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Partial Differential Equations

Boundary singularities of positive solutions of some nonlinear elliptic equations

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Abstract

We study the behavior near x_0 of any positive solution of (E) $-\Delta u = u^q$ in Ω which vanishes on $\partial\Omega \setminus \{x_0\}$, where $\Omega \subset \mathbb{R}^N$ is a smooth domain, $q \geqslant (N+1)/(N-1)$ and $x_0 \in \partial\Omega$. Our results are based upon a priori estimates of solutions of (E) and existence, non-existence and uniqueness results for solutions of some nonlinear elliptic equations on the upper-half unit sphere. ***To cite this article: M.-F. Bidaut-Véron et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Singularités au bord de solutions positives d'équations elliptiques non-linéaires. Nous étudions le comportement quand x tend vers x_0 de toute solution positive de (E) $-\Delta u = u^q$ dans Ω qui s'annule sur $\partial\Omega \setminus \{x_0\}$, où $\Omega \subset \mathbb{R}^N$ est un domaine régulier, $q \geqslant (N+1)/(N-1)$ et $x_0 \in \partial\Omega$. Nos résultats sont fondés sur des estimations a priori des solutions de (E), et des résultats d'existence, de non existence et d'unicité de solutions de certaines équations elliptiques non linéaires sur la demi-sphère unité. ***Pour citer cet article : M.-F. Bidaut-Véron et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit Ω un ouvert régulier de $\mathbb{R}^N$, $N \geqslant 4$, tel que $0 \in \partial\Omega$. Étant donné $q > 1$, nous considérons une fonction $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ qui vérifie (3). Nous nous intéressons à la description du comportement de u au voisinage de 0.

Nous distinguerons les trois valeurs critiques de q données par (4). Si $1 < q < q_1$, le comportement en 0 des solutions est décrit dans [4] ; aussi supposerons-nous le plus souvent $q \geqslant q_1$. Si u est une solution de (3) dans $\mathbb{R}^N_+$ de la forme $u(x) = u(r,\sigma) = r^{-2/(q-1)}\omega(\sigma)$, alors ω vérifie l'équation (6). Dans ce cas, nous avons le résultat suivant :

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Théorème 0.1.

(i) *Si* $1 < q \leqslant q_1$, *le problème* (3) *n'admet aucune solution.*
(ii) *Si* $q_1 < q < q_3$, (3) *admet une unique solution, notée* ω_0.
(iii) *Si* $q \geqslant q_3$, (3) *n'admet aucune solution.*

Le résultat d'unicité décrit en (ii) est en fait un cas particulier d'un résultat plus général :

Théorème 0.2. *Pour tous* $1 < q \leqslant q_3$ *et* $\lambda \in \mathbb{R}$, *il existe au plus une solution positive de* (7).

Ce résultat demeure si, dans (7), S_+^{N-1} est remplacé par une boule dans $\mathbb{R}^N$, et Δ' par le laplacien ordinaire.

Par simplicité, nous pouvons supposer que $\partial\mathbb{R}_+^N$ est l'hyperplan tangent à Ω en 0. Le théorème ci-dessous donne une classification des singularités isolées du problème (3) :

Théorème 0.3. *Soit* $q \geqslant q_1$, *avec* $q \neq q_2$. *Supposons que la solution u du problème* (3) *vérifie*

$$0 \leqslant u(x) \leqslant C|x|^{-2/(q-1)} \quad \forall x \in \Omega \cap B_a(0), \tag{1}$$

pour $C, a > 0$. *Si* $q_1 \leqslant q < q_3$, *ou bien u est continue en* 0, *ou bien*

$$u(r,\sigma) = \begin{cases} r^{-(N-1)}\big(\log(1/r)\big)^{(1-N)/2}\big(k_N\sigma_1 + \mathrm{o}(1)\big) & si\ q = q_1, \\ r^{-2/(q-1)}\big(\omega_0(\sigma) + \mathrm{o}(1)\big) & si\ q_1 < q < q_3, \end{cases} \tag{2}$$

lorsque $r \to 0$, *uniformément par rapport à* $\sigma \in S_+^{N-1}$; k_N *est une constante qui dépend seulement de* N.

