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Partial Differential Equations

On a class of singular Gierer–Meinhardt systems arising in morphogenesis

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Abstract

We study the existence or the nonexistence of classical solutions to a singular Gierer–Meinhardt system with Dirichlet boundary condition. The main feature of our model is that the activator and the inhibitor have different sources given by general nonlinearities. Additional regularity and uniqueness results are established for the one-dimensional case. ***To cite this article: M. Ghergu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sur une classe de systèmes singuliers de Gierer–Meinhardt avec applications en morphogenèse. On étudie l'existence ou la non-existence des solutions classiques pour une classe de systèmes singuliers de Gierer–Meinhardt avec condition de Dirichlet sur le bord. La caractéristique de notre modèle réside dans la présence de sources différentes pour l'activateur et aussi pour l'inhibiteur. Des propriétés supplémentaires de régularité et d'unicité sont établies en dimension 1. ***Pour citer cet article : M. Ghergu, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ un ouvert borné et régulier. On considère le système de Gierer–Meinhardt

$$\begin{cases} \Delta u - \alpha u + \dfrac{f(u)}{g(v)} + \rho(x) = 0, \quad u > 0 & \text{dans } \Omega, \\ \Delta v - \beta v + \dfrac{h(u)}{k(v)} = 0, \quad v > 0 & \text{dans } \Omega, \\ u = 0, \quad v = 0 & \text{sur } \partial\Omega, \end{cases} \tag{1}$$

où $\alpha, \beta > 0$, $0 \leqslant \rho \in C^{0,\gamma}(\Omega)$ $(0 < \gamma < 1)$, $\rho \not\equiv 0$ et $f, g, h, k \in C^{0,\gamma}[0,\infty)$ sont des fonctions non négatives et décroissantes, telles que $g(0) = k(0) = 0$. Soit $\Phi : [0,1) \to [0,\infty)$ l'application définie par $\Phi(t) =$

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$2^{-1/2}\int_0^t(\int_\tau^1 1/k(\theta)\,\mathrm{d}\theta)^{-1/2}\,\mathrm{d}\tau$. On désigne par $\Psi:[0,a)\to[0,1)$ l'inverse de Φ, où $a=\lim_{t\to1}\Phi(t)$. On démontre d'abord le résultat suivant de non existence :

Théorème 0.1. *On suppose que*

$$\int_0^a \frac{tf(mt)}{g(M\Psi(t))}\,\mathrm{d}t=+\infty,$$

pour tous $0<m<1<M$. *Alors le systéme* (1) *n'admet pas de solutions classiques.*

Introduisons maintenant les conditions suivantes :

(A_1) $\frac{f(t_1)}{h(t_1)}-\frac{g(t_2)}{k(t_2)}\leqslant 0$, pour tous $t_1\geqslant t_2>0$;
(A_2) $k:(0,\infty)\to[0,\infty)$ est une application croissante de classe C^1 telle que $\lim_{t\to+\infty}\int_0^t k(\tau)\,\mathrm{d}\tau/h(t+c)=+\infty$, pour tout $c>0$.

Théorème 0.2. *Supposons que les hypothèses* (A_1), (A_2) *soient satisfaites. Alors le système* (1) *admet des solutions classiques.*

On considère maintenant le système

$$\begin{cases}\Delta u-\alpha u+\dfrac{u^p}{v^q}+\rho(x)=0, & u>0 \quad \text{dans } \Omega,\\ \Delta v-\beta v+\dfrac{u^{p+\sigma}}{v^{q+\sigma}}=0, & v>0 \quad \text{dans } \Omega,\\ u=v=0 & \text{sur } \partial\Omega,\end{cases}\tag{2}$$

où $\sigma\geqslant 0$.

