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Partial Differential Equations

Convergence of a ferromagnetic film model

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Abstract

In this Note, we present a Γ-convergence type result for ferromagnetic films. We propose a model of films for which we could ensure the strong convergence of minimizers when the exchange parameter vanishes. In this model, the plate thickness is kept constant and the magnetization stays constant in the thickness of the film. ***To cite this article: F. Alouges, S. Labbé, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***
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Résumé

Convergence pour un modèle de film mince en ferromagnétisme. Dans cette Note, nous présentons un résultat de Γ-convergence pour les films de matériaux ferromagnétiques. Nous proposons un modèle pour lequel il est possible d'assurer une convergence forte des minimiseurs quand le paramètre d'échange tend vers zéro. Dans ce modèle, l'épaisseur du film est considérée comme constante et l'aimantation est contrainte à rester constante dans l'épaisseur du film. ***Pour citer cet article : F. Alouges, S. Labbé, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***
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Version française abrégée

Le but de cette Note est de présenter un résultat obtenu par les auteurs dans [1] concernant l'étude mathématique d'un modèle asymptotique pour les films ferromagnétiques. Le modèle du micromagnétisme a été introduit par W.F. Brown [5] pour décrire le comportement de l'aimantation dans les matériaux ferromagnétiques. Cette approche thermodynamique s'appuie sur une description énergétique de ces matériaux. Dans la version statique de cette théorie, qui nous intéresse pour cette Note, les états d'équilibre sont les minimiseurs de l'énergie totale (3) dont les contributions sont données par (4), (5) et (6). On s'intéresse alors au problème de minimisation suivant :

$$\begin{cases} \text{Trouver } m_\varepsilon \text{ dans } M^1(\Omega) \text{ tel que} \\ E_\varepsilon(m_\varepsilon) = \displaystyle\min_{m\in M^1(\Omega)} \bigl(E_\varepsilon(m)\bigr), \end{cases} \tag{1}$$

pour des ensembles de configurations admissibles $M^1(\Omega)$ inclus dans $H^1(\Omega;\mathbb{S}^2)$ où $\Omega = \omega \times]0,\eta[$ est le domaine de $\mathbb{R}^3$ contenant le matériau ferromagnétique.

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Plusieurs études ont déja été effectuées sur des problèmes du même type. Dans le cas d'un paramètre d'échange fixé mais pour une épaisseur η tendant vers zéro, la convergence des minimiseurs a été étudiée par G. Carbou [6] et G. Gioia et R.D. James [14]. Pour ce modèle, les minimiseurs du problème limite sont des états constants sur la plaque, ce qui ne rend pas compte de la répartition en domaines réguliers observée [16] et décrite par la construction de van den Berg [22]. Par la suite, deux autres études furent effectuées. La première par A. Desimone, R. Kohn, S. Müller and F. Otto [9–12]. Dans ce modèle, l'épaisseur et la constante d'échange tendent vers zéro ; sans entrer dans les détails, pour une relation bien choisie entre les deux paramètres, les auteurs montrent un résultat de Γ-convergence de l'énergie pour la topologie faible de $L^2(\omega)$.

Un second modèle, moins proche de la physique a ensuite été étudié par T. Rivière et S. Serfaty [19,20] puis F. Alouges, T. Rivière et S. Serfaty [2] ; ici, l'épaisseur η est infinie et ce modèle rend compte de phénomènes physiques importants tels que les structures en 'cross-tie'. Il permet de plus d'assurer une convergence forte des minimiseurs.

Dans cette Note, nous présentons un modèle intermédiaire entre les deux familles précédentes. L'objectif est d'avoir, comme dans le premier cas, une modélisation proche de la physique mais par contre, comme dans le second modèle, de pouvoir assurer une convergence forte des minimiseurs. Dans la suite de la note nous nous intéressons au problème de minimisation (1) quand le paramètre d'échange ε tend vers 0. La modélisation de la petite épaisseur du film est obtenue en forçant les solutions recherchées à être constantes dans l'épaisseur de la plaque. Pour cela, nous considérons

$$M^1(\Omega)=\left\{m\in H^1\big(\Omega;\mathbb{S}^2\big)\text{ tel que }\frac{\partial m}{\partial z}=0\right\},$$
$$K^0(\Omega)=\left\{m=(m_1,m_2,0)\in L^2(\Omega)\text{ tel que }\big|m(x)\big|\leqslant 1\text{ p.p. }x\text{ dans }\Omega,\ \text{ et tel que }\frac{\partial m}{\partial z}=0\right\}$$

et

$$M^0(\Omega)=\big\{m\in K^0\text{ tel que }\big|m(x)\big|=1\text{ p.p. dans }\Omega\big\}$$

et le problème limite

$$\begin{cases}\text{Trouver } m_0 \text{ dans } K^0(\Omega)\text{ tel que}\\ E_0(m_0)=\min\limits_{m\in K^0(\Omega)}\big(E_0(m)\big).\end{cases}\tag{2}$$

