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C. R. Acad. Sci. Paris, Ser. I 344 (2007) 127–133
COMPTES RENDUS
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Numerical Analysis

Dupire-like identities for complex options

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Received 22 May 2006; accepted after revision 21 November 2006

Available online 29 December 2006

Presented by Alain Bensoussan

Abstract

Dupire's identity is very useful to compute all financial options based on a single asset at once and also for the calibration of models. We show that it is not limited to European options based on a single Brownian driven asset. By using the adjoint equations of the financial models we extend the concept to barrier options, Lévy driven options, basket options and partially to stochastic volatility models. The technique does not work for American and Asian options. The analytic derivations of these Dupire-like formulae is tested numerically and excellent agreement is found proving henceforth that the method is also numerically feasible. ***To cite this article: O. Pironneau, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Identités de Dupire pour quelques modèles d'option. L'identité de Dupire est très utile pour calculer tous les prix d'options sur un seul sous-jacent par une unique résolution des équations aux dérivées partielles et aussi pour la calibration des modèles en mathématiques financières. Nous montrons ici comment obtenir de telles identités dans quelques cas plus complexes que le cadre traité par Dupire lui-même : options barrières, options paniers et options modélisées par des modèles à volatilité stochastique ou par des processus de Lévy. Les formules sont aussi testées sur des exemples numériques et une très bonne précision est obtenue. ***Pour citer cet article : O. Pironneau, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Le modèle mathématique pour le pricing d'option correspond en général à une situation idéale et doit donc être « calibré » en pratique par rapport aux observations boursières [4,2]. Il est bien connu que cette calibration ne peut se faire que dans la même classe ; ainsi pour ajuster la volatilité locale σ dans un modèle de Black–Scholes (1) pour des options européennes par exemple sur un sousjaçant S, il faudra observer d'autres options européennes sur le même sous-jacent différentes seulement par le strike K et la maturité T. Le problème est donc de trouver une surface $\sigma(S,t)$ telle que $J := \sum_i |C_{K_i,T_i}(S_0,t_0) - C_{di}|^2$ soit minimal, où C_{di} sont les observations de l'option C_{K_i,T_i} au jour t_0 alors que le sousjaçant vaut S_0. Les méthodes itératives d'ajustement de σ pour rendre J minimal doivent donc résoudre (1) pour tous les différents K_i ce qui est très coûteux. Dupire [5] a donné une identité qui permet de résoudre cette

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doi:10.1016/j.crma.2006.11.032

difficulté en montrant que $\widetilde{C}_{S_0,t_0}(K,T) := C_{K,T}(S_0,t_0)$ vérifie aussi une équation aux dérivées partielles (EDP) en K, T où S_0, t_0 sont maintenant les paramètres. Il existe plusieurs démonstrations de cette identité soit en utilisant la fonction de Green du problème soit par l'équation de Kolmogorov en revenant à la formulation stochastique du problème. Nous donnons ici une troisième démonstration en utilisant l'EDP adjointe du problème. Cette démonstration a l'avantage de s'étendre à d'autres cas plus complexes, comme les options barrières (Proposition 1.2), les options basées sur des processus de saut (Proposition 1.3) et les options paniers 2.1. Toutefois dans ce dernier cas ainsi que pour les modèles à volatilité stochastique, on obtient un outil numérique équivalent mais pas exactement une identité à la Dupire.

1. Dupire's equation and the adjoint state

Models for the pricing of options must be either refined or *calibrated* to fit market observations. For instance, the Black–Scholes partial differential equation for European call options [8]

$$\partial_t C + \frac{1}{2}\sigma^2 S^2 \partial_{SS} C + rS\partial_S C - rC = 0, \quad C(S,T) = (S-K)^+ \quad \forall S, t \in Q \tag{1}$$

may be extended by using a volatility surface $\sigma(S,t)$ in place of a fixed value in $Q := \mathcal{R}^+ \times (0,T)$. During the life of the call some data are available, namely the values of calls on S, with possibly different parameters $C_i := C_{K_i,T_i}(S_{t_0}, t_0)$, $i \in I$. Calibration can be done by least squares on σ so as to fit the data; however one is faced with the costly problem of integrating (1) for each different $K_i \in I$. Dupire [5] observed that an equation for $C_{S,t}(K,T)$ can be derived where S, t are now parameters:

$$\partial_\tau C - \frac{1}{2}\sigma^2 K^2 \partial_{KK} C + rK\partial_K C = 0, \quad C(K,t_0) = (S-K)^+ \quad \forall K > 0, \tau \geqslant t_0. \tag{2}$$

Now calibration can be done with (2) integrated once only over $(t_0, \max_I T_i)$ with $S = S_0$.