Théorème 0.3. *On suppose que* $p,q\geqslant 0$ *satisfont* $p-q<1$. *Alors le système* (2) *admet des solutions classiques pour tout* $\sigma\geqslant 0$. *De plus, pour chaque solution* (u,v) *de* (2) *on a les propriétés suivantes* :

(i) *il existe les constantes positives* c_1 *et* c_2 *telles que* $c_1d(x)\leqslant u(x),\ v(x)\leqslant c_2d(x)$ *dans* Ω ;
(ii) $u,v\in C^2(\Omega)\cap C^{1,1+p-q}(\overline{\Omega})$ *si* $-1<p-q<0$;
(iii) $u,v\in C^2(\overline{\Omega})$ *si* $0\leqslant p-q<1$.

1. Introduction

Nonlinear systems of Gierer–Meinhardt type have been intensively studied in the last few decades in view of the understanding of some basic phenomena arising in morphogenesis. We refer to the pioneering works by Turing [12] and Gierer and Meinhardt [5,8], as well as to the recent papers of Choi and McKenna [1,2], Ni et al. [9–11], Wei and Winter [13,14].

Let $\Omega\subset\mathbb{R}^N$ ($N\geqslant 1$) be a bounded domain with smooth boundary. In this Note we are concerned with stationary Gierer–Meinhardt systems for wide classes of nonlinearities, subject to homogeneous Dirichlet boundary conditions. More exactly, we study the following elliptic system:

$$\begin{cases}\Delta u-\alpha u+\dfrac{f(u)}{g(v)}+\rho(x)=0, & u>0 \quad \text{in } \Omega,\\ \Delta v-\beta v+\dfrac{h(u)}{k(v)}=0, & v>0 \quad \text{in } \Omega,\\ u=0,\quad v=0 & \text{on } \partial\Omega,\end{cases}\tag{3}$$

where $\alpha,\ \beta>0$, $0\leqslant\rho\in C^{0,\gamma}(\Omega)$ $(0<\gamma<1)$, $\rho\not\equiv 0$ and $f,g,h,k\in C^{0,\gamma}[0,\infty)$ are nonnegative and nonincreasing functions such that $g(0)=k(0)=0$. This last assumption on g and k, together with the Dirichlet conditions on $\partial\Omega$ make the system singular at the boundary.

The main feature of this Note is that we assume that both the activator and the inhibitor have different source terms, that is, the mappings $t \mapsto f(t)/h(t)$ and $t \mapsto g(t)/k(t)$ are not constant on $(0, \infty)$. Our study is motivated by some questions addressed in Choi and McKenna [1,2] or Kim [6,7] related to the existence, nonexistence or even uniqueness of classical solutions for the model system

$$\begin{cases} \Delta u - \alpha u + \dfrac{u^p}{v^q} + \rho(x) = 0 & \text{in } \Omega, \\ \Delta v - \beta v + \dfrac{u^r}{v^s} = 0 & \text{in } \Omega, \\ u = 0, \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

In [1,6] it is assumed that the activator and inhibitor have common sources and the approach rely on Schauder's fixed point theorem through a decouplization of the system. More precisely, subtracting the two equations of (4) we obtain in the case $p = r$ and $q = s$ a linear equation in $w = u - v$. This is suitable to obtain a priori estimates in order to control the map whose fixed points are solutions of (4).

In Choi and McKenna [2] it is obtained the existence of radially symmetric solutions of (4) in the case $\Omega = (0, 1)$ or $\Omega = B_1 \subset \mathbb{R}^2$ and $p = r > 1$, $q = 1$, $s = 0$. In [2] a priori bounds are obtained via sharp estimates of the associated Green's function.

2. Main results

Basic auxiliary results in our approach are the comparison principle established in [3, Lemma 2.1] and the following property, which is a direct consequence of the maximum principle:

Lemma 2.1. *Let $k \in C(0, \infty)$ be a positive nondecreasing function and $a_1, a_2 \in C(\Omega)$ with $0 < a_2 \leqslant a_1$ in Ω. Assume that there exist $\beta > 0$, $v_1, v_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $v_1, v_2 > 0$ in Ω, $v_2 \leqslant v_1$ on $\partial\Omega$ and*

$$\Delta v_1 - \beta v_1 + \frac{a_1(x)}{k(v_1)} \leqslant 0 \leqslant \Delta v_2 - \beta v_2 + \frac{a_2(x)}{k(v_2)} \quad \text{in } \Omega.$$

Then $v_2 \leqslant v_1$ in Ω.

