +\infty, \end{cases} \tag{3}$$

and $\lambda_1(\Omega^*, \gamma \operatorname{Id}, v_0^*, V_0^*) \leqslant \lambda_1(\Omega, A, v, V)$.

In the case when Ω is not a ball, we obtain a more precise comparison result for the same constraints on the coefficients of L:

Theorem 1.2. *Under the notations of Theorem 1.1, assume that $\Omega \in \mathcal{C}$ is not a ball and let $\overline{M}_A \geqslant \underline{m}_\Lambda > 0$, $\overline{M}_v \geqslant 0$ and $\overline{M}_V \geqslant 0$ be such that*

$$\|A\|_{W^{1,\infty}(\Omega,\mathcal{S}_n(\mathbb{R}))} \leqslant \overline{M}_A, \quad \operatorname*{ess\,inf}_{\Omega} \Lambda \geqslant \underline{m}_\Lambda, \quad \|v\|_{L^\infty(\Omega,\mathbb{R}^n)} \leqslant \overline{M}_v \quad \text{and} \quad \|V\|_{L^\infty(\Omega)} \leqslant \overline{M}_V.$$

Then there exists a positive constant $\theta = \theta(\Omega, n, \overline{M}_A, \underline{m}_\Lambda, \overline{M}_v, \overline{M}_V) > 0$ depending only on Ω, n, $\overline{M}_A$, $\underline{m}_\Lambda$, $\overline{M}_v$ and $\overline{M}_V$, such that if $\lambda_1(\Omega, A, v, V) > 0$, then there exist three radially symmetric $C^\infty(\overline{\Omega^*})$ fields $\Lambda^* > 0$, $\omega^* \geqslant 0$,

$\overline{V}^* \leqslant 0$ *and a nonpositive radially symmetric* $L^\infty(\Omega^*)$ *field* V^*, *which satisfy* (2), $\mu_{V^*} = \mu_{-V^-}$, $V^* \leqslant \overline{V}^* \leqslant 0$ *and are such that, for* $v^* = \omega^* e_r$ *in* $\overline{\Omega^*}\backslash\{0\}$,

$$\lambda_1\big(\Omega^*, \Lambda^* \mathrm{Id}, v^*, V^*\big) \leqslant \lambda_1\big(\Omega^*, \Lambda^* \mathrm{Id}, v^*, \overline{V}^*\big) \leqslant \frac{\lambda_1(\Omega, A, v, V)}{1+\theta}.$$

When $n \geqslant 2$, we also prove an optimization result with constraints on the determinant and another symmetric function of the eigenvalues of A. If $A \in \mathcal{S}_n(\mathbb{R})$, if $p \in \{1, \ldots, n-1\}$ and if $\lambda_1[A] \leqslant \cdots \leqslant \lambda_n[A]$ denote the eigenvalues of A, then define

$$\sigma_p(A) = \sum_{I \subset \{1,\ldots,n\}, \mathrm{card}(I)=p} \Big(\prod_{i \in I} \lambda_i[A]\Big).$$

In particular, $\sigma_1(A)$ is the trace of A. Our optimization result is then as follows:

Theorem 1.3. *Assume* $n \geqslant 2$. *Let* $\Omega \in \mathcal{C}$, $A \in W^{1,\infty}(\Omega, \mathcal{S}_n(\mathbb{R}))$, $v \in L^\infty(\Omega, \mathbb{R}^n)$, $V \in C(\overline{\Omega})$ *and let* $p \in \{1, \ldots, n-1\}$, $\omega > 0$, $\sigma > 0$. *Assume that*

$$\det\big(A(x)\big) \geqslant \omega, \quad \sigma_p\big(A(x)\big) \leqslant \sigma \quad \textit{for all } x \in \overline{\Omega},$$

and that $\lambda_1(\Omega, A, v, V) \geqslant 0$. *Then, there are two positive numbers* $0 < a_1 \leqslant a_2$ *which only depend on n, p, ω and σ, such that, for all* $\varepsilon > 0$, *there exist a matrix field* $A^* \in C^\infty(\overline{\Omega^*}\backslash\{0\}, \mathcal{S}_n(\mathbb{R}))$, *two radially symmetric* $C^\infty(\overline{\Omega^*})$ *fields* $\omega^* \geqslant 0$ *and* $\overline{V}^* \leqslant 0$, *and a nonpositive radially symmetric* $L^\infty(\Omega^*)$ *field* V^*, *such that, for* $v^* = \omega^* e_r$ *in* $\overline{\Omega^*}\backslash\{0\}$,

$$\begin{cases} A \geqslant a_1 \mathrm{Id} \quad \textit{in } \Omega, \quad A^* \geqslant a_1 \mathrm{Id} \quad \textit{in } \Omega^*, \quad \det\big(A^*(x)\big) = \omega, \quad \sigma_p(A^*(x)) = \sigma \quad \textit{for all } x \in \overline{\Omega^*}\backslash\{0\}, \\ \|v^*\|_{L^\infty(\Omega^*, \mathbb{R}^n)} \leqslant \|v\|_{L^\infty(\Omega, \mathbb{R}^n)}, \quad \|v^*\|_{L^2(\Omega^*)} = \|v\|_{L^2(\Omega)}, \\ \mu_{|\overline{V}^*|} \leqslant \mu_{V^-}, \quad \mu_{V^*} = \mu_{-V^-}, \quad V^* \leqslant \overline{V}^* \leqslant 0 \end{cases}$$

and

$$\lambda_1\big(\Omega^*, A^*, v^*, V^*\big) \leqslant \lambda_1\big(\Omega^*, A^*, v^*, \overline{V}^*\big) \leqslant \lambda_1(\Omega, A, v, V) + \varepsilon.$$