Dupire's proof [5] is not easy to generalize to more complex options. Our purpose here is to show that the proof is in fact quite general and applies to most linear option models; the proof also shows why there is little hope of finding the equivalent for American options.

Proposition 1.1 *(Dupire). Let v be solution in $\mathcal{R}^+ \times (t_0, T)$ of*

$$\partial_t v - \frac{1}{2}\sigma^2 S^2 \partial_{SS} v + rS\partial_S v = 0, \quad v(S,t_0) = (S_0 - S)^+ \tag{3}$$

then $C_{K,T}(S_0,t_0) = v(K,T)$ and $p = \partial_{SS} v$ is solution of the adjoint of (1):

$$\partial_t p - \partial_{SS}\left(\frac{\sigma^2 S^2}{2} p\right) + \partial_S(rSp) + rp = 0, \quad p(t_0) = \delta(S - S_0). \tag{4}$$

Proof. An integration by parts in time and Green's formula in space applied to (1) multiplied by p and integrated over Q yields

$$C(S_0,t_0) = \int_0^\infty C_T\, p(T)\,\mathrm{d}S + \int_{t_0}^T \left[p\frac{\sigma^2 S^2}{2}\partial_S C - C\partial_S\left(\frac{\sigma^2 S^2}{2}p\right) + prSC\right]_0^\infty \tag{5}$$

where $C_T(S)$ is the payoff. By the properties of C and p at zero and infinity the last term vanishes. Let v be the double primitive of p, i.e. $\partial_{SS} v = p$ then, (4) integrated twice becomes (3) for an appropriate choice of the integration constants. □

Binary call options: The payoff is one monetary unit if $S_T > K$ and zero otherwise. It can be treated by the same method. The adjoint equation is integrated once only and the result is:

$$u(S_0,t_0) = \int_0^\infty u_T \partial_S w\,\mathrm{d}S = -\int_0^\infty w\partial_S u_T\,\mathrm{d}S + [u_T w]_0^\infty = -w(K,T) \quad \text{with}$$

$$\partial_t w - \partial_S\left(\frac{\sigma^2 S^2}{2}\partial_S w\right) + rS\partial_S w + rw = 0, \quad w(t_0) = 1_{S>S_0} - 1. \tag{6}$$

1.1. Dupire's equation for barrier options

Consider a European barrier option which stops to exist if $S_t \notin (S_m e^{rt}, S_M e^{rt})$. The change of variable $c(s,t) = e^{-rt}C(se^{rt}, t)$ brings

$$\begin{aligned}&\partial_t c + \frac{1}{2}\sigma^2 s^2 \partial_{ss} c = 0,\\ &c(s,T) = \left(s - Ke^{-rT}\right)^+, \quad c(S_m,t) = c(S_M,t) = 0, \quad \forall s \in (S_m, S_M)\ \forall t \in (0,T).\end{aligned} \tag{7}$$

Proposition 1.2. *Assume that $S_m < Ke^{-rT} < S_M$. If c verifies (7) then*

$$c(s_0,t_0) = v\left(Ke^{-rT}, T\right) + \left(S_M - Ke^{-rT}\right)^+ \partial_S v|_{S_M,T} \tag{8}$$

where v is the solution of

$$\begin{aligned}&\partial_t v - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v = 0, \quad v(s,t_0) = (s_0 - S)^+\\ &v(S_m,t) = (s_0 - S_m)^+, \quad v(S_M,t) = 0 \quad \forall t \in (t_0,T).\end{aligned} \tag{9}$$

Proof. Now the boundary conditions in (4) with $r = 0$ are $p(S_m,t) = p(S_M,t) = 0$ for all t. For v, the double primitive of p, it translates into $\partial_{ss} v = 0$ which in turn implies $\partial_t v = 0$ which means that v is constant at the barriers. Naturally $C(S_0,t_0)$ can be recovered from c by choosing $s_0 = S_0 e^{rt_0}$. □

Numerical results: A finite difference method implicit in time of order one (Euler's scheme) is used for (1) and (9); the parameters are $K = 100$, $r = 0.06$, $\sigma = 0.4$, 200 time steps and 250 mesh points for S. The accuracy (Fig. 1) is excellent.

