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Mathematical Analysis

Scaled asymptotics for q-orthogonal polynomials

Mourad E.H. Ismail [a], Ruiming Zhang [b]

[a] *Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA*
[b] *School of Mathematics, Guangxi Normal University, Guilin City, Guangxi 541004, PR China*

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Abstract

We summarize results of a forthcoming paper on Plancherel–Rotach asymptotic expansions for the q^{-1}-Hermite, q-Laguerre and Stieltjes–Wigert polynomials. The asymptotics in the bulk exhibit chaotic behavior when a certain variable is irrational. In the rational case the main terms in the asymptotic expansion involve theta functions. ***To cite this article: M.E.H. Ismail, R. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Asymptotiques dilatés pour q-polynômes orthogonaux. Nous résumons des résultats d'un article à venir sur les expansions asymptotiques de Plancherel–Rotach pour les polynômes q^{-1}-Hermite, q-Laguerre et de Stieltjes–Wigert. Le comportement asymptotique est en général chaotique lorsqu'une certaine variable est irrationnelle. Dans le cas rationnel, les termes principaux de l'expansion asymptotique comportent des fonctions théta. ***Pour citer cet article : M.E.H. Ismail, R. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Les comportements asymptotiques de Plancherel–Rotach sont ceux d'une suite de polynômes orthogonaux sur une partie non bornée de $\mathbb{R}$, où x est mis à l'échelle par la plus grande racine $x_{n,1}$. Une premiere mise à l'échelle est pour les grandes valeurs, $x = x_{n,1}\cosh\theta$, for $\theta \geqslant \epsilon > 0$. Une autre est la mise à l'échelle du gros des valeurs, $x = x_{n,1}\cos\theta$, for $\pi - \epsilon \geqslant \theta \geqslant \epsilon > 0$. Et la troisième est pour les valeurs limites, lorsque x n'est pas borné au delà de $x_{n,1}$, et alors on pose $x = x_{n,1} - ctn^{-\alpha}$, où c est une constante, t un paramètre, et α un exposant bien choisi. Les comportements asymptotiques polynomiaux de Hermitte et Laguerre sont traités dans [17] et [9]. Le terme principal dans l'expansion asymptotique au voisinage de la plus grande racine comporte la fontction de Airy. On trouve des résultats semblables pour des polynômes orthogonaux selon $\exp(-\pi(x))$, où $\pi(x)$ est un polynôme, dans [14] et [5].

Nous prenons les notations usuelles pour les q-séries, comme dans [6] et [9]. Les polynômes q-orthogonaux traités ici sont les polynômes q^{-1}-Hermite $\{h_n(x|q)\}$, les polynômes de Stieltjes–Wigert $\{S_n(x;q)\}$, et les polynômes

E-mail addresses: ismail@math.ucf.edu (M.E.H. Ismail), ruimingzhang@yahoo.com (R. Zhang).

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q-Laguerre $\{L_n^\alpha(x;q)\}$, [9]. Dans notre étude, le rôle de la fonction de Airy est joué par la fonction de Ramanujan, introduite dans les travaux de Ramanujan [15].

La mise à l'échelle que nous avons prise est très différente de $x = x_{n,1} - ctn^{-\alpha}$, qui conduit généralement à des comportements asymtotiques dans le cas limite comprenant la fonction de Airy. Notre mise à l'échelle est logarithmique, avec $x = (q^{-nt}u - q^{nt}/u)/2$ ou $q^{-nt}u$. C'est un autre régime, qui conduit à la présence de la fonction A_q dans les comportements asymptotiques du cas limite. Ceux du cas général classique comprenent des fonctions trigonométriques. Les nôtres comprennent des fonctions théta et présentent un comportement chaotique lorsque t est irrationnel et un comportement périodique lorsque t est rationnel. Il s'agit d'un comportement nouveau, qui semble n'apparaître que pour les polynômes q-orthogonaux.

Nos résultats sont les Théorèmes 2.1, 2.2 et 2.3 énoncés dans la Section 2.