Furthermore, the matrix field A^* *is defined, for all* $x \in \overline{\Omega^*}\backslash\{0\}$, *by*:

$$A^*(x)x \cdot x = a_1|x|^2 \quad \textit{and} \quad A^*(x)y \cdot y = a_2|y|^2 \quad \textit{for all } y \perp x.$$

Lastly, there exist two radially symmetric bounded functions $\omega_0^* \geqslant 0$ *and* $V_0^* \leqslant 0$ *in* Ω^* *satisfying* (3) *and* $\lambda_1(\Omega^*, A^*, v_0^*, V_0^*) \leqslant \lambda_1(\Omega, A, v, V)$, *where* $v_0^* = \omega_0^* e_r$ *in* Ω^*.

Observe that, although A^* does not belong to $W^{1,\infty}(\Omega^*)$ in general (unless $a_1 = a_2$), we can still define the principal eigenvalue of $-\mathrm{div}(A^*\nabla) + v^* \cdot \nabla + V^*$ in this context. Indeed, for $\tilde{A}^* = a_1 \mathrm{Id}$ in $\overline{\Omega^*}$, the principal eigenfunction φ^* of the operator $-\mathrm{div}(\tilde{A}^*\nabla) + v^* \cdot \nabla + V^*$ is radially symmetric and belongs to all $W^{2,r}(\Omega^*)$ spaces for all $1 \leqslant r < +\infty$. Hence, $A^*\nabla\varphi^* = \tilde{A}^*\nabla\varphi^* = a_1\nabla\varphi^*$. We therefore call $\lambda_1(\Omega^*, A^*, v^*, V^*) = \lambda_1(\Omega^*, \tilde{A}^*, v^*, V^*)$.

As an application of Theorem 1.1, we prove the following 'Faber–Krahn' type inequality:

Theorem 1.4. *Let* $\Omega \in \mathcal{C}$ *which is not a ball,* $\overline{M}_A \geqslant \underline{m}_\Lambda > 0$, $\tau_1 \geqslant 0$ *and* $\tau_2 \geqslant 0$. *Consider* $A \in W^{1,\infty}(\Omega, \mathcal{S}_n(\mathbb{R}))$, $\Lambda \in L^\infty_+(\Omega)$, $v \in L^\infty(\Omega, \mathbb{R}^n)$ *and* $V \in L^\infty(\Omega)$ *satisfying*

$$\|A\|_{W^{1,\infty}(\Omega, \mathcal{S}_n(\mathbb{R}))} \leqslant \overline{M}_A, \quad \operatorname*{ess\,inf}_{\Omega} \Lambda \geqslant \underline{m}_\Lambda, \quad A \geqslant \Lambda \mathrm{Id} \quad \textit{a.e. in } \Omega,$$

$$\|v\|_{L^\infty(\Omega, \mathbb{R}^n)} \leqslant \tau_1 \quad \textit{and} \quad \|V\|_{L^\infty(\Omega, \mathbb{R})} \leqslant \tau_2.$$

Then there exist a positive constant $\eta = \eta(\Omega, n, \overline{M}_A, \underline{m}_\Lambda, \tau_1) > 0$ *only depending on* $\Omega, n, \overline{M}_A, \underline{m}_\Lambda$ *and* τ_1, *and a radial* $C^\infty(\overline{\Omega^*})$ *function* $\Lambda^* > 0$ *such that*

$$\operatorname*{ess\,inf}_{\Omega} \Lambda \leqslant \min_{\overline{\Omega^*}} \Lambda^* \leqslant \max_{\overline{\Omega^*}} \Lambda^* \leqslant \operatorname*{ess\,sup}_{\Omega} \Lambda, \qquad \big\|(\Lambda^*)^{-1}\big\|_{L^1(\Omega^*)} = \big\|\Lambda^{-1}\big\|_{L^1(\Omega)}$$

and

$$\lambda_1(\Omega^*, \Lambda^* \mathrm{Id}, \tau_1 e_r, -\tau_2) \leqslant \lambda_1(\Omega, A, v, V) - \eta.$$

