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C. R. Acad. Sci. Paris, Ser. I 344 (2007) 135–140
COMPTES RENDUS MATHEMATIQUE
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Mathematical Problems in Mechanics

A strictly hyperbolic equilibrium phase transition model

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Received 17 October 2006; accepted 7 November 2006

Available online 11 December 2006

Presented by Philippe G. Ciarlet

Abstract

This Note is concerned with the strict hyperbolicity of the compressible Euler equations equipped with an equation of state that describes the thermodynamical equilibrium between the liquid phase and the vapor phase of a fluid. The proof is valid for a very wide class of fluids. The argument only relies on smoothness assumptions and on the classical thermodynamical stability assumptions, that requires a definite negative Hessian matrix for each phase entropy as a function of the specific volume and internal energy. ***To cite this article: G. Allaire et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Un modèle strictement hyperbolique de changement de phase à l'équilibre. Cette Note a pour but de démontrer la stricte hyperbolicité des équations d'Euler de la mécanique des fluides compressible lorsque le système est fermé par une équation d'état qui décrit l'équilibre thermodynamique d'un fluide entre sa phase liquide et sa phase vapeur. La preuve que nous proposons est valable pour une large classe de fluide. En effet, outre une hypothèse de régularité, les seules hypothèses nécessaires sont celles qui qualifient classiquement la stabilité d'un corps pur homogène : chaque phase doit être munie d'une entropie dont la matrice hessienne est définie négative relativement aux variables de volume et d'énergie interne spécifiques. ***Pour citer cet article : G. Allaire et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Nous proposons dans cette Note de démontrer le caractère strictement hyperbolique des équations d'Euler fermées par une loi d'état décrivant l'équilibre liquide–vapeur à saturation. Ce travail est motivé par l'étude du caractère bien-posé des systèmes utilisés pour la description des phénomènes de changement de phase liquide–vapeur [1,2,5,10–12,14].

Le système diphasique que nous considérons est constitué de deux phases $\alpha = 1, 2$ représentant respectivement la vapeur et le liquide. Chaque phase est supposée munie d'une loi d'état $s_\alpha \colon \boldsymbol{w}_\alpha = (\tau_\alpha, \varepsilon_\alpha) \mapsto s_\alpha$ où τ_α, ε_α, s_α sont respectivement le volume, l'énergie interne et l'entropie spécifiques de la phase $\alpha = 1, 2$. Nous supposerons que les

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.11.008

hypothèses classiques de stabilité d'un corps pur données par les inégalités (2) sont vérifiées pour s_α, de même que la positivité des variables d'états $\tau_\alpha \geqslant 0$ et $\varepsilon_\alpha \geqslant 0$, $\alpha = 1, 2$. Nous nous donnons la loi d'état de notre système diphasique par la donnée d'une fonction entropie spécifique $(\boldsymbol{w}_1, \boldsymbol{w}_2, y) \mapsto \sigma$ définie classiquement (voir [4]) par la relation (3), où y est la fraction de masse de la phase 1. Enfin, si l'on note $\boldsymbol{w} = (\tau, \varepsilon)$ le volume et l'énergie interne spécifiques du système diphasique, alors par additivité des volumes et des énergies, pour un état $\boldsymbol{w} = (\tau, \varepsilon) \in \mathcal{C} = \{\tau \geqslant 0, \varepsilon \geqslant 0\}$, on a $(\boldsymbol{w}_1, \boldsymbol{w}_2, y) \in \mathcal{Q} = \{0 \leqslant \tau_\alpha \leqslant \tau,\ 0 \leqslant \varepsilon_\alpha \leqslant \varepsilon,\ 0 \leqslant y \leqslant 1 \mid \boldsymbol{w} = y\boldsymbol{w}_1 + (1-y)\boldsymbol{w}_2\}$.