1.2. Dupire's equation for options on Lévy driven assets

If a Poisson–Lévy process is used in the Back–Scholes model, as in [3], a term appears in the right-hand side of (1):

$$\int_{\mathcal{R}} \left(C\left(Se^y, t\right) - C(S,t) - S\left(e^y - 1\right)\frac{\partial C}{\partial S}\right)k(y)\,dy \tag{10}$$

where k, the kernel of the process is usually singular at the origin and decaying fast at infinity.

When multiplied by p and integrated in S, with $z = Se^{-y}$ this term can be transformed into an integral of $C(S,t)\chi(S,t)$ for an appropriate χ leading to the integro-differential equation (11) for p. The resulting equation is again integrated twice and leads to the following result (derived earlier by a different method by Achdou [1]):

Proposition 1.3. *A Poisson–Lévy driven option $C_{K,T}(S,t)$, solution of (1) with (10) on the right-hand side, verifies Dupire's identity*

$$C_{K,T}(S_0,t_0) = v(K,T)$$

where v is the solution of

$$\begin{aligned}&\partial_t v - \frac{\sigma^2 S^2}{2}\partial_{SS} v + rS\partial_S v - \int_{\mathcal{R}} \left(e^y\left(v\left(Se^{-y},t\right) - v(S,t)\right) + \left(e^y - 1\right)S\partial_S v(S,t)\right)k(y)\,dy = 0,\\ &v(S,t_0) = (S_0 - S)^+, \quad v(0,t) = S_0, \quad v(S,t) \to 0 \quad \textit{when } S \to \infty,\ \forall t \in (t_0,T).\end{aligned} \tag{11}$$