Par ailleurs, pour des raisons techniques, nous supposerons aussi que ω est un ouvert borné de $\mathbb{R}^2$ dont le bord est analytique. Le résultat obtenu est résumé dans le théorème suivant :

Théorème 0.1. *Il existe $H_c>0$ tel que si $|H_{\rm ext}|\leqslant H_c$, et $(m_\varepsilon)_\varepsilon\subset M^1(\Omega)$ est une famille de minimiseurs de E_ε sur $M^1(\Omega)$, il existe m_0 dans $M^0(\Omega)$, minimiseur de E_0, tel que (à l'extraction d'une sous-suite près) on ait*

$$m_\varepsilon\to m_0,\quad \textit{dans}\ \bigcap_{1\leqslant p<\infty}L^p(\Omega)$$

et

$$\lim_{\varepsilon\to 0}E_\varepsilon(m_\varepsilon)=E_0(m_0).$$

Pour prouver ce théorème nous utilisons des techniques de Γ-convergence [4].

1. Introduction

The aim of this Note is to present the results obtained in [1] by the authors, which concerns the asymptotics of the micromagnetic model as the exchange constant vanishes of vertically invariant configurations on cylindrical domains. The model of micromagnetism was introduced by W.F. Brown [5] in order to describe the behavior of the magnetization distribution in ferromagnetic materials.

It is expressed as a competition between various energy contributions which depend on the magnetization, a vector field m of constant magnitude (in the remainder of the Note, we consider the dimensionless problem for which this magnitude is equal to one). More precisely, calling Ω the three dimensional bounded domain representing the ferromagnetic sample, and $m : \Omega \mapsto \mathbb{S}^2$ (the unit sphere of $\mathbb{R}^3$) the magnetization field, one associates to m the micromagnetic energy $E(m)$ defined by

$$E(m) = E_{\mathrm{ex}}(m) + E_{\mathrm{d}}(m) + E_{\mathrm{ext}}(m), \tag{3}$$

where these three terms stand respectively for the exchange energy, the stray-field energy and the external energy. For the exchange energy one usually takes

$$E_{\mathrm{ex}}(m) = \varepsilon^2 \int_{\Omega} |\nabla m|^2 \,\mathrm{d}x. \tag{4}$$

The stray-field energy is defined by

$$E_{\mathrm{d}}(m) = \int_{\mathbb{R}^3} \left|\nabla \varphi(m)\right|^2 \mathrm{d}x, \tag{5}$$

where $\varphi(m)$ is the solution, in sense of distributions on $\mathbb{R}^3$ of

$$\begin{bmatrix} \Delta\varphi(m) = -\operatorname{div}(\widetilde{m}), \\ \varphi \text{ vanishes at } \infty. \end{bmatrix}$$

Here, $\widetilde{m}$ is the zero extension of m on the whole space defined by

$$\widetilde{m}(x) = \begin{cases} m(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Eventually, the external energy (or Zeeman energy) is given by

$$E_{\mathrm{ext}}(m) = -2 \int_{\Omega} H_{\mathrm{ext}} \cdot m \,\mathrm{d}x. \tag{6}$$

Then, the equilibrium states of a micromagnetic system are the minimizers of the energy $E(m)$ under the non-convex constraint that the magnitude of the magnetization is equal to one almost everywhere in Ω.

This problem has been the subject of many recent mathematical papers, both analytical and numerical (see, for a non-exhaustive list of examples, [3,7,11,12,15]).

Here, we focus on the case where $\Omega = \omega \times]0, \eta[$ is a vertical cylinder and where ω is a bounded subset of $\mathbb{R}^2$ with analytic boundary.