1. Introduction

The Plancherel–Rotach asymptotics refers to the asymptotics of a sequence of orthogonal polynomials on an unbounded subset of $\mathbb{R}$ where x is scaled by the largest zero $x_{n,1}$. One scaling is at the tail, $x = x_{n,1}\cosh\theta$, for $M \geqslant \theta \geqslant \epsilon > 0$. Another is the bulk scaling $x = x_{n,1}\cos\theta$, for $\pi - \epsilon \geqslant \theta \geqslant \epsilon > 0$. A third is the soft edge asymptotics when x is bounded away from $x_{n,1}$, so we let $x = x_{n,1} - ctn^{-\alpha}$, where c is a constant, t is a parameter and the exponent $\alpha > 0$ is chosen judiciously. The Hermite and Laguerre polynomial asymptotics are in [17] and [9]. The main term in the asymptotic expansion around the largest zero contains the Airy function. Similar results hold for polynomials orthogonal with respect to $\exp(-\pi(x))$, $\pi(x)$ is a polynomial, [14,5], and this is manifested in the universality principle, [4] and [13].

We follow the standard notation for q-series as in [6,9].

The q-orthogonal polynomials we treat the q^{-1}-Hermite polynomials $\{h_n(x|q)\}$ of [10], the Stieltjes–Wigert polynomials $\{S_n(x;q)\}$, and the q-Laguerre polynomials $\{L_n^\alpha(x;q)\}$, [9],

$$h_n(\sinh\xi|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)}(-1)^k \mathrm{e}^{(n-2k)\xi}, \qquad S_n(x;q) = \sum_{k=0}^{n} \frac{q^{k^2}(-x)^k}{(q;q)_k(q;q)_{n-k}}, \tag{1}$$

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{q^{k^2+\alpha k}}{(q^{\alpha+1};q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x)^k. \tag{2}$$

These polynomials satisfy three term recurrence relations whose coefficients grow exponentially. This makes the corresponding moment problem indeterminate, that is the measure with respect to which the polynomials are orthogonal is not unique. We believe these three examples represent the typical cases of polynomials with exponentially growing recursion coefficients. The zeros of the Stieltjes–Wigert and q-Laguerre polynomials are positive while the q^{-1}-Hermite polynomials have positive and negative zeros.

In our analysis the role of the Airy function is played by the Ramanujan function,

$$A_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}(-z)^k, \tag{3}$$

which appeared in Ramanujan's work [15]. The Rogers–Ramanujan identities give infinite product representations for $A(-1)$ and $A(-q)$, see for example [1]. Ramanujan [15] conjectured certain patterns of the zeros of A_q and some these patterns have been recently proved in [2,3,11]. See also [7]. Another function which will appear in our analysis is the theta function

$$\Theta(z|q) := \sum_{n=-\infty}^{\infty} q^{n^2} z^n = \left(q^2, -qz, -q/z; q^2\right)_\infty, \tag{4}$$

and the infinite product representation is the Jacobi triple product identity.

The scaling used in our approach is very different from $x = x_{n,1} - ctn^{-\alpha}$ and the explanation may be related to the equilibrium measure [16] which is a measure for the density of zeros. The scaling $x = x_{n,1} - ctn^{-\alpha}$ usually leads to edge asymptotics involving the Airy function. We believe this will happen when the equilibrium measure $\sigma(x)\,\mathrm{d}x$

behaves like $\sqrt{x-a}$ and $\sqrt{b-x}$ as $x \to a^+$ or $x \to b^-$, when σ is supported on a single interval $[a,b]$. Our scaling is a logarithmic scaling, namely $x = (q^{-nt}u - q^{nt}/u)/2$ or $q^{-nt}u$. The reason is that the zeros of the polynomials in our study are well-separated, that for this scaling is the ratio of two consecutive zeros is at least q^{-c} for some fixed $c > 0$. This completely different behavior results in the presence of the A_q function in the edge asymptotics. The classical bulk asymptotics involve trigonometric functions. Our bulk asymptotics are dramatically different because they involve theta functions and exhibit a chaotic behavior when t is irrational and periodic behavior when t is rational. This is a completely new phenomenon which seems to appear only in q-orthogonal polynomials.