We first state a nonexistence result for classical solutions of the system (3). The main idea is to evaluate the asymptotic behavior of v in the second equation of (3). This will be then used in the first equation of the system and by classical arguments we obtain the desired nonexistence result.

Let $\Phi : [0, 1) \to [0, \infty)$ be the mapping defined by

$$\Phi(t) = \int_0^t \frac{1}{\sqrt{2\int_\tau^1 (1/k(\theta))\, d\theta}}\, d\tau, \quad 0 \leqslant t < 1.$$

Denote by $\Psi : [0, a) \to [0, 1)$ the inverse of Φ, where $a = \lim_{t \to 1} \Phi(t)$. Our nonexistence result is the following:

Theorem 2.2. *Assume that*

$$\int_0^a \frac{tf(mt)}{g(M\Psi(t))}\, dt = +\infty, \tag{5}$$

for all $0 < m < 1 < M$. Then the system (3) has no classical solutions.

Sketch of the proof. If (u, v) is a classical solution of (3) then v verifies $\Delta v - \beta v + \frac{C}{k(v)} \geqslant 0$ in Ω, where $C = \max_{x \in \overline{\Omega}} h(u(x)) > 0$. The weak maximum principle implies that $u(x) \geqslant m\, \mathrm{dist}(x, \partial\Omega)$ in Ω, for some $m > 0$ small enough. Let φ_1 be the normalized first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$ and consider $c > 0$ such that $c\varphi_1 \leqslant \min\{a, \mathrm{dist}(x, \partial\Omega)\}$ in Ω.

The main point is to show that there exists $M > 1$ large enough such that $\bar{v} = M\Psi(c\varphi_1)$ satisfies

$$\Delta\bar{v} - \beta\bar{v} + \frac{C}{k(\bar{v})} \leqslant 0 \quad \text{in } \Omega. \tag{6}$$

Then, by Lemma 2.1 it follows that $v \leqslant \bar{v}$ in Ω. Next, we consider the approximated problem

$$\begin{cases} \Delta w - \alpha w + \dfrac{f(m\,\mathrm{dist}(x,\partial\Omega))}{g(M\Psi(c\varphi_1)) + \varepsilon} = 0 & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

It is obvious that $\overline{w} = u$ is a super-solution of (7) while $\underline{w} = 0$ is a sub-solution. By classical results, the problem (7) has a unique positive solution $w_\varepsilon \in C^2(\overline{\Omega})$ such that $w_\varepsilon \leqslant u$ in Ω. To raise a contradiction, we multiply by φ_1 in (7) and then we integrate over Ω. □

If $k(t) = t^s$, $s > 0$, condition (5) can be written more explicitly by describing the asymptotic behavior of Ψ.

Corollary 2.3. *Assume that $k(t) = t^s$, $s > 0$, and one of the following conditions hold*:

(i) $s > 1$ *and*

$$\int_0^a \frac{tf(mt)}{g(Mt^{2/(1+s)})}\,\mathrm{d}t = +\infty,$$

for all $0 < m < 1 < M$;

(ii) $s = 1$ *and*

$$\int_0^{\min\{a,1/2\}} \frac{tf(mt)}{g(Mt\sqrt{-\ln t})}\,\mathrm{d}t = +\infty,$$

for all $0 < m < 1 < M$;

(iii) $0 < s < 1$ *and*

$$\int_0^a \frac{tf(mt)}{g(Mt)}\,\mathrm{d}t = +\infty,$$

for all $0 < m < 1 < M$.

Then the system (3) *has no positive classical solutions.*

In the case of pure powers in the nonlinearities we have the following nonexistence result:

Corollary 2.4. *Let $p, q, r, s > 0$ be such that one of the following conditions hold*:

(i) $s > 1$ *and* $2q \geqslant (s+1)(p+2)$;
(ii) $s = 1$ *and* $q > p + 2$;
(iii) $0 < s < 1$ *and* $q \geqslant p + 2$.