Si l'on suppose qu'à chaque instant le système atteint instantanément l'équilibre entre le liquide et la vapeur, alors l'application du second principe de la thermodynamique nous indique que pour une valeur $\boldsymbol{w} = (\tau, \varepsilon) \in \mathcal{C}$ donnée, la composition $(\boldsymbol{w}_1^*, \boldsymbol{w}_2^*, y^*)$ du système sera telle que l'entropie $\sigma(\boldsymbol{w}_1, \boldsymbol{w}_2, y)$ est maximale. Suivant [2], nous traduisons cette hypothèse en dotant le système diphasique d'une nouvelle fonction entropie $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ définie par la relation (4) qui réalise la maximisation de σ à $\boldsymbol{w}$ donné. La stricte concavité de $(\boldsymbol{w}_1, \boldsymbol{w}_2, y) \mapsto \sigma$ et un argument basé sur l'inf-convolution de fonctions convexes nous assure que la définition (4) est consistante et que $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ est concave [2,10,11].

Il est bien connu que la stricte positivité de la vitesse du son, définie en (1), assure la stricte hyperbolicité du système des équations d'Euler : nous démontrons ici qu'effectivement c est strictement positive. Notons que la concavité de $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ implique que $c^2 \geqslant 0$ et donc que le système est toujours à caractéristiques réelles comme montré dans [1,2] et [14] où sont présentés des calculs pour le cas où les fluides $\alpha = 1, 2$ sont des stiffened gaz. Il est néanmoins nécessaire de poursuivre l'analyse car la concavité non-stricte de $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ ne permet pas de conclure. En effet, on ne peut a priori écarter des cas où la vitesse du son s'annulerait de la même manière que pour les systèmes étudiés par [5–8] menant à une jacobienne associée au flux non-diagonalisable bien qu'à spectre réel.

Afin de mener à bien cette analyse nous commençons par rappeler trois propositions. Tout d'abord la Proposition 2.1, qui est un résultat classique de thermodynamique [2,4] (utilisé à des fins de simulation numérique par [5–8,12]), décrit les états d'équilibre du système en optimisant la composition $(\boldsymbol{w}_1, \boldsymbol{w}_2, y)$. Ensuite la Proposition 2.2 fournit une interprétation de cet équilibre via une construction géométrique de l'entropie d'équilibre s^{eq} basée sur l'analyse des bitangentes des graphes associées aux fonctions $\boldsymbol{w}_\alpha \mapsto s_\alpha$. On définit grâce à cette proposition les états du système pour lesquels il y a coexistence de vapeur et de liquide à l'équilibre (états à saturation). Enfin, la Proposition 2.3 établit l'équivalence de l'épigraphe de $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ avec l'enveloppe concave de l'ensemble $\{(\boldsymbol{w}, s) \mid s \leqslant \max[s_1(\boldsymbol{w}), s_2(\boldsymbol{w})]\}$. Cette dernière proposition établit que le graphe de $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ dans les zones d'états à saturation est constitué de segments de l'espace (τ, ε, s) sur lesquels la pression, la température et le potentiel chimique sont constants. Ceci prouve que $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ n'est effectivement pas strictement concave.

Grâce à une hypothèse de régularité par morceaux (voir [3]) de s^{eq}, nous démontrons ensuite le Théorème 2.4 qui examine la matrice hessienne de $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ dans les zones d'états à saturation, ainsi que la valeur de la pression et de la température du système. Finalement, en réinjectant ces résultats dans la définition de la vitesse du son (1), on conclut au Théorème 3.1 que la vitesse du son ne peut pas s'annuler.

1. Introduction

The goal of the present Note is to prove that the compressible Euler system equipped with an equation of state (EOS) that describes the liquid–vapor thermodynamical equilibrium is strictly hyperbolic. This result is valid when the EOS for the liquid and the vapor phases satisfy the classical thermodynamical stability hypotheses for a pure homogeneous fluid. This work has been motivated by the well-posedness study of two-phase systems used to model liquid–vapor phase change phenomena.