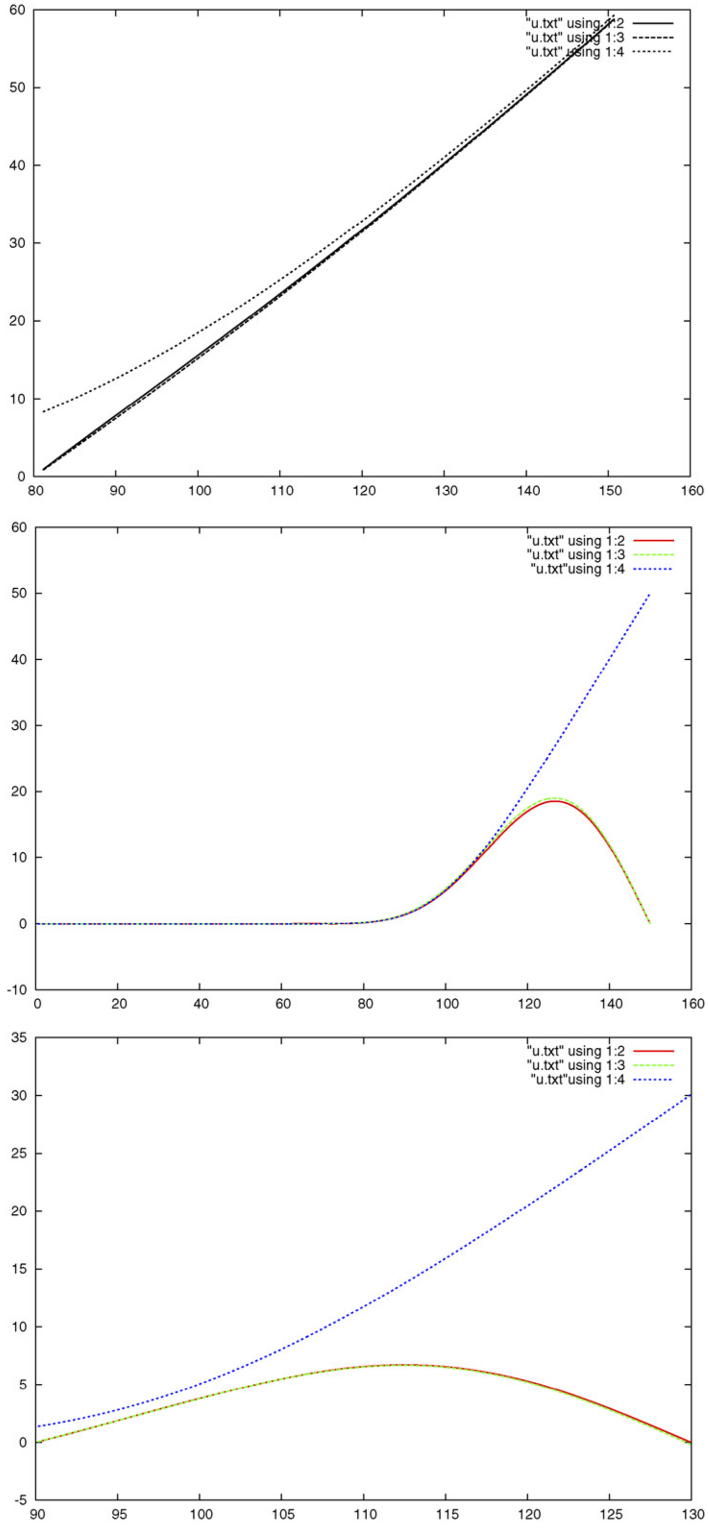

Fig. 1. European option with $K = 100$, $r = 0$ one barrier at 80 (top) or one at 150 (middle) and two barriers at 90 and 130 (bottom); the 3 curves are C by solving (7), C by the Dupire formula (8) and the unconstrained C (no barrier) computed by the Black–Scholes analytic formula. For the first 2 cases $T = 1$, for the last one $T = 0.1$ because the curves are too flat at $T = 1$.

2. Dupire identities for bi-dimensional problems

2.1. Basket options

An option on two assets would be modeled by

$$\partial_t C + \sum_1^2 \left(\frac{\sigma_i^2}{2} S_i^2 \partial_{S_i S_i} C + r S_i \partial_{S_i} C \right) - 2q S_1 S_2 \partial_{S_1 S_2} C - rC = 0, \quad C(T) = (S_1 + S_2 - K)^+ \tag{12}$$

with $q = -\frac{1}{2} q' \sigma_1 \sigma_2$ where $q' \, \mathrm{d}t = \mathrm{d}\mathbf{E}(W_1 W_2)$ is the instantaneous correlation between the processes driving S_1 and S_2 in the models where for simplicity we have assumed that the drifts are both equal to r.

So let p be solution of the adjoint equation with $p(t_0) = \delta(S_1 - S_{01})\delta(S_2 - S_{02})$. As before

$$C_{K,T}(S_{01}, S_{02}, t_0) = \int_{\mathcal{R}^{+2}} p(S_1, S_2, T)(S_1 + S_2 - K)^+ \, \mathrm{d}S_1 \, \mathrm{d}S_2. \tag{13}$$

To remove the Dirac singularities in (13) let us seek first a w such that $p = \partial_{S_1 S_2} w$.Then, when σ_i does not depend on S_j, $j \neq i$,

$$\partial_t w - \frac{1}{2} \sum_1^2 \partial_{S_i} \left(\sigma_i^2 S_i^2 \partial_{S_i} w \right) + 2q S_1 S_2 \partial_{S_1 S_2} w + r S_1 \partial_{S_1} w + r S_2 \partial_{S_2} w + r w = 0,$$
$$w(S_1, S_2, t_0) = \big(1 - H(S_1 - S_{01})\big)\big(1 - H(S_2 - S_{02})\big). \tag{14}$$

It corresponds to a special choice for the integration constant which gives an exponential decay at infinity of w; H is the Heaviside function; finally the integral above can be integrated by parts and the following is found:

Proposition 2.1. *If r is function of t only and each σ_i is a function of S_i and t only, $i = 1, 2$, then the basket option solution of* (12) *is given by*

$$C_{K,T}(S_{01}, S_{02}, t_0) = \int_{S_1 + S_2 = K} \frac{w(S_1, S_2, T)}{\sqrt{2}} + \int_K^\infty w(S_1, 0, T) \, \mathrm{d}S_1 + \int_K^\infty w(0, S_2, T) \, \mathrm{d}S_2. \tag{15}$$

$H(z)$ being the Heaviside function ($= z^+/z$), w the solution of

$$\partial_t w - \sum_1^2 \left(\partial_{S_i} \left(\frac{\sigma_i^2}{2} S_i^2 \partial_{S_i} w \right) - r S_i \partial_{S_i} w \right) + 2q S_1 S_2 \partial_{S_1 S_2} w + r w = 0,$$
$$w(S_1, S_2, t_0) = \big(1 - H(S_1 - S_{01})\big)\big(1 - H(S_2 - S_{02})\big). \tag{16}$$