In the following we assume that u is a complex number, $u \neq 0$. We also assume that $\chi(n) = \chi_1(n)$ is the principal character modulo 2, that is

$$\chi(n) = 1 \quad \text{if } 2 \nmid n \quad \text{and} \quad \chi(n) = 0 \quad \text{if } 2 \mid n.$$

Consider the set $\mathcal{R} = \{\{n\theta\}\colon\ n \in \mathbb{N}\}$, and $\{x\}$ is the fractional part of x. If $\theta = p/r \in \mathbb{Q}$ with $r > 0, (p,r) = 1$ then $\mathcal{R} = \{0, 1/r, \dots, (r-1)/r\}$. Thus for any $\lambda \in \mathcal{R}$, there is a subsequence $\{n_j\}_{j \geqslant 1}$ of $\mathbb{N}$ that $\{n_j\theta\} = \lambda$. If θ is irrational, then the set $\mathcal{R}$ is uniformly distributed in the interval $(0,1)$.

2. Main results

We shall always assume $0 < q < 1, t > 0$, and $u \neq 0$. The proofs of the results stated in this section are proved in [12] using the explicit forms (1), (2) and a discrete analogue of the Laplace asymptotic method for integrals.

Theorem 2.1. *Let* $\sinh \xi_n := [q^{-nt}u - q^{nt}/u]/2$, *and assume that* $t > 0$. *For* $t \geqslant \frac{1}{2}$ *we have*

$$h_n(\sinh \xi_n|q) = u^n q^{-n^2 t}\bigl\{A_q\bigl(u^{-2}q^{n(2t-1)}\bigr) + r_n\bigr\}, \tag{5}$$

and r_n *is majorized by*

$$|r_n| \leqslant \frac{4(-q^3;q)_\infty A_q(-|u|^{-2}q^{n(2t-1)})}{(q;q)_\infty^2}\bigl(q^{n/2} + q^{(4t-1)n^2/4}|u|^{-2\lfloor n/2\rfloor - 2}\bigr). \tag{6}$$

If $t \in \mathbb{Q}$, *and* $t < 1/2$ *then the set* $\mathbb{S}_H := \{\{n(1-2t)\}\colon\ n \in \mathbb{N}\}$ *has a finite number of points in* $[0,1)$. *Let* $\lambda \in \mathbb{S}_H$, *then there are infinitely many positive integers* n *such that*

$$n(1-2t) = m + \lambda \tag{7}$$

where m *is a positive integer dependent on* n *and* $0 \leqslant \lambda < 1$. *For such* n *and as* $n \to \infty$ *we have*

$$\frac{(q;q)_\infty(-1)^{\lfloor m/2\rfloor}h_n(\sinh \xi_n|q)}{u^{n-2\lfloor m/2\rfloor}q^{-n^2t-\lfloor m/2\rfloor(\lfloor (m+1)/2)\rfloor+\lambda)}} = \Theta\bigl(-u^2q^{\chi(m)+\lambda}|q\bigr) + r(n), \tag{8}$$

with the error estimate

$$\begin{aligned}\bigl|r(n)\bigr| \leqslant{}& \frac{16(-q^3;q)_\infty^2}{(q;q)_\infty^3}\Theta\bigl(|u|^2q^{\chi(m)+\lambda}|q\bigr)\biggl[q^{(1-2t)n/4} \\ &+ |u|^{2+2\lfloor(1-2t)n/4\rfloor}q^{(1-2t)^2n^2/16} + \frac{q^{(1+2t)^2n^2/32}}{|u|^{2\lfloor(1+2t)n/4\rfloor+2}}\biggr].\end{aligned} \tag{9}$$

When $1-2t$ *is a positive irrational number, then given an arbitrary real number* $\beta \in [0,1)$, *there are infinitely many positive integers* n *the and corresponding integers* m *such that*

$$n(1-2t) = m + \beta + \gamma_n, \quad \text{with } |\gamma_n| \leqslant \frac{3}{n},\ \text{so } m = \lfloor(1-2t)n\rfloor \text{ for large enough } n. \tag{10}$$

In this case for sufficiently large n *satisfying* (9) *we have the asymptotic result*

$$h_n(\sinh \xi_n|q) = \frac{\{\Theta(-u^2q^{\chi(m)+\beta}|q) + \mathcal{O}(\frac{\log n}{n})\}}{(-1)^{\lfloor m/2\rfloor}u^{2\lfloor m/2\rfloor - n}q^{n^2t+\lfloor m/2\rfloor((1-2t)n-\lfloor m/2\rfloor)}(q;q)_\infty}. \tag{11}$$