2. Equilibrium phase transition model

Before describing the two-phase system we shall examine here, let us first briefly recall a few statements of classical thermodynamics (see [4]). We consider transformations of a pure fluid whose state is described by the variable $\boldsymbol{w}(\tau, \varepsilon)$ lying in the cone $\mathcal{C} = \{\tau \geqslant 0, \varepsilon \geqslant 0\}$, where τ and ε are respectively the specific volume and the specific internal energy. We denote by $s : \boldsymbol{w} \mapsto s(\boldsymbol{w})$ the specific entropy of the system. In the following we suppose that s is a $\mathscr{C}^2$ regular function and we shall use the notations

$$s_\varepsilon = \left.\frac{\partial s}{\partial \varepsilon}\right|_\tau, \quad s_\tau = \left.\frac{\partial s}{\partial \tau}\right|_\varepsilon, \quad s_{\varepsilon\varepsilon} = \left.\frac{\partial^2 s}{\partial \varepsilon^2}\right|_\tau, \quad s_{\tau\tau} = \left.\frac{\partial^2 s}{\partial \tau^2}\right|_\varepsilon, \quad s_{\tau\varepsilon} = \frac{\partial^2 s}{\partial \tau \partial \varepsilon}.$$

The temperature T, the pressure P, the free enthalpy g and the speed of sound c are defined by

$$(T,P)=(1/s_\varepsilon, s_\tau/s_\varepsilon),\qquad g=\varepsilon+P\tau-Ts,\qquad c^2=-\tau^2T\,[\,P,\quad -1\,]\begin{bmatrix} s_{\varepsilon\varepsilon} & s_{\tau\varepsilon}\\ s_{\tau\varepsilon} & s_{\tau\tau}\end{bmatrix}\begin{bmatrix} P\\ -1\end{bmatrix}. \tag{1}$$

Thermodynamics characterizes the derivatives of s: first, temperature and pressure strict positivity require s to be a strictly increasing function of ε and τ; secondly, a stability assumption is enforced by assuming a definite negative Hessian matrix for s. Therefore for all $\boldsymbol{w}\in\mathcal{C}$, we have

$$\text{(A) } s_\varepsilon>0,\ s_\tau>0,\qquad \text{(B) } s_{\varepsilon\varepsilon}s_{\tau\tau}>(s_{\tau\varepsilon})^2,\qquad \text{(C) } s_{\varepsilon\varepsilon}<0 \text{ (or equivalently } s_{\tau\tau}<0). \tag{2}$$

Let us note that the above relation implies that $\boldsymbol{w}\mapsto s$ is strictly concave. Finally, we also assume that the entropy admits an upper bound in the vicinity of the line $\tau=0$ and the line $\varepsilon=0$.

We now turn to the description of the two-phase system investigated in the present Note: the system consists of two fluids $\alpha=1,2$ that represent the vapor phase and the liquid phase. Each phase is equipped with a complete EOS $s_\alpha:\boldsymbol{w}_\alpha=(\tau_\alpha,\varepsilon_\alpha)\mapsto s_\alpha$, where τ_α, ε_α and s_α denote respectively the specific volume, internal energy and entropy of the phase $\alpha=1,2$. In the following we suppose that s_α is $\mathscr{C}^2$ and verifies the inequalities (2) for $\alpha=1,2$. The system composition is characterized by the mass fraction $y\in[0,1]$ of the phase 1. We note $\boldsymbol{w}=(\tau,\varepsilon)\in\mathcal{C}$ the fluid system global specific volume and specific internal energy. The additivity of the volume and the energy implies that

$$(\boldsymbol{w}_1,\boldsymbol{w}_2,y)\in\mathcal{Q}(\boldsymbol{w})=\big\{0\leqslant\tau_\alpha\leqslant\tau,\ 0\leqslant\varepsilon_\alpha\leqslant\varepsilon,\ 0\leqslant y\leqslant 1 \mid \boldsymbol{w}=y\boldsymbol{w}_1+(1-y)\boldsymbol{w}_2\big\}.$$

Following classical thermodynamics (see [4]) we define an entropy for a phase mixture:

$$\sigma(\boldsymbol{w}_1,\boldsymbol{w}_2,y)=ys_1(\boldsymbol{w}_1)+(1-y)s_2(\boldsymbol{w}_2)\quad \text{with } (\boldsymbol{w}_1,\boldsymbol{w}_2,y)\in\mathcal{C}\times\mathcal{C}\times[0,1]. \tag{3}$$