Remark 1. If q depends on t only (which happens only when the σ_i are themselves like that, or if q is 'calibrated') then (16) can be integrated further and set $w = \partial_{S_1 S_2} v$ and then

$$C_{K,T}(S_{01}, S_{02}, t_0) = \int_{S_1 + S_2 = K} \frac{\partial_{S_1 S_2} v}{\sqrt{2}} - \partial_{S_2} v(K, 0) - \partial_{S_1} v(0, K)$$

where v is the solution of

$$\partial_t v - \sum_1^2 \left(\frac{\sigma_i^2}{2} S_i^2 \partial_{S_i S_i} v - (r + 2q) S_i \partial_{S_i} v \right) - 2q S_1 S_2 \partial_{S_1 S_2} v - (r + 2q) v = 0,$$
$$v(S_1, S_2, t_0) = (S_{01} - S_1)^+ (S_{02} - S_2)^+. \tag{17}$$

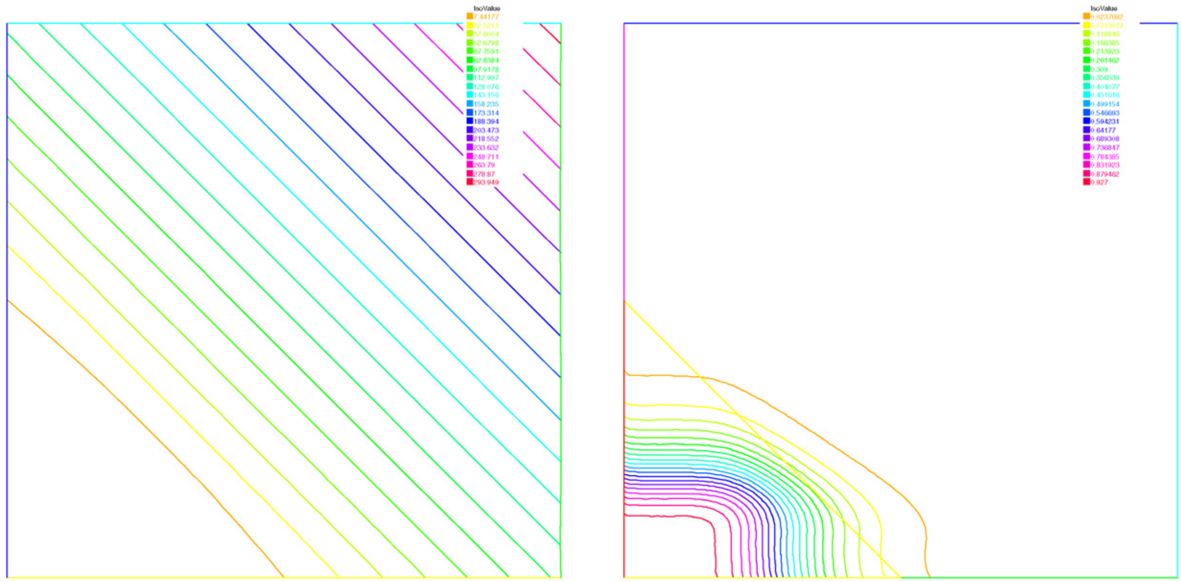

Fig. 2. Left: Iso-value lines of C computed by (12). Right: Iso-value lines of w computed by (16); the yellow line is $S_1 + S_2 = K$.

Table 1
Comparison between direct calculation of C, the basket call on S_1, S_2, based on (12) – lines :c- and C computed by solving the Dupire equation (16) – lines :d

$S_1 \backslash S_2$	20	50	80	110	140
20:c	−0.097	−0.089	5.61	31.57	61.49
20:d	1.1e-07	0.0065	5.88	32.11	61.67
50:c	4.32	31.50	61.49	91.49	31.57
50:d	4.39	31.52	61.81	91.44	32.11
80:c	61.49	91.49	121.49	61.49	5.61
80:d	61.70	91.99	121.62	61.81	5.88
110:c	121.49	151.49	91.5	31.50	−0.089
110:d	122.23	151.86	91.99	31.52	0.0065
140:c	181.49	121.5	61.49	4.3	−0.097
140:d	181.60	122.2	61.69	4.3	0