Si $q \geqslant q_3$, *u est continue en* 0.

L'estimation a priori (1) est obtenue pour $q_1 \leqslant q < q_2$:

Théorème 0.4. *Si* $q_1 \leqslant q < q_2$, *toute solution u de* (3) *vérifie* (1) *pour* $C = C(N, q, \Omega) > 0$.

Les démonstrations détaillées sont présentées dans [1].

1. Introduction and main results

Let Ω be a smooth open subset of $\mathbb{R}^N$, $N \geqslant 4$, such that $0 \in \partial\Omega$ and let $q > 1$. Assume that $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of

$$\begin{cases} -\Delta u = u^q & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{3}$$

Our goal in this Note is to describe the behavior of u in a neighborhood of 0.

This problem has similar features with the case where $x_0 \in \Omega$, which has been studied by Gidas and Spruck [7]. In our case, we encounter three critical values of q in describing the local behavior of u:

$$q_1 := \frac{N+1}{N-1}, \quad q_2 := \frac{N+2}{N-2} \quad \text{and} \quad q_3 := \frac{N+1}{N-3}. \tag{4}$$

When $1 < q < q_1$, it is proved in [4] that for every solution u of (3) there exists $\alpha \geqslant 0$ (depending on N and u) such that

$$u(x) = \alpha|x|^{-N}\rho(x)\big(1 + \mathrm{o}(1)\big) \quad \text{as } x \to 0, \tag{5}$$

where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$, $\forall x \in \Omega$. For this reason, we shall mainly restrict ourselves to $q \geqslant q_1$.

Let us first consider the case where $\Omega = \mathbb{R}^N_+$ and we look for solutions of (3) of the form $u(x) = u(r,\sigma) = r^{-2/(q-1)}\omega(\sigma)$, where $r = |x|$ and $\sigma \in S^{N-1}_+$. An easy computation shows that ω must satisfy

$$\begin{cases} -\Delta'\omega = \ell_{N,q}\omega + \omega^q & \text{in } S^{N-1}_+, \\ \omega \geqslant 0 & \text{in } S^{N-1}_+, \\ \omega = 0 & \text{on } \partial S^{N-1}_+, \end{cases} \tag{6}$$

where Δ' denotes the Laplacian in S^{N-1} and $\ell_{N,q} = \frac{2(N-q(N-2))}{(q-1)^2}$. Concerning Eq. (6), we prove

Theorem 1.1.

(i) *If* $1 < q \leqslant q_1$, *then* (6) *admits no positive solution.*
(ii) *If* $q_1 < q < q_3$, *then* (6) *admits a unique positive solution.*
(iii) *If* $q \geqslant q_3$, *then* (6) *admits no positive solution.*

One of the main ingredients in the proof of Theorem 1.1 (ii) is the following

Theorem 1.2. *If* $1 < q \leqslant q_3$ *and* $\lambda \in \mathbb{R}$, *then there exists at most one positive solution of*

$$\begin{cases} -\Delta' v = \lambda v + v^q & \text{in } S^{N-1}_+, \\ v = 0 & \text{on } \partial S^{N-1}_+. \end{cases} \tag{7}$$

We now return to the case where $\Omega \subset \mathbb{R}^N$ is an arbitrary smooth set such that $0 \in \partial\Omega$. For simplicity, we may assume that $\partial\mathbb{R}^N_+$ is the tangent hyperplane of Ω at 0. Using Theorem 1.2, we provide a classification of isolated singularities of solutions of (3):