According to the second law of thermodynamics, for a given global state $\boldsymbol{w}=(\tau,\varepsilon)\in\mathcal{C}$ the equilibrium composition parameters $(\boldsymbol{w}_1^*,\boldsymbol{w}_2^*,y^*)$ are characterized as maximizer of $(\boldsymbol{w}_1,\boldsymbol{w}_2,y)\mapsto\sigma$. Following Ph. Helluy and T. Barberon in [2], if we consider that the fluid always instantaneously reaches equilibrium then we provide the system with a new entropy definition:

$$s^{\mathrm{eq}}(\boldsymbol{w})=\max_{(\boldsymbol{w}_1,\boldsymbol{w}_2,y)\in\mathcal{Q}(\boldsymbol{w})}\sigma(\boldsymbol{w}_1,\boldsymbol{w}_2,y). \tag{4}$$

Ph. Helluy and N. Seguin show in [11] that the optimization problem (4) is equivalent to perform an inf-convolution between two convex functions. This result ensures that $\boldsymbol{w}\mapsto s^{\mathrm{eq}}$ is always concave and therefore that the mixture speed of sound satisfies $c\geqslant 0$. Numerical computations of s^{eq} for the special cases of two stiffened gases are available in [1]. Under hypothesis (2), the maximization in (4) admits a solution which is given by the following result (see [2,10]):

Proposition 2.1 *(Extremum principle).* $(\boldsymbol{w}_1,\boldsymbol{w}_2,y)\mapsto\sigma$ *is a* $\mathscr{C}^2$ *concave function over* $\mathcal{C}\times\mathcal{C}\times[0,1]$ *and for a given* $\boldsymbol{w}\in\mathcal{C}$ *the equilibrium state* $(\boldsymbol{w}_1^*,\boldsymbol{w}_2^*,y^*)\in\mathcal{Q}(\boldsymbol{w})$ *is the solution of the optimization problem* (4) *if and only if*

$$\begin{cases} y^*=1,\\ s_1(\boldsymbol{w})>s_2(\boldsymbol{w})\end{cases}\quad \textit{or}\quad \begin{cases} y^*=0,\\ s_1(\boldsymbol{w})<s_2(\boldsymbol{w})\end{cases}\quad \textit{or}\quad (\mathrm{MZ})\begin{cases} 0<y^*<1,\\ \left(\dfrac{1}{T_1},\dfrac{P_1}{T_1},\dfrac{g_1}{T_1}\right)(\boldsymbol{w}_1^*)=\left(\dfrac{1}{T_2},\dfrac{P_2}{T_2},\dfrac{g_2}{T_2}\right)(\boldsymbol{w}_2^*).\end{cases}$$

We observe that in many cases the maximum is reached at a point on the boundary $y^*\in\{0,1\}$ of the set $\mathcal{Q}(\boldsymbol{w})$.

In the following we assume that:

$$0<y^*<1\quad\Rightarrow\quad\big\{\tau_1^*\neq\tau_2^*,\varepsilon_1^*\neq\varepsilon_2^*,s_1(\boldsymbol{w}_1^*)\neq s_2(\boldsymbol{w}_2^*)\big\}. \tag{5}$$

This is a property for a liquid–vapor phase transition as described, for example, in [4, p. 228].

The optimization procedure (4) also reads as a convexification of $\boldsymbol{w}\mapsto\max\{s_1(\boldsymbol{w}),s_2(\boldsymbol{w})\}$. This approach allows to define the saturation state for the fluid (see [4]) and has been exploited in [5–8,12] within a numerical simulation framework. Following [4–8,12] we recall below the geometrical result that characterizes the solution of (4).

Proposition 2.2 *(Bitangent plane). Let* $\mathcal{S}_\alpha$ *be the surface of* $\boldsymbol{w}\mapsto s_\alpha$ *in the* $(\boldsymbol{w},s)$ *space. Let* $\boldsymbol{w}\in\mathcal{C}$*, if* $(\boldsymbol{w}_1^*,\boldsymbol{w}_2^*,y^*)\in\mathcal{Q}(\boldsymbol{w})$ *maximizes* σ *and* $0<y^*<1$*, then there is an unique plane, called 'bitangent plane', tangent to* $\mathcal{S}_1$ *at the* *point* $(\boldsymbol{w}_1^*, s_1^*)$ *and to* $\mathcal{S}_2$ *at the point* $(\boldsymbol{w}_2^*, s_2^*)$*. For such* $(\boldsymbol{w}_1^*, \boldsymbol{w}_2^*, y^*)$*, the state* $\boldsymbol{w} = y^*\boldsymbol{w}_1^* + (1-y^*)\boldsymbol{w}_2^*$ *is called a 'saturated state'.*

Consequently solving (4) leads either to $y^* \in \{0,1\}$ or to find the bitangent plane to $\mathcal{S}_1$ and $\mathcal{S}_2$.