The scaling for the Stieltjes–Wigert and q-Laguerre polynomials is

$$x_n(t,u) = q^{-nt}u. \tag{12}$$

Theorem 2.2. *For $t \geqslant 2$ the Stieltjes–Wigert polynomials satisfy*

$$S\big(x_n(t,u);q\big) = \frac{(-u)^n q^{n^2(1-t)}}{(q;q)_\infty}\big\{A_q\big(q^{n(t-2)}u^{-1}\big) + r(n)\big\}, \tag{13}$$

together with

$$\big|r(n)\big| \leqslant \frac{2(-q^3;q)_\infty}{(q;q)_\infty} A_q\big(-q^{n(t-2)}/|u|\big)\big\{q^{n/2} + q^{n^2(2t-3)/4}|u|^{-1-\lfloor n/2\rfloor}\big\}. \tag{14}$$

When $2-t$ is a positive rational number let $\mathbb{S} = \{\{n(2-t)\}\colon\ n \in \mathbb{N}\}$. The elements of $\mathbb{S}$ are positive and belong to the interval $[0,1)$. For any $\lambda \in \mathbb{S}$ there are infinitely many positive integers n such that

$$n(2-t) = m + \lambda, \quad \text{so that } m = \big\lfloor (2-t)n\big\rfloor, \text{ for sufficiently large } n. \tag{15}$$

With this notation we have the asymptotic result

$$S\big(x_n(t,u);q\big) = \frac{q^{n^2(1-t)-\lfloor m/2\rfloor(\lfloor m/2\rfloor+\chi(m)+\lambda)}}{(-u)^{\lfloor m/2\rfloor-n}(q;q)_\infty^2}\big\{\Theta\big(-q^{\lambda+\chi(m)}u|q\big) + r(n)\big\}, \tag{16}$$

and the error term satisfies:

$$\big|r(n)\big| \leqslant \frac{8(-q^3;q)_\infty^2}{(q;q)_\infty^2}\Theta\big(|u|q^{\lambda+\chi(m)}|q\big)\big\{q^{\min\{(2-t)n/4,tn/4\}} + |u|^{\lfloor m/4\rfloor+1}q^{m^2/16} + |u|^{-1-\lfloor tn/4\rfloor}q^{t^2n^2/32}\big\}. \tag{17}$$

When $2-t$ is a positive irrational number, and given an arbitrary real number $\beta \in [0,1)$, there are infinitely many positive integers n and their associated m such that

$$n(2-t) = m + \beta + \gamma_n, \quad \text{with } |\gamma_n| \leqslant 3/n, \text{ and } m = \big\lfloor (2-t)n\big\rfloor. \tag{18}$$

For $\beta \in [0,1)$, and n as above, the following asymptotic relationship holds

$$S\big(x_n(t,u);q\big) = \frac{(-u)^{n-\lfloor m/2\rfloor}\{\Theta(-uq^{\chi(m)+\beta}|q) + \mathcal{O}(\frac{\log n}{n})\}}{(q;q)_\infty^2 q^{n^2(t-1)+(n(2-t)-\lfloor m/2\rfloor)\lfloor m/2\rfloor}}, \tag{19}$$

for sufficiently large n, satisfying (17).

Theorem 2.3. *When $t \geqslant 2$ the q-Laguerre polynomials satisfy*

$$L_n^{(\alpha)}\big(x_n(t,u);q\big) = \frac{(-u)^n q^{n^2(1-t)+\alpha n}}{(q;q)_\infty}\big\{A_q\big(q^{n(t-2)-\alpha}u^{-1}\big) + r(n)\big\}, \tag{20}$$

where

$$\big|r(n)\big| \leqslant \frac{8(-q^2;q)_\infty^2 A_q(-|u|^{-1}q^{(t-2)n-\alpha})}{(q;q)_\infty^4 q^{\alpha+1}} \times \big(q^{n/2} + q^{n^2(2t-3)/4-\alpha n/2}|u|^{-1-\lfloor n/2\rfloor}\big), \quad \text{for } n \geqslant \alpha+1. \tag{21}$$