Theorem 1.3. *Let* $q \geqslant q_1$, $q \neq q_2$, *and let* u *be a solution of* (3). *Assume that* u *satisfies*

$$0 \leqslant u(x) \leqslant C|x|^{-2/(q-1)} \quad \forall x \in \Omega \cap B_a(0), \tag{8}$$

for some $C, a > 0$. *If* $q_1 \leqslant q < q_3$, *then either* u *is continuous at* 0 *or*

$$u(r,\sigma) = \begin{cases} r^{-(N-1)}\big(\log(1/r)\big)^{(1-N)/2}\big(k_N\sigma_1 + \mathrm{o}(1)\big) & \text{if } q = q_1, \\ r^{-2/(q-1)}\big(\omega_0(\sigma) + \mathrm{o}(1)\big) & \text{if } q_1 < q < q_3, \end{cases} \tag{9}$$

as $r \to 0$, *uniformly with respect to* $\sigma \in S^{N-1}_+$; k_N *denotes a constant depending only on* N *and* ω_0 *is the unique positive solution of* (6).

If $q \geqslant q_3$, *then* u *is continuous at* 0.

Remark 1. We do not know whether Theorem 1.3 is true when $q = q_2$. In this case, the equation is conformally invariant and thus other techniques are required. If $\Omega = \mathbb{R}^N_+$, then it can be proved that any solution of (3) depends only on the variables $r = |x|$ and $\theta = \cos^{-1}(x_1/|x|)$.

The next result establishes the existence of an *a priori* estimate for the solutions of (3). According to Theorem 1.4 below, assumption (8) is always fulfilled when $q_1 \leqslant q < q_2$:

Theorem 1.4. *Let* $q_1 \leqslant q < q_2$ *and let* u *be a solution of* (3). *Then,*

$$0 \leqslant u(x) \leqslant C\rho(x)|x|^{-2/(q-1)-1} \quad \forall x \in \Omega \cap B_1(0), \tag{10}$$

where C *depends on* N, q *and* Ω.

Remark 2. According to the Doob Theorem [6], any positive superharmonic function v in Ω satisfies $\int_\Omega |\Delta v|\rho < \infty$ and admits a boundary trace, which is a Radon measure on $\partial\Omega$. If u is a solution of (3), then its trace must be of the form $k\delta_{x_0}$, for some $k \geqslant 0$. We may have $k > 0$ if $1 < q < q_1$ (see [2]), but k is necessarily equal to 0 if $q \geqslant q_1$. Indeed, by the maximum principle, u satisfies $u \geqslant kP_\Omega(x,0)$, where P_Ω denotes the Poisson potential of Ω. Since $u^q \in L^1_\rho(\Omega)$ (by the Doob Theorem), we must have $k = 0$ if $q \geqslant q_1$.

Detailed proofs will appear in [1].

2. Sketch of the proofs

Proof of Theorem 1.1. Assertion (i) is proved by multiplying (6) by $\phi(\sigma) = \sigma_1$. Note that ϕ is the first eigenfunction of $-\Delta'$ on S^{N-1}_+, with eigenvalue $\lambda_1 = N - 1$. Integrating the resulting expression over S^{N-1}_+, and using the fact that $1 < q \leqslant q_1 \Rightarrow \ell_{N,q} \geqslant \lambda_1$, we obtain (i).

The existence part in (ii) is obtained by using the Mountain Pass Theorem; the uniqueness is a consequence of Theorem 1.2.

Assertion (iii) can be deduced from the following Pohožaev-type identity:

Proposition 2.1. *Assume $q > 1$. Then, any solution of* (7) *satisfies*

$$\frac{N-3}{q+1}(q-q_3)\int_{S^{N-1}_+} |\nabla' v|^2\phi\,\mathrm{d}\sigma - \frac{(N-1)(q-1)}{q+1}\left(\lambda + \frac{N-1}{q-1}\right)\int_{S^{N-1}_+} v^2\phi\,\mathrm{d}\sigma = -\int_{\partial S^{N-1}_+} |\nabla' v|^2\,\mathrm{d}\tau.$$

This identity is obtained by computing the divergence of the vector field $P = \langle\nabla'\phi, \nabla' v\rangle\nabla' v$, where ∇' is the gradient on S^{N-1}, and then using the fact that the first eigenfunction satisfies $D^2\phi + \phi g_0 = 0$, where g_0 is the tensor of the standard metric on S^{N-1}. In order to establish (iii), it suffices to observe that $\ell_{N,q} \leqslant -\frac{N-1}{q-1} \Leftrightarrow q \geqslant q_3$. □