The derivation of reduced models in which the thickness η vanishes has been the aim of some mathematical papers in the last decade [2,6,8,20]. These models are usually analyzed by means of Γ-convergence from the original problem. For instance, it has been proved by G. Gioia, R.D. James [14] and G. Carbou [6] that in the case where A, and H_{ext} are kept constant, the unique equilibrium state of the limiting model as η tends to 0 is a constant horizontal state which only depends on the shape of the boundary of ω. Unfortunately, this model is not able to reproduce the non-uniform states observed experimentally in thin films [16]. Indeed, those states consist in domains in which the magnetization is smooth separated by smooth discontinuity lines. Therefore, in order to model this type of behavior, A. DeSimone, R. Kohn, S. Müller and F. Otto have written a series of papers [9,11–13] deriving the model obtained when A and H_{ext} vanish together with η at a proper scale. This model is a rigorous derivation of the complete three dimensional micromagnetic model, but its mathematical analysis faces several difficulties. In particular, the local constraint $|m| = 1$ a.e. in the magnetic domain is lost during the converging process, and due to the weak convergence of the magnetization, is replaced by the convexified one $|m| \leqslant 1$ a.e. at the limit (the strong L^p convergence of the magnetization is still an open problem for this model). As a consequence, it still does not fully explain the pictures observed experimentally which are described by the so-called van den Berg construction [22]: in fact, the theory given in [9,11] is a first order analysis of the asymptotic model (zero order in the exchange constant ε). The following order in the energy is given by the behavior of the magnetization inside the walls. This has been studied by T. Rivière and S. Serfaty [19,20] and F. Alouges, T. Rivière and S. Serfaty [2], on less physical models corresponding to $\eta = \infty$ and where the magnetization does not depend on the vertical coordinate. In [19,20] the magnetization is, moreover, constrained to be in the horizontal plane, which avoids vortices. On the other hand, in [2], the vertical component of the magnetization is only penalized and so, one can observe vortices (Bloch lines).

In [1], we consider the domain to be cylindrical and fixed. We also force the configurations to be vertically invariant. This permits us to make an analysis similar to those of [9,11] at first order and, moreover, to go one step further in the spirit of [2,19,20]. We also think strongly that this model could be of interest for the thick films physical situation, as suggested by the numerical computations given in [18].

2. The main results

We consider the situation where the thickness η is kept fixed and the magnetization is constrained to be vertically invariant in Ω. Calling

$$M^1(\Omega) = \left\{ m \in H^1(\Omega; \mathbb{S}^2) \text{ such that } \frac{\partial m}{\partial z} = 0 \right\},$$

we are interested in the limit, as ε vanishes, of the following minimization problem (P_ε):

$$(P_\varepsilon) \qquad \min_{m \in M^1(\Omega)} \big(E_\varepsilon(m)\big).$$

where for all $\varepsilon \geqslant 0$, E_ε is the micromagnetic energy with exchange parameter ε^2

$$E_\varepsilon(m) = \varepsilon^2 \int_\Omega |\nabla m|^2 + \int_{\mathbb{R}^3} \big|H_\mathrm{d}(m)\big|^2 - 2 \int_\Omega H_\mathrm{ext} \cdot m,$$

and $H_\mathrm{ext} = (H_{\mathrm{ext},1}, H_{\mathrm{ext},2}, 0)$ is a (constant) horizontal vector-field.

In the remainder of the Note, we also call

$$K^0(\Omega) = \left\{ m = (m_1, m_2, 0) \in L^2(\Omega) \text{ such that } \big|m(x)\big| \leqslant 1 \text{ a.e. } x \in \Omega, \text{ and such that } \frac{\partial m}{\partial z} = 0 \right\},$$

and

$$M^0(\Omega) = \big\{ m \in K^0 \text{ such that } \big|m(x)\big| = 1 \text{ a.e. in } \Omega \big\}.$$

Theorem 2.1. *There exists $H_c > 0$ such that if $|H_\mathrm{ext}| \leqslant H_\mathrm{c}$ and $(m_\varepsilon)_\varepsilon \subset M^1(\Omega)$ is a family of minimizers of E_ε in $M^1(\Omega)$, there exists $m_0 \in M^0(\Omega)$ which is a minimizer of E_0 on $K^0(\Omega)$, such that* (*up to the extraction of a subsequence*)

$$m_\varepsilon \longrightarrow m_0 \quad \textit{strongly in} \bigcap_{1 \leqslant p < \infty} L^p(\Omega),$$

and

$$\lim_{\varepsilon \to 0} E_\varepsilon(m_\varepsilon) = E_0(m_0).$$

As for Γ-convergence type results [4], the proof follows several steps that are described hereafter. We first prove a lower semi-continuity property:

