Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 344 (2007) 121–126
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Dynamical Systems

Liénard systems and potential-Hamiltonian decomposition I – methodology

Jacques Demongeot [a,b], Nicolas Glade [b], Loic Forest [b]

[a] *Institut Universitaire de France, France*
[b] *TIMC-IMAG UMR CNRS 5525, University J. Fourier Grenoble, faculté de médecine, 38700 La Tronche, France*

Received 16 November 2004; accepted after revision 12 October 2006

Presented by Yves Meyer

Abstract

Following the Hodge decomposition of regular vector fields we can decompose the second member of any Liénard system into 2 (non-unique) polynomials, the first corresponding to potential and the second to Hamiltonian dynamics. This polynomial Hodge decomposition is called potential-Hamiltonian, denoted PH-decomposition, and we give it for any polynomial differential system of dimension 2. We will give in a future Note an algorithm expliciting the PH-decomposition in the neighborhood of particular orbits, like a limit-cycle for Liénard systems, the method being applicable for any polynomial differential system of dimension 2. ***To cite this article: J. Demongeot et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Systèmes de Liénard et décomposition potentielle-hamiltonienne I – méthodologie. Un système de Liénard est un système différentiel du second ordre, du type : $\mathrm{d}x/\mathrm{d}t = y, \mathrm{d}y/\mathrm{d}t = -g(x) + yf(x)$, où g et f sont des polynômes. Un tel système est susceptible d'être décomposé, de manière non unique, en 2 parties polynomiales, l'une potentielle et l'autre hamiltonienne, c'est-à-dire qu'il existe 2 polynômes P et H vérifiant : $\mathrm{d}x/\mathrm{d}t = -\partial P/\partial x + \partial H/\partial y, \mathrm{d}y/\mathrm{d}t = -\partial P/\partial y - \partial H/\partial x$. On montre, en utilisant la décomposition de Hodge des champs de vecteurs réguliers, que le second membre d'un tel système est décomposable en 2 polynômes, l'un correspondant à une dynamique de gradient et l'autre à une dynamique hamiltonienne. Cette décomposition de Hodge polynomiale est appelée potentielle-hamiltonienne, notée PH-décomposition, et nous en donnons la formule pour tout système différentiel polynomial du plan. Nous donnerons, dans une Note ultérieure, un algorithme permettant d'obtenir une formule explicite de la PH-décomposition au voisinage d'orbites particulières, telles qu'un cycle-limite dans le cas des systèmes de Liénard, la méthode étant applicable à tout système différentiel polynomial du plan. ***Pour citer cet article : J. Demongeot et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Les systèmes de Liénard sont des Equations Différentielles Ordinaires de Dimension 2 (2D-EDO) du type : $\mathrm{d}x/\mathrm{d}t = y$, $\mathrm{d}y/\mathrm{d}t = -g(x) + yf(x)$, où f et g sont des polynômes.

E-mail addresses: jacques.demongeot@imag.fr (J. Demongeot), nicolas.glade@imag.fr (N. Glade), loic.forest@imag.fr (L. Forest).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.10.016

Le système de van der Pol (cf. Fig. 1) en est un bon exemple [2–4], utilisé pour modéliser de nombreux systèmes biologiques régulés, dont le système cardiaque. Les polynômes g et f d'un système de van der Pol sont définis par : $g(x) = x$ et $f(x) = \mu(1 - x^2)$.

L'unique cycle-limite d'un système de van der Pol, lorsqu'il existe, est une courbe non algébrique difficile à approximer et il peut être très utile, dans les applications, d'en avoir une estimation polynomiale. Nous proposons donc, de manière générale, une décomposition des EDO régulières, dite potentielle-hamiltonienne, dont l'existence est fondée théoriquement sur la décomposition de Hodge [5–8].

Une ODE est dite potentiel-hamiltonienne décomposable, s'il existe un couple de polynômes (P, H), tel que : $\mathrm{d}x/\mathrm{d}t = -\partial P/\partial x + \partial H/\partial y$, $\mathrm{d}y/\mathrm{d}t = -\partial P/\partial y - \partial H/\partial x$.

La partie du flot de vecteurs vitesse définie par P correspond à une dynamique de descente de plus fort gradient sur la surface représentative de P et la partie définie par H correspond à une dynamique sur les courbes de niveau de la surface représentative de H.

Un exemple de système hamiltonien pur classique est celui du système de Lotka–Volterra, utilisé pour modéliser les interactions proie/prédateur [9]. Des exemples de systèmes différentiels purs potentiels sont les systèmes de type n-switch, à second membre défini par une cinétique de Hill compétitive (équation (4)), pour lesquels on peut obtenir explicitement la formule de définition de P et donc tracer la surface correspondante et en localiser les minima, qui sont les états stationnaires du n-switch (cf. Fig. 3). Un exemple de système polynomial mixte potentiel-hamiltonien est le système chimique de Lotka, pour lequel il est facile d'exhiber P et H [22].