Proof of Theorem 1.2. We first notice that any positive solution of (7) depends only on the variable $\theta = \cos^{-1}(x_1/|x|) \in [0, \pi/2]$; this follows from a straightforward adaptation of the Gidas–Ni–Nirenberg moving plane method to S^{N-1}_+ (see [9]). Thus, v satisfies

$$\begin{cases} v'' + (N-2)\cot\theta\, v' + \lambda v + v^q = 0 & \text{in } (0, \pi/2), \\ v'(0) = 0, \quad v(\pi/2) = 0. \end{cases} \tag{11}$$

Let $w(\theta) := \sin^\alpha\theta\, v(\theta)$, where $\alpha > 0$. By choosing $\alpha = 2(N-2)/(q+3)$, then w satisfies

$$\big(w'(\pi/2)\big)^2 = \int_0^{\pi/2} G'(\theta) w^2(\theta)\,\mathrm{d}\theta, \tag{12}$$

where G is a function of the form $G(\theta) = \sin^{\beta'}\theta(\alpha_1\sin^2\theta + \alpha_2)$; the parameters $\alpha_1, \beta' \in \mathbb{R}$ and $\alpha_2 \geqslant 0$ can be explicitly computed in terms of λ, N and q.

Assume, by contradiction, that v_1 and v_2 are two distinct solutions of (11). Then,

$$\int_0^{\pi/2} v_1 v_2\big(v_2^{q-1} - v_1^{q-1}\big)\,\mathrm{d}\theta = 0. \tag{13}$$

Therefore, their graphs must intersect at some $\theta_0 \in (0, \pi/2)$. We claim that v_1 and v_2 intersect at least twice in $(0, \pi/2)$. If there is only one intersection point, then assuming $v_2(0) > v_1(0)$ it can be shown that there exists $\gamma \geqslant (\frac{w_2'(\pi/2)}{w_1'(\pi/2)})^2$ such that the function $\theta \mapsto G'(\theta)(w_2^2(\theta) - \gamma w_1^2(\theta))$ is nonnegative in $(0, \pi/2)$. Thus, by Eq. (12),

$$0 < \int_0^{\pi/2} G'(\theta)\big(w_2^2(\theta) - \gamma w_1^2(\theta)\big)\,\mathrm{d}\theta = \big(w_2'(\pi/2)\big)^2 - \gamma\big(w_1'(\pi/2)\big)^2 \leqslant 0.$$

This is a contradiction. Therefore, v_1 and v_2 must intersect at least twice. This fact leads to another contradiction by using the Shooting Method (see [8]). Thus, $v_1 = v_2$ in $(0, \pi/2)$. □

Remark 3. The method above follows the lines of the proof of Kwong and Li [8].

Proof of Theorem 1.3. It follows from methods developed in [7] and [3]. For simplicity, we shall assume that $a = 1$ and $\partial\Omega \cap B_1 = \partial\mathbb{R}^N_+ \cap B_1$. We set

$$w(t,\sigma) = r^{2/(q-1)}u(r,\sigma), \quad t = \log(1/r) \in (0,\infty) \times S^{N-1}_+ := Q.$$

Then, w satisfies

$$w_{tt} - \left(N - 2\frac{q+1}{q-1}\right)w_t + \Delta' w + \ell_{N,q}w + w^q = 0 \quad \text{in } Q \tag{14}$$

and w vanishes on $(0,\infty) \times \partial S^{N-1}_+$. Since w is uniformly bounded on Q, standard *a priori* estimates for elliptic problems yield

$$\left|\partial_t^k \nabla'^j w\right| \leqslant M_{k,j} \quad \text{in } (1,\infty) \times S^{N-1}_+$$

for any integers $k, j \geqslant 0$, where ∇'^j stands for the covariant derivative on S^{N-1}. Thus, the trajectory $\mathcal{T}_w = \{w(t,\cdot)\colon\ t \geqslant 1\}$ is relatively compact in $C^2(\overline{S^{N-1}_+})$. Multiplying (14) by w_t and integrating over S^{N-1}_+, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t) = \left(N - 2\frac{q+1}{q-1}\right)\int_{S^{N-1}_+} w_t^2\,\mathrm{d}\sigma, \tag{15}$$

where

$$H(t) := \frac{1}{2}\int_{S^{N-1}_+}\left(w_t^2 - |\nabla' v|^2 - \ell_{N,q}w^2 + \frac{2}{q+1}w^{q+1}\right)\mathrm{d}\sigma.$$