We now recall a classical result (see [4,12]) that connects the optimization problem (4) and the bitangent plane construction of Proposition 2.2.

Proposition 2.3 *(Concave hull). Let* $\mathcal{S}$ *be the graph of* s^{eq} *in the* $(\boldsymbol{w}, s)$ *space. The set* $\mathcal{S}$ *is the concave hull of the set* $\{(\boldsymbol{w}, s) \mid s \leqslant \max[s_1(\boldsymbol{w}), s_2(\boldsymbol{w})]\}$*. Therefore for every 'saturated state'* $\boldsymbol{w} \in \mathcal{C}$ *the surface* $\mathcal{S}$ *contains a segment passing through the point* $(\boldsymbol{w}, s^{\mathrm{eq}}(\boldsymbol{w}))$*. Along this segment* P*,* T *and* g *are constant.*

In full generality $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ is not $\mathscr{C}^2$ but merely $\mathscr{C}^1$ [3]. We assume that it is piecewise $\mathscr{C}^2$ in the sense that the mixture zone defined by (MZ) is a $\mathscr{C}^2$ manifold with a boundary which is a $\mathscr{C}^1$ closed loop curve. We introduce a new technical result in the following theorem that investigates some of the properties of $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$:

Theorem 2.4. *Let* $\boldsymbol{w} \in \mathcal{C}$ *be a 'saturated state', then*

(I) *there exists a* single *couple of points* $\mathbf{M}_1^* = (\boldsymbol{w}_1^*, s_1^* = s_1(\boldsymbol{w}_1^*)) \in \mathcal{S}_1$ *and* $\mathbf{M}_2^* = (\boldsymbol{w}_2^*, s_2^* = s_2(\boldsymbol{w}_2^*)) \in \mathcal{S}_2$ *such that* $\mathbf{M} = (\boldsymbol{w}, s^{\mathrm{eq}}(\boldsymbol{w}))$ *belongs to the line segment* $\mathfrak{r}_{12} = (\mathbf{M}_1^*, \mathbf{M}_2^*) = \{y\mathbf{M}_1^* + (1-y)\mathbf{M}_2^* \mid y \in [0,1]\}$.

Moreover, for every point of the straight line segment $\mathfrak{r}_{12}$ *we have*

$$\text{(II)}\quad \begin{cases} s^{\mathrm{eq}}_{\tau\varepsilon}(\varepsilon_1^* - \varepsilon_2^*) + s^{\mathrm{eq}}_{\tau\tau}(\tau_1^* - \tau_2^*) = 0, \\ s^{\mathrm{eq}}_{\varepsilon\varepsilon}(\varepsilon_1^* - \varepsilon_2^*) + s^{\mathrm{eq}}_{\tau\varepsilon}(\tau_1^* - \tau_2^*) = 0, \end{cases} \qquad s^{\mathrm{eq}}_{\tau\tau}s^{\mathrm{eq}}_{\varepsilon\varepsilon} = \left(s^{\mathrm{eq}}_{\tau\varepsilon}\right)^2, \quad s^{\mathrm{eq}}_{\tau\tau} = \left(\frac{\varepsilon_1^* - \varepsilon_2^*}{\tau_1^* - \tau_2^*}\right)^2 s^{\mathrm{eq}}_{\varepsilon\varepsilon};$$

$$\text{(III)}\quad s^{\mathrm{eq}}_{\tau\tau} < 0, \quad s^{\mathrm{eq}}_{\varepsilon\varepsilon} < 0, \quad s^{\mathrm{eq}}_{\tau\varepsilon} \neq 0;$$

$$\text{(IV)}\quad P \neq -\frac{\varepsilon_1^* - \varepsilon_2^*}{\tau_1^* - \tau_2^*}, \qquad T \neq \frac{\varepsilon_1^* - \varepsilon_2^*}{s_1^* - s_2^*}.$$