Numerical results: Two programs were written in the freefem language [6], one to solve (12) and one for (16). Both use a finite element method of order 1 on triangles and Euler's implicit time scheme. Mesh adaptivity was used for (12) and the numerical scheme was applied to $\widetilde{C} = C - S_1 - S_2 + K\mathrm{e}^{-r(T-t)}$ so as to have a decaying function at infinity. The following data are used:

$$\sigma_1 = \sigma_2 = 0.3, \quad q = 0.02, \quad r = 0.05, \quad K = 100, \quad T = 0.3. \tag{18}$$

The computational domain is $(0, 300) \times (0, 300)$ and the time step is 0.02. The level lines of $\widetilde{C}(0, \cdot)$ are shown on Fig. 2. Eq. (16) was solved by the same method with various values for S_{01}, S_{02} shown in Table 1. Fig. 2 also shows the level lines of w but at $T = 1$.

2.2. Orstein–Uhlenbeck's stochastic volatility model

Following [3] and [2] consider

$$\partial_t C + \frac{1}{2}|y|^2 S^2 \partial_{SS} C + \rho\beta S|y|\partial_{Sy} C + \frac{1}{2}\beta^2 \partial_{yy} C + rS\partial_S C + \gamma(y,t)\partial_y C - rC = 0 \tag{19}$$

for all $S > 0$, all y and $t \in (0, T)$; in [7] $\gamma = \alpha(m - y) - \beta\lambda y^2$. Boundary conditions are

$$C(S, y, T) = (S - K)^+, \qquad \lim_{|y|\to\infty} \partial_y C = 0.$$

The adjoint equation is

$$\partial_t p - \frac{1}{2}\partial_{SS}\big(S^2 y^2 p\big) - \frac{1}{2}\partial_{yy}\big(\beta^2 p\big) - \partial_{Sy}\big(\rho\beta S|y]p\big) + r\partial_S(Sp) + \partial_y(\gamma p) + rp = 0. \tag{20}$$

If the parameters β, γ, ρ are functions of y and t only then it is possible to integrate the adjoint equation twice in S. Let w be such that $\partial_{SS} w = p$ then

$$\partial_t w - \frac{1}{2} S^2 y^2 \partial_{SS} w - \frac{1}{2}\partial_{yy}\big(\beta^2 w\big) + rS\partial_S w + \partial_y(\gamma w) - \partial_y\big(\rho\beta|y|(S\partial_S w - w)\big) = 0. \tag{21}$$

Unfortunately this is as far as one can go because the second term cannot be integrated in y. If the initial condition in (20) is $p(T_0) = \delta(S - S_0)\phi(y)$ then only an integral of C can be obtained:

$$\int_{\mathcal{R}} C(S_0, y, t_0)\phi(y)\,\mathrm{d}y = \int_{\mathcal{R}^+\times\mathcal{R}} (S - K)^+ \partial_{SS} w(S, y, T)\,\mathrm{d}S\,\mathrm{d}y = \int_{\mathcal{R}} w(K, y, T)\,\mathrm{d}y. \tag{22}$$

Acknowledgements

The author happily express his gratitude to Antoine Conze of Nexgenfs Bank who suggested the topic and showed a solution using Kolmogorov's equation for a stochastic volatility model.

References

[1] Y. Achdou, An inverse problem for a parabolic variational inequality with an integro-differential operator arising in the calibration of Lévy processes with American options, 2006, in press.
[2] Y. Achdou, O. Pironneau, Numerical Methods for Option Pricing, SIAM, Philadelphia, USA, 2005.
[3] R. Cont, P. Tankov, Financial Modelling with Jump Processes, Chapman and Hall, 2003.
[4] S. Crépey, Calibration of the local volatility in a generalized Black–Scholes model using Tikhonov regularization, SIAM J. Math. Anal. 34 (5) (2003) 1183–1206.
[5] B. Dupire, Pricing with a smile, Risk (1994) 18–20.
[6] F. Hecht, O. Pironneau, A. Le Yaric, K. Ohtsuka, freefem++ documentation, http://www.freefem.org.
[7] E. Stein, J. Stein, Stock price distributions with stochastic volatility: an analytic approach, The Review of Financial Studies 4 (4) (1991) 727–752.
[8] P. Wilmott, S. Howison, J. Dewynne, The Mathematics of Financial Derivatives, Cambridge University Press, Cambridge, 1995 (A student introduction).