Then the system (4) *has no positive classical solutions.*

We are now concerned with existence results for the general system (3). Set

$$A(t_1, t_2) = \frac{f(t_1)}{h(t_1)} - \frac{g(t_2)}{k(t_2)}, \quad \text{for all } t_1, t_2 > 0.$$

We assume that A fulfills

(A$_1$) $A(t_1, t_2) \leqslant 0$ for all $t_1 \geqslant t_2 > 0$.

We also suppose that

(A$_2$) $k \in C^1(0,\infty)$ is a nonnegative and nondecreasing function such that

$$\lim_{t\to+\infty} \frac{\int_0^t k(\tau)\,\mathrm{d}\tau}{h(t+c)} = +\infty, \quad \text{for all } c > 0.$$

Here are some examples of nonlinearities that fulfill (A$_1$) and (A$_2$):

(i) $f(t) = t^p$, $g(t) = t^q$, $h(t) = t^r$, $k(t) = t^s$, $t \geqslant 0$, $p, q, r, s > 0$, $r - p = s - q \geqslant 0$ and $p - q < 1$;
(ii) $f(t) = \ln(1 + t^p)$, $g(t) = \mathrm{e}^{t^q} - 1$, $h(t) = t^p$ and $k(t) = t^q$, $t \geqslant 0$, $p, q > 0$, $p - q < 1$;
(iii) $f(t) = \log(1 + at)$, $g(t) = \log(1 + t)$, $h(t) = at$ and $k(t) = t$, $t \geqslant 0$, $a \geqslant 1$.

We give in what follows a general method to construct nonlinearities f, g, h, k that verify (A$_1$) and (A$_2$). Let $f, g, h, k : [0,\infty) \to [0,\infty)$ be nondecreasing functions such that k and h verify (A$_2$) and at least one of the above assumptions hold.

(a) $fk = gh$ and the mapping $(0,\infty) \ni t \mapsto f(t)/h(t)$ is nonincreasing;
(b) there exists $m > 0$ such that $f(t)/h(t) \leqslant m \leqslant g(t)/k(t)$, for all $t > 0$.

Then A verifies (A$_1$).

For instance, the mappings given in example (i) satisfy the condition (a) while the mappings given in example (ii) verify the condition (b).

Theorem 2.5. *Assume that the hypotheses* (A$_1$), (A$_2$) *are fulfilled. Then the system* (3) *has classical solutions.*

Sketch of the proof. Solutions of (3) are obtained by considering the regularized system

$$\begin{cases} \Delta u - \alpha u + \dfrac{f(u+\varepsilon)}{g(v+\varepsilon)} + \rho(x) = 0 & \text{in } \Omega, \\ \Delta v - \beta v + \dfrac{h(u+\varepsilon)}{k(v+\varepsilon)} = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

The existence of positive classical solutions to (8) is obtained by topological degree arguments. The main difficulty is to prove that for all $u_\varepsilon, v_\varepsilon \in C(\overline{\Omega}) \cap C^2(\Omega)$ that verify (8), there exists $M > 0$ which does not depend on ε such that $\max\{\|u_\varepsilon\|_\infty, \|v_\varepsilon\|_\infty\} \leqslant M$. Moreover, by the weak maximum principle and Lemma 2.1 we deduce that $\zeta(x) \leqslant u_\varepsilon(x)$ and $\xi(x) \leqslant v_\varepsilon(x)$ in Ω, where $\zeta, \xi \in C^2(\overline{\Omega})$ are the unique solutions of

$$\begin{cases} \Delta\zeta - \alpha\zeta + \rho(x) = 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\xi - \beta\xi + \dfrac{h(\zeta)}{k(\xi+1)} = 0 & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, by standard Hölder and Schauder estimates, the sequence $\{(u_\varepsilon, v_\varepsilon)\}_{0<\varepsilon<1}$ converges (up to a subsequence) in $C^2_{\mathrm{loc}}(\Omega) \times C^2_{\mathrm{loc}}(\Omega)$ to $(u, v) \in C^2(\Omega) \times C^2(\Omega)$. The continuity up to the boundary of the solution (u, v) is then obtained in a classical way. $\square$