Proof. (I) The existence of the segment $\mathfrak{r}_{12}$ follows from Proposition 2.3. We prove the uniqueness: s_α is strictly concave and increasing according to τ and ε then there is a bijection with (P, T) and $\boldsymbol{w}_\alpha$. If $\tilde{\mathfrak{r}}_{12} = ((\widetilde{\boldsymbol{w}}_1^*, \tilde{s}_1^*), (\widetilde{\boldsymbol{w}}_2^*, \tilde{s}_2^*))$ is such that $(\boldsymbol{w}, s^{\mathrm{eq}}(\boldsymbol{w})) \in \mathfrak{r}_{12} \cap \tilde{\mathfrak{r}}_{12}$, as (P, T, g) are constant along $\mathfrak{r}_{12}$ and $\tilde{\mathfrak{r}}_{12}$, we have $\boldsymbol{w}_\alpha^* = \widetilde{\mathbf{w}}_\alpha^*$ and consequently $\mathfrak{r}_{12} = \tilde{\mathfrak{r}}_{12}$.

(II) The jump of specific volume, energy and entropy (5) implies that for every point $(\boldsymbol{w}, s^{\mathrm{eq}}(\boldsymbol{w}))$ in the saturation zone we have $0 < y^* < 1$, $\tau_1^* \neq \tau_2^*$, $\varepsilon_1^* \neq \varepsilon_2^*$, $s_1^* \neq s_2^*$. Along $\mathfrak{r}_{12}(\boldsymbol{w})$, (P, T, g) are constant, then we have

$$0 = \mathrm{d}(P/T) = s^{\mathrm{eq}}_{\tau\varepsilon}(\varepsilon_1^* - \varepsilon_2^*) + s^{\mathrm{eq}}_{\tau\tau}(\tau_1^* - \tau_2^*) \quad \text{and} \quad 0 = \mathrm{d}(1/T) = s^{\mathrm{eq}}_{\varepsilon\varepsilon}(\varepsilon_1^* - \varepsilon_2^*) + s^{\mathrm{eq}}_{\tau\varepsilon}(\tau_1^* - \tau_2^*).$$

(III) By contradiction: let $\overline{\boldsymbol{w}}$ be a saturated state such that $s^{\mathrm{eq}}_{\tau\tau}(\overline{\boldsymbol{w}}) = 0$. By relations (II), the Hessian matrix is null, i.e. $\mathrm{d}^2 s^{\mathrm{eq}}(\overline{\boldsymbol{w}}) = \left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)$. We note $(P_\alpha, T_\alpha, g_\alpha)(\overline{\boldsymbol{w}}) = (\overline{P}, \overline{T}, \bar{g})$ for $\alpha = 1, 2$. We consider a regular $\mathscr{C}^2$ curve in $\mathcal{S}$ parameterized by $t \in [-1, 1] \mapsto (\boldsymbol{w}, \gamma = s^{\mathrm{eq}}(\boldsymbol{w}))(t)$ such that $\boldsymbol{w}(0) = \overline{\boldsymbol{w}}$. We have

$$\gamma''(0) = \mathrm{d}s^{\mathrm{eq}}(\overline{\boldsymbol{w}})\frac{\mathrm{d}^2\boldsymbol{w}(0)}{\mathrm{d}t^2} + \frac{\mathrm{d}\boldsymbol{w}(0)}{\mathrm{d}t}^{\mathrm{T}} \mathrm{d}^2 s^{\mathrm{eq}}(\overline{\boldsymbol{w}})\frac{\mathrm{d}\boldsymbol{w}(0)}{\mathrm{d}t} = \mathrm{d}s^{\mathrm{eq}}(\overline{\boldsymbol{w}})\frac{\mathrm{d}^2\boldsymbol{w}(0)}{\mathrm{d}t^2}.$$