When t is rational and $t<2$ we define the set $\mathbb{S}$ as in Theorem 2.2 *and proceed as in* (14). *For $n \geqslant 8[(2+|\alpha|)/t + |\alpha|/(2-t)]$ we have*

$$\frac{L_n^{(\alpha)}(x_n(t,u);q)(q;q)_\infty^2(-u)^{\lfloor m/2\rfloor-n}}{q^{n^2(1-t)+\alpha n-\lfloor m/2\rfloor(\lfloor m/2\rfloor+\chi(m)+\lambda+\alpha)}} = \Theta\big(-uq^{\chi(m)+\lambda+\alpha}|q\big) + r(n), \tag{22}$$

and

$$\big|r(n)\big| \leqslant \frac{32(-q^2;q)^3\Theta(|u|q^{\chi(m)+\lambda+\alpha}|q)}{(q;q)_\infty^5} \times \big\{q^{\min(nt,n(2-t))/4} + |u|^{-1-\lfloor nt/4\rfloor}q^{n^2t^2/32} + |u|^{\lfloor (2-t)n/4\rfloor+1}q^{(2-t)^2n^2/32}\big\}. \tag{23}$$

When $2 - t > 0$ is irrational, and given $\beta \in [0, 1)$, there are infinitely many positive integers n and their associated m such that (17) hold. For any given $\beta \in [0, 1)$, the q-Laguerre polynomials have the limiting behavior

$$L_n^{(\alpha)}\big(x_n(t,u);q\big) = \frac{(-u)^{n-\lfloor m/2\rfloor}\{\Theta(-uq^{\alpha+\beta+\chi(m)}|q) + \mathcal{O}(\frac{\log n}{n})\}}{(q;q)_\infty^2 q^{n^2(t-1)-\alpha n+\lfloor m/2\rfloor(\alpha+(2-t)n-\lfloor m/2\rfloor)}}, \tag{24}$$

for sufficiently large n.

The cases $t = 1/2$ of Theorem 2.1 and $t = 2$ of Theorems 2.2 and 2.3 are in [8].

The asymptotic formulas in Theorems 2.1–2.3 have more symmetry when written in terms of orthonormal functions. We will record only the q^{-1}-Hermite polynomial case. A weight function is, [9],

$$w_H(x|q) = q^{1/8}\sqrt{\frac{-2}{\pi \ln q}}\exp\left(\frac{2}{\log q}\Big[\log(x+\sqrt{x^2+1})\Big]^2\right). \tag{25}$$

Eqs. (5), (8) and (11) become

$$\sqrt{w_H(\sinh\xi_n|q)}\tilde{h}_n(\sinh\xi_n|q) = \sqrt{\frac{w_H(\sinh u|q)}{(q;q)_n}}u^{n(1-2t)}q^{n^2(t-1/2)^2+n/4} \times \big\{A_q\big(u^{-2}q^{n(2t-1)}\big) + r_n\big\}, \tag{26}$$

$$\sqrt{\frac{w_H(\sinh\xi_n|q)(q;q)_n}{w_H(\sinh u|q)}}\tilde{h}_n(\sinh\xi_n|q) = \frac{(-1)^{\lfloor m/2\rfloor}}{(q;q)_\infty}u^{\lambda+\chi(m)}q^{(\{m/2\}+\lambda/2)^2+n/4} \times \big[\Theta\big(-u^2q^{\chi(m)+\lambda}|q\big) + r(n)\big], \tag{27}$$

and

$$\sqrt{\frac{w_H(\sinh\xi_n|q)\,(q;q)_n}{w_H(\sinh u|q)}}\,\tilde{h}_n(\sinh\xi_n|q) = \frac{(-1)^{\lfloor m/2\rfloor}}{(q;q)_\infty}\,u^{\beta+\gamma_n+\chi(m)}\,q^{(\{m/2\}+(\beta+\gamma_n)/2)^2+n/4} \times\left\{\Theta\big(-u^2q^{\chi(m)+\beta}|q\big) + \mathcal{O}\left(\frac{\log n}{n}\right)\right\}, \tag{28}$$

respectively.

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