The next result concerns the following system:

$$\begin{cases} \Delta u - \alpha u + \dfrac{u^p}{v^q} + \rho(x) = 0, \ u > 0 & \text{in } \Omega, \\ \Delta v - \beta v + \dfrac{u^{p+\sigma}}{v^{q+\sigma}} = 0, \ v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where σ is a nonnegative real number.

Theorem 2.6. *Assume that $p, q \geqslant 0$ satisfy $p - q < 1$. Then the system* (9) *has classical solutions for all $\sigma \geqslant 0$. Moreover, for any solution (u, v) of* (9) *we have*

(i) *there exist $c_1, c_2 > 0$ such that $c_1 d(x) \leqslant u, v \leqslant c_2 d(x)$ in Ω;*
(ii) *$u, v \in C^2(\Omega) \cap C^{1,1+p-q}(\overline{\Omega})$ if $-1 < p - q < 0$;*
(iii) *$u, v \in C^2(\overline{\Omega})$ if $0 \leqslant p - q < 1$.*

In the one-dimensional case we are able to prove the uniqueness of the solution to system (9). Our approach is inspired by the methods in [1] in which a C^2-regularity of the solution up to the boundary is needed. So, we restrict our attention to the case $0 < q \leqslant p \leqslant 1$.

Theorem 2.7. *Let $\Omega = (0, 1)$, $0 < q \leqslant p \leqslant 1$ and $\sigma \geqslant 0$. Then the system* (9) *has a unique solution $(u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$.*

We refer to Ghergu and Rădulescu [4] for complete proofs and further results.

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References

[1] Y.S. Choi, P.J. McKenna, A singular Gierer–Meinhardt system of elliptic equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (2000) 503–522.
[2] Y.S. Choi, P.J. McKenna, A singular Gierer–Meinhardt system of elliptic equations: the classical case, Nonlinear Anal. 55 (2003) 521–541.
[3] M. Ghergu, V. Rădulescu, On a class of sublinear singular elliptic problems with convection term, J. Math. Anal. Appl. 311 (2005) 635–646.
[4] M. Ghergu, V. Rădulescu, A singular Gierer–Meinhardt system with different source terms, in preparation.
[5] A. Gierer, H. Meinhardt, A theory of biological pattern formation, Kybernetik 12 (1972) 30–39.
[6] E.H. Kim, A class of singular Gierer–Meinhardt systems of elliptic boundary value problems, Nonlinear Anal. 59 (2004) 305–318.
[7] E.H. Kim, Singular Gierer–Meinhardt systems of elliptic boundary value problems, J. Math. Anal. Appl. 308 (2005) 1–10.
[8] H. Meinhardt, A. Gierer, Generation and regeneration of sequence of structures during morphogenesis, J. Theoret. Biol. 85 (1980) 429–450.
[9] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, Notices Amer. Math. Soc. 45 (1998) 9–18.
[10] W.-M. Ni, K. Suzuki, I. Takagi, The dynamics of a kinetics activato-inhibitor system, J. Differential Equations 229 (2006) 426–465.
[11] W.-M. Ni, J. Wei, On positive solutions concentrating on spheres for the Gierer Meinhardt system, J. Differential Equations 221 (2006) 158–189.
[12] A.M. Turing, The chemical basis of morphogenesis, Philos. Trans. Roy. Soc. London B 237 (1952) 37–72.
[13] J. Wei, M. Winter, Existence and stability analysis of asymmetric for the Gierer Meinhardt system, J. Math. Pures Appl. 83 (2004) 433–476.
[14] J. Wei, M. Winter, Spikes for the Gierer–Meinhardt system in two dimensions: the strong coupling case, J. Differential Equations 178 (2002) 478–518.