Pour généraliser la décomposition ci-dessus, on montre un théorème donnant une formule générale de décomposition potentielle-hamiltonienne (non unique), pour des ODE du type : $\mathrm{d}x/\mathrm{d}t = f(x, y)$, $\mathrm{d}y/\mathrm{d}t = g(x, y)$, où f et g sont des polynômes. A partir de cette formule générale, nous donnerons, dans deux prochaines notes, un algorithme permettant d'obtenir la décomposition potentielle-hamiltonienne dans le cas des systèmes de Liénard, puis de l'appliquer à des problèmes biologiques, dans lesquels il existe un grand intérêt à décomposer le flot en une partie potentielle (dont les paramètres sont responsables d'une modulation de fréquence d'un signal biologique, dans le cas d'une trajectoire du système de type cycle-limite) et une partie hamiltonienne (dont les paramètres sont responsables d'une modulation d'amplitude).

1. Introduction

Liénard systems [12] are 2-dimensional ordinary differential equations (2D ODEs): $\mathrm{d}x/\mathrm{d}t = y$, $\mathrm{d}y/\mathrm{d}t = -g(x) + yf(x)$, where g and f are polynomials. The use of Liénard systems is universal in biological modeling, especially for physiological processes. A simple example of Liénard system is given by the van der Pol system used to model cardiac and respiratory rhythms [20,21,15], where:

$$g(x) = x \quad \text{and} \quad f(x) = \mu\left(1 - x^2\right). \tag{1}$$

For $\mu > 0$, it presents a limit-cycle bifurcating from the stationary state at the origin, which becomes progressively (as the anharmonic parameter μ increases) far from a circle, and reciprocally becomes circular when μ tends to 0 (Fig. 1).

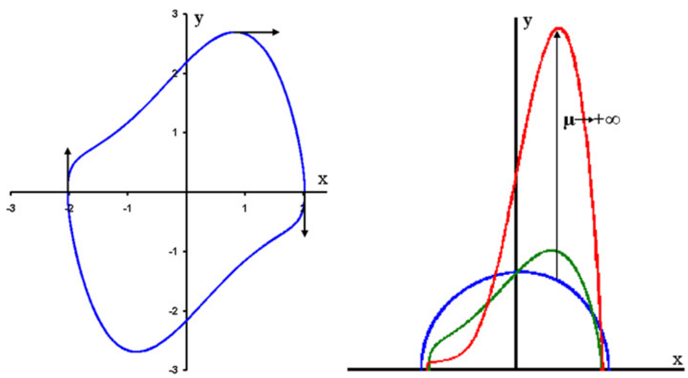


Fig. 1. Van der Pol system with $\mu = 1$ (left) and μ tending to infinity in the upper phase plane (right).

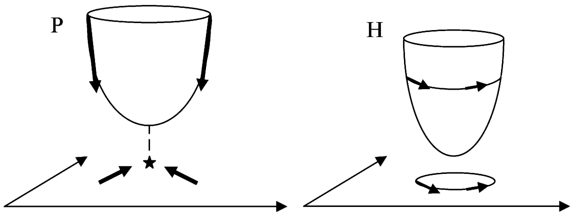


Fig. 2. Schematic representation of the gradient (potential P) flow (left) and of the conservative (Hamiltonian H) flow (right).

For many applications, it is essential to approximate the limit-cycle (a non-algebraic closed curve) by a polynomial curve. A manner to do it is to decompose the second member of the van der Pol into 2 polynomial parts, a potential one and a Hamiltonian one, closed contour lines of the Hamiltonian being candidates for estimating the limit-cycle.

2. Potential-Hamiltonian decomposition and Hodge decomposition

A 2D ODE $\mathrm{d}x/\mathrm{d}t = f(x, y)$, $\mathrm{d}y/\mathrm{d}t = g(x, y)$ is called potential-Hamiltonian decomposable (PH-decomposable), if and only if there is at least one couple of polynomials (P, H) such as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial P}{\partial x} + \frac{\partial H}{\partial y} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\partial P}{\partial y} - \frac{\partial H}{\partial x}. \tag{2}$$

The potential flow is the projection on the 2D phase plane of a steepest descent velocity vector field on the surface defined by the potential P, and the Hamiltonian flow the vector field tangential to contour lines of the surface generated by H (Fig. 2). The Hodge decomposition [10,8,3,14] ensures the existence of regular P and H functions (e.g. L^2 or C^1) for sufficiently smooth differential systems, e.g. ODE's having a second member of the same level of regularity L^2 or C^1. We try here to obtain explicit expressions (if possible polynomials) for P and H. As example, we have the Lotka–Volterra system, a pure Hamiltonian system, introduced by V. Volterra to account for the fluctuations observed in the struggle for life between a prey population of size x and a predator population of size y [22]:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(a - by) = \frac{\partial H}{\partial Y} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = y(cx - d) = -\frac{\partial H}{\partial X}, \quad \text{with } H(X, Y) = -c\mathrm{e}^X + dX - b\mathrm{e}^Y + aY \tag{3}$$

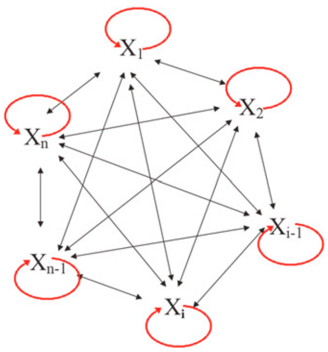
by changing variables $X = \log(x)$ and $Y = \log(y)$ (population 'affinities'). The trajectories are just contour lines of H.