Moreover there exist $\mathscr{C}^2$ smooth functions $t \mapsto (y_\alpha^*, \boldsymbol{w}_\alpha^*)(t)$ such that

$$(\boldsymbol{w}, \gamma)(t) = \sum_\alpha y_\alpha^*\big(\boldsymbol{w}_\alpha^*, s_\alpha^*(\boldsymbol{w}_\alpha^*)\big)(t)$$

where $y_1^* = y^*$ and $y_2^* = 1 - y^*$. We have

$$\frac{\mathrm{d}^2\boldsymbol{w}}{\mathrm{d}t^2} = \sum_\alpha\left(\frac{\mathrm{d}^2 y_\alpha^*}{\mathrm{d}t^2}\boldsymbol{w}_\alpha^* + 2\frac{\mathrm{d}y_\alpha^*}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{w}_\alpha^*}{\mathrm{d}t} + y_\alpha^*\frac{\mathrm{d}^2\boldsymbol{w}_\alpha^*}{\mathrm{d}t^2}\right)$$

and

$$\gamma''(t) = \sum_\alpha \left[\frac{\mathrm{d}^2 y_\alpha^*}{\mathrm{d}t^2} s_\alpha^* + 2\frac{\mathrm{d}y_\alpha^*}{\mathrm{d}t}\, \mathrm{d}s_\alpha^* \frac{\mathrm{d}\boldsymbol{w}_\alpha^*}{\mathrm{d}t} + y_\alpha^*\, \mathrm{d}s_\alpha^* \frac{\mathrm{d}^2 \boldsymbol{w}_\alpha^*}{\mathrm{d}t^2} + y_\alpha^* \left(\frac{\mathrm{d}\boldsymbol{w}_\alpha^*}{\mathrm{d}t}\right)^{\mathrm{T}} \mathrm{d}^2 s_\alpha^* \frac{\mathrm{d}\boldsymbol{w}_\alpha^*}{\mathrm{d}t}\right].$$

This implies

$$\gamma''(0) = \mathrm{d}s^{\mathrm{eq}}(\overline{\boldsymbol{w}}) \frac{\mathrm{d}^2 \boldsymbol{w}(0)}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 y^*}{\mathrm{d}t^2}\left(\frac{g_1}{T_1} - \frac{g_2}{T_2}\right) + \sum_\alpha y_\alpha^* \left(\frac{\mathrm{d}\boldsymbol{w}_\alpha^*}{\mathrm{d}t}\right)^{\mathrm{T}} \mathrm{d}^2 s_\alpha^* \frac{\mathrm{d}\mathbf{w}_\alpha^*}{\mathrm{d}t}$$

since

$$\mathrm{d}s_\alpha^*\big(\boldsymbol{w}_\alpha^*(0)\big) = \mathrm{d}s^{\mathrm{eq}}\big(\boldsymbol{w}(0)\big)(1/\overline{T}, \overline{P}/\overline{T}).$$

Consequently, as $g_\alpha/T_\alpha = \bar{g}/\overline{T}$, we have

$$\sum_\alpha y_\alpha^* \left(\frac{\mathrm{d}\boldsymbol{w}_\alpha^*}{\mathrm{d}t}\right)^{\mathrm{T}} \mathrm{d}^2 s_\alpha^* \left(\frac{\mathrm{d}\boldsymbol{w}_\alpha^*}{\mathrm{d}t}\right) = 0,$$

which is impossible.

(IV) This point follows from $0 = g_1 - g_2 = (\varepsilon_1^* - \varepsilon_2^*) + P(\tau_1^* - \tau_2^*) - T(s_1^* - s_2^*)$. □

Remark 1. Theorem 2.4 ensures that it is not possible to have more than one solution $(\boldsymbol{w}_1^*, \boldsymbol{w}_2^*, y^*)$, therefore existence of solution of (4) ensures uniqueness.

3. Euler system with equilibrium phase change EOS

The main result is the following theorem:

Theorem 3.1. *We consider an entropy function s^{eq} as defined by Theorem* 2.4 *or Proposition* 2.1. *For smooth solutions, the one-dimensional compressible Euler system equipped with the liquid–vapor equilibrium EOS $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ reads*

$$\partial_t \begin{pmatrix} \rho \\ u \\ \varepsilon \end{pmatrix} + \begin{pmatrix} u & \rho & 0 \\ \frac{1}{\rho}\frac{\partial P}{\partial \rho}\big|_\varepsilon & u & \frac{1}{\rho}\frac{\partial P}{\partial \varepsilon}\big|_\rho \\ 0 & \frac{P}{\rho} & u \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{with } \rho = \frac{1}{\tau},\ P = \frac{s_\tau^{\mathrm{eq}}}{s_\varepsilon^{\mathrm{eq}}}. \tag{6}$$