Notice that, in Theorem 1.4, contrary to what happens in Theorems 1.1, 1.2 and 1.3, we make no assumption on the sign of $\lambda_1(\Omega, A, v, V)$. Notice also that in Theorems 1.1, 1.2 and 1.4, the functions Λ and Λ^* are not necessarily constant, which means that the operator $\operatorname{div}(\Lambda^*\nabla)$ is not equal to the Laplacian up to a multiplicative constant in general. These theorems also extend some results which could be derived implicitly from comparison results for elliptic problems in the case when Λ is constant and $V \geqslant 0$, by means of Schwarz symmetrization (see [1,10]). Actually, all the results of the present Note are new even for symmetric operators, that is when $v = 0$, and also in dimension $n = 1$ (except Theorem 1.3, the statement of which makes sense only when $n \geqslant 2$). To sum up, all these results mean that minimizing the principal eigenvalue of operators of the type $L = -\operatorname{div}(A\nabla) + v \cdot \nabla + V$ in Ω under the abovementioned constraints is the same as minimizing the principal eigenvalue of some operators L^* in the ball Ω^* with smooth and radially symmetric coefficients. However, we do not solve the optimization problems in the ball, we plan to come back to this issue in a forthcoming paper.

2. Sketch of the proofs

The proofs of all the previous results rely on a rearrangement technique, which generalizes the one used in [4,5], and is of independent interest. Let $\Omega \in \mathcal{C}$ and $R > 0$ be the radius of Ω^*. Consider $A_\Omega \in C^1(\overline{\Omega}, \mathcal{S}_n(\mathbb{R}))$ and $\Lambda_\Omega \in C^1(\overline{\Omega})$ positive in Ω such that $A_\Omega \geqslant \Lambda_\Omega \operatorname{Id}$ in $\overline{\Omega}$. Let $\psi \in C^1(\overline{\Omega})$ be positive and analytic in Ω, vanishing on $\partial\Omega$ and such that $\nabla\psi \neq 0$ everywhere on $\partial\Omega$ and $\operatorname{div}(A\nabla\psi) := f$ is a nonzero polynomial in Ω. For all $0 \leqslant a < M := \max_{\overline{\Omega}} \psi$, define $\rho(a) \in (0, R]$ such that $|\Omega_a| = |B_{\rho(a)}|$, where $\Omega_a = \{x \in \Omega;\ \psi(x) > a\}$ and B_s denotes the Euclidean ball of $\mathbb{R}^n$ with center 0 and radius $s > 0$. Set also $\rho(M) = 0$. The function $\rho : [0, M] \to [0, R]$ is continuous, one-to-one and onto. Since ψ is real analytic and its gradient does not vanish on $\partial\Omega$, the set Z of its critical values is finite, denote $Y = [0, M] \setminus Z$. For all $x \in \Omega^*$ such that $|x| \in \rho(Y)$, define

$$\hat{\Lambda}(x) = \frac{\int_{\partial\Omega_{\rho^{-1}(|x|)}} |\nabla\psi(y)|^{-1}\, d\sigma}{\int_{\partial\Omega_{\rho^{-1}(|x|)}} \Lambda_\Omega(y)^{-1} |\nabla\psi(y)|^{-1}\, d\sigma} > 0, \tag{4}$$

where $\rho^{-1} : [0, R] \to [0, M]$ denotes the reciprocal of the function ρ. The function $\hat{\Lambda}$ is defined almost everywhere in Ω^*. We now define $\tilde{\psi}$ as the radially decreasing function in $H_0^1(\Omega^*) \cap W^{1,\infty}(\Omega^*)$ such that

$$\int_{\Omega_a} \operatorname{div}(A\nabla\psi)(x)\, dx = \int_{B_{\rho(a)}} \operatorname{div}\big(\hat{\Lambda}\nabla\tilde{\psi}\big)(x)\, dx$$

for all $a \in [0, M)$. The key inequality satisfied by $\tilde{\psi}$ is that, for all $x \in \overline{\Omega^*}$,

$$\tilde{\psi}(x) \geqslant \rho^{-1}\big(|x|\big).$$

The proof of this inequality (and an improved version when Ω is not a ball) relies, in particular, on the co-area formula and the classical isoperimetric inequality in $\mathbb{R}^n$. If $v_\Omega \in C(\overline{\Omega}, \mathbb{R}^n)$ and $V_\Omega \in C(\overline{\Omega})$, radial rearrangements $\hat{v}$ and $\widehat{V}$ of v_Ω and V_Ω are defined, by formulas analogous to (4), and a pointwise comparison between $-\operatorname{div}(A_\Omega\nabla\psi) + v_\Omega \cdot \nabla\psi + V_\Omega\psi$ in Ω and $-\operatorname{div}(\hat{\Lambda}\nabla\tilde{\psi}) + \hat{v} \cdot \nabla\tilde{\psi} + \widehat{V}\tilde{\psi}$ in Ω^* is obtained.

To prove Theorems 1.1 and 1.2, the data A, Λ, v and V, as well as the corresponding eigenfunction, are approximated by sufficiently smooth fields, in order to apply our rearrangement technique to these smooth approximations. All these proofs involve many technicalities, and we refer to [6] for complete proofs and other statements.

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