3. A pure potential system, the *n*-switch model

A n-switch model is used to formalize fully connected interaction networks with only inhibitory interactions. Such a system has been proposed in [1] to model genetic and metabolic regulations and is encountered in plant growth [7,6,17,19], embryogenesis [11,9] and neural networks [23,18], in which hormones (like auxine), proteins (like transduction peptide) or neuro-transmitters (like Gaba) secreted by a part of the system (e.g. apex in plants) inhibit the growth or activity of the other parts (e.g. cotyledonary buds in plants) (cf. Fig. 3). We can represent each inhibition by a Hill competition with a cooperativity coefficient c, a maximal reaction rate σ and affinity coefficients a_i's as in the following system of equations, where x_i denotes the concentration (or the size, or the amount) of the component X_i in inhibitory interaction with all other components of the n-switch system [1]:

$$\forall i = 1, \dots, n, \quad \frac{\mathrm{d}x_i}{\mathrm{d}t} = -\nu x_i + \sigma a_i x_i^c \Big/ \left(1 + \sum_{j=1}^{n} a_j x_j^c\right) + \alpha_i. \tag{4}$$

If we consider a n-switch with $c = 1$, and $\forall i = 1, \dots, n$, $\alpha_i = 0$ and $a_i = 1 : \mathrm{d}x_i/\mathrm{d}t = -\nu x_i + \sigma x_i/(1 + \sum_{j=1}^{n} x_j)$, with $\sigma > \nu$, then the non-zero stationary states have $m \geqslant 1$ components x_i^* equal to $(\sigma - \nu)/m\nu > 0$ and the others equal to 0. The Jacobian matrix in these stationary states has only one non-zero eigenvalue $\lambda = \nu(\nu - \sigma)/\sigma < 0$ (cf. [1] for a stability analysis in the general case). Studying the stability basins topography is easier if we decompose the flow. By changing variables $y_i = (x_i)^{1/2}$ we see that the system is purely gradient with: $P(y) = \nu \sum_i y_i^2/4 - \sigma \log(1 + \sum_i a_i y_i^{2c})/4c - \sum_i \alpha_i \log y_i/2$ (cf. in Fig. 3 a 2-switch with 3 stable stationary states). Replacing Hill by allosteric inhibition, the system remains pure potential: $\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\nu x_i + \sigma x_i((1 + x_i)^{c-1} \prod_{j \neq i}(1 + x_j)^c +$



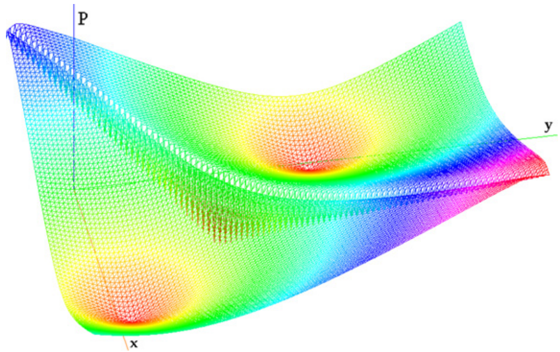


Fig. 3. Fully connected inhibitory interactions (except eventually auto-loops) of a n-switch (left) and potential P, if $n = 2$ (right).

$a_i(1+a_ix_i)^{c-1}\prod_{j\neq i}(1+a_jx_j)^c)/(\prod_{k=1}^n(1+x_k)^c+\prod_{k=1}^n(1+a_kx_k)^c)-\alpha_i$ and $\mathrm{d}y_i/\mathrm{d}t = -\partial P/\partial y_i$, with: $P(y) = \nu\sum_{i=1}^n y_i^2/4-\sigma\log(\prod_{i=1}^n(1+y_i^2)^c+\prod_{i=1}^n(1+a_iy_i^2)^c)/4c-\sum_{i=1}^n\alpha_i\log y_i/2$, where $y_i=(x_i)^{1/2}$ (cf. [4,5] for a first proof).

We represent on Fig. 3 (right) the surface of the potential P for a 2-switch ($\nu = 1$, $c = 2$, $\sigma = 2$, $a_i = 0.1$, $\alpha_i = 1$) over the square $[-5, 35] \times [-5, 