This system is strictly hyperbolic, *i.e. it admits a complete set of real eigenvalues and eigenvectors on the cone $\mathcal{C}$.*

Proof. The Jacobian matrix of the flux has three eigenvalues $u - c$, u, $u + c$, where c is the speed of sound given by (1). We only have to check that $c > 0$. We distinguish two regions:

Pure Phase Regions $(y \in \{0, 1\})$: $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ satisfies (2) then $c^2 > -\tau^2 T(P s_{\varepsilon\tau}^{\mathrm{eq}} - s_{\tau\tau}^{\mathrm{eq}})^2 / s_{\tau\tau}^{\mathrm{eq}} \geqslant 0$.

Mixture Zone $(0 < y < 1)$: in Theorem 2.4 we proved that $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ does not satisfies (2)-B but however we have $c^2 = -\tau^2 T(s_{\tau\varepsilon}^{\mathrm{eq}})^2 (P + \frac{(\varepsilon_1^* - \varepsilon_2^*)}{(\tau_1^* - \tau_2^*)})^2 / s_{\tau\tau}^{\mathrm{eq}}$ then $c^2 > 0$. □

Corollary 3.2. *The MTT-equilibrium system defined in* [5], *the Euler system provided with the EOS defined in* [2,10,11] *or defined in* [12] *or still defined in* [14] *are strictly hyperbolic.*

Remark 2. We emphasize that the strict positivity of the speed of sound does not seem totally trivial. First, the function $\boldsymbol{w} \mapsto s^{\mathrm{eq}}$ is concave (but not strictly concave) and its Hessian matrix is not definite negative. Concavity provides that $c \geqslant 0$, but we do not have a priori informations that ensures that $c > 0$. Secondly, for some systems [5–8] the loss of strict concavity of the entropy indeed leads to a zero speed of sound: in this case hyperbolicity is lost because the system eigenvalues are real but the Jacobian matrix of the flux is no longer diagonalizable. This situation leads to ill-posed problems (see, for example, [13]) where uniqueness is lost within the classical class of entropy solutions.

Remark 3. Theorem 2.4 implies that: 1) the flux function in the Euler equations is only piecewise regular. 2) The functions $\boldsymbol{w} \mapsto (P, T)(\boldsymbol{w})$ are piecewise $\mathscr{C}^1$ regular: their derivatives are discontinuous when the material changes from a pure phase state to a diphasic state. 3) The function $\boldsymbol{w} \mapsto c$ is piecewise $\mathscr{C}^0$ regular: the speed of sound is discontinuous across $y \in \{0, 1\}$ regions and $0 < y < 1$ region.

Remark 4. Explicit computations of the speed of sound for the Euler system (6) are available for the special case of two perfect gases in [1,2,10–12] when $c_{v_1} = c_{v_2} = 1$. Explicit computations are also available for the case of two stiffened gases in [14]. Let us underline that for two perfect gas EOS, namely $\boldsymbol{w}_\alpha \mapsto s_\alpha = c_{v_\alpha} \log(\varepsilon_\alpha \tau_\alpha^{\gamma_\alpha - 1})$, the hypothesis of a jump of specific volume, energy and entropy across the interface (5), we used here, implies respectively that $c_{v_1}(\gamma_1 - 1) \neq c_{v_2}(\gamma_2 - 1)$, $c_{v_1} \neq c_{v_2}$ and $c_{v_1}\gamma_1 \neq c_{v_2}\gamma_2$.

4. Conclusion

We have proved the strict hyperbolicity of the compressible Euler system equipped with an EOS that describes the liquid–vapor thermodynamical equilibrium of a fluid. The proof is valid for a wide range of EOS. This result is a step towards the well-posedness study of compressible fluids systems with phase change. This work provides a solid ground for the simulation of compressible phase transition phenomena. Let us mention for example the computations of the equilibrium entropy s^{eq} for the case of two perfect gases that has been recently achieved in [9] and its extension to more general EOS that is ongoing work. This will help building general numerical solvers in the lines of [1,2,5,7, 8,10,11,14].

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