Since $q \neq q_2$, we know that $N - 2(q+1)/(q-1) \neq 0$. Thus, iterated energy estimates imply that $w_t(t,\cdot), w_{tt}(t,\cdot) \to 0$ in $L^2(S^{N-1}_+)$ as $t \to \infty$. Therefore, the limit set Γ_w of $\mathcal{T}$ is a connected subset of the set of solutions of (6). By Theorem 1.1, we deduce that

$$\Gamma_w = \begin{cases} \{0\} & \text{if } q = q_1 \text{ or } q \geqslant q_3, \\ \{0\} \text{ or } \{\omega_0\} & \text{if } q_1 < q < q_3. \end{cases}$$

Then, a linearization argument as in [3] leads to the conclusion if $q > q_1$.

We now consider the case $q = q_1$; we borrow some ideas from [2] and [11]. We first prove, by ODE techniques, that

$$X(t) := \int_{S^{N-1}_+} w(t,\cdot)\phi\,\mathrm{d}\sigma \leqslant Ct^{-(N-1)/2}. \tag{16}$$

Using (8) and the boundary Harnack inequality (see [5]), we derive

$$0 \leqslant w(t,\sigma) \leqslant Ct^{-(N-1)/2} \quad \text{in } (1,\infty) \times S^{N-1}_+. \tag{17}$$

Set $\eta(t,\sigma) := t^{(N-1)/2}w(t,\sigma)$. We verify as above that the limit set Γ_η in $C^2(\overline{S^{N-1}_+})$ of the trajectory $\mathcal{T}_\eta$ of η is an interval of the form $\{\kappa\phi\colon\ 0 \leqslant \kappa_0 \leqslant \kappa \leqslant \kappa_1\}$. In order to show that $\mathcal{T}_\eta$ is reduced to a single point, we prove that $\|r(t,\cdot)\|_{L^2} \leqslant Ct^{-1}$, where

$$r(t,\cdot) := \eta(t,\cdot) - z(t)\phi \quad \text{and} \quad z(t) = \int_{S^{N-1}_+} \eta(t,\cdot)\phi\,\mathrm{d}\sigma.$$

Writing the equation satisfied by z as a non-homogeneous second order linear ODE, we prove that either $z(t) \to 0$, which implies that u is continuous at 0, or $z(t) \to \tilde{k}_N$ as $t \to \infty$, for some constant depending only on N. □

Proof of Theorem 1.4. It is an application of the Doubling Lemma Method introduced in [10], from which we derive the following local estimate:

Lemma 2.1. *Let $1 < q < q_2$ and let u be a solution of* (3). *Then, for every $x_0 \in \partial\Omega \setminus \{0\}$ and $0 < R < |x_0|$, we have*

$$0 \leqslant u(x) \leqslant C\big(R - |x - x_0|\big)^{-2/(q-1)} \quad \forall x \in B_R(x_0) \cap \Omega, \tag{18}$$

for some constant $C > 0$ depending only on Ω.

Apply this lemma with $x_0 \in \partial\Omega \setminus \{0\}$ and $R = |x_0|/2$. Using elliptic regularity theory, we obtain

$$0 \leqslant u(x) \leqslant C\rho(x)|x|^{-2/(q-1)-1} \quad \forall x \in \Omega \text{ such that } 0 < \rho(x) < |x|/2.$$

If $\rho(x) \geqslant |x|/2$, then we use Gidas–Spruck's internal estimates (see [7]). We thus obtain (10). □

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