Theorem 2.2 *(Lower semi-continuity). Let $(m_\varepsilon)_{\varepsilon > 0} \subset M^1(\Omega)$, there exist $m_0 \in K^0(\Omega)$ such that* (*up to a subsequence*)

$$\lim_{\varepsilon \to 0} m_\varepsilon = m_0 \quad \textit{weakly in } L^p(\Omega),$$

and

$$\liminf_{\varepsilon \to 0} E_\varepsilon(m_\varepsilon) \geqslant E_0(m_0).$$

Next, we have to proceed to the construction of the so-called 'recovery sequence' which is much more difficult. For that construction we first show (Theorem 2.3) that for small enough external fields, there exists a non saturated minimizer m_0 which is analytic on Ω. Then, considering solutions of the form $m_0' = m_0 + \nabla^{\perp}\phi$, we show (Theorem 2.4) that there exists a subanalytic minimizer of E_0 in $M^0(\Omega)$. For that, we adapt the proof of a recent result by E. Trélat [21] on subanalytic solutions to Hamilton–Jacobi equations. It turns out that it is much more delicate in our situation since the analytic minimizer m_0 of Theorem 2.3 is not analytic up to the boundary of Ω but only in its interior.

Theorem 2.3. *There exists $H_c > 0$ such that if $|H_{\mathrm{ext}}| \leqslant H_c$, there exists $m_0 \in K_0(\Omega)$ a minimizer of E_0 in $K^0(\Omega)$ which is analytic in Ω. Moreover, m_0 satisfies $\|m_0\|_{L^\infty} < 1$.*

Theorem 2.4 *(Existence of subanalytic minimizers of E_0 in M^0). Under the hypotheses of Theorem 2.3, there exists $\overline{m}_0 \in M^0(\Omega)$ which minimizes E_0 in $K^0(\Omega)$ and is subanalytic in Ω.*

The regularity result given by Theorem 2.4 permits the construction of sequences of $M^1(\Omega)$ which converge to minimizers of E_0 at a given rate.

Theorem 2.5 *(Energy estimation). Let $m_0 \in M^0(\Omega)$ a subanalytic minimizer of E_0 in $K^0(\Omega)$. There exists $H_c > 0$ such that if $|H_{\mathrm{ext}}| \leqslant H_c$, there exists $C_1 > 0$ and a sequence $(m_\varepsilon)_\varepsilon \subset M^1(\Omega)$ such that*

$$m_\varepsilon \longrightarrow m_0 \quad \text{strongly in} \bigcap_{1\leqslant p<\infty} L^p(\Omega)$$

and

$$0 \leqslant \limsup_{\varepsilon\to 0} \frac{E_\varepsilon(m_\varepsilon) - E_0(m_0)}{\varepsilon^{4/3}} \leqslant C_1 \int_\Sigma \big|[m_0]\big|^2 \sigma(x) \tag{7}$$

where Σ is the singular set of m_0 and $[m_0]$ stands for the jump of m_0 through Σ.

In fact, it turns out that the energy estimation (7) is sufficient to get the strong L^p convergence, since applying the averaging lemmas methods developed in [17], one can prove the following theorem:

Theorem 2.6 *(Strong convergence). Let $m_0 \in K^0$ a minimizer of E_0, if $(m_\varepsilon)_{\varepsilon>0} \subset M^1(\Omega)$ is such that*

$$0 \leqslant E_\varepsilon(m_\varepsilon) - E_0(m_0) \leqslant C\varepsilon^{4/3},$$

for some constant $C > 0$, then there exists $m_0' \in M^0(\Omega)$ such that $E_0(m_0) = E_0(m_0')$ (m_0' is also a minimizer of E_0), and up to the extraction of a subsequence,

$$m_\varepsilon \longrightarrow m_0' \quad \text{strongly in} \bigcap_{1\leqslant p<\infty} L^p(\Omega).$$

That Theorem 2.1 follows from Theorems 2.2, 2.5, and 2.6 is standard. The complete proof of all these results can be found in [1].

3. Conclusion

In this Note we gave a result of strong convergence of minimizers of a model for ferromagnetic films. The proof, based upon the use of Γ-convergence techniques, pseudo-differential calculus, subanalytic solutions for Hamilton–Jacobi equations and averaging lemmas, gives a description of limiting minimizers which are close to the ones seen in experimental situations (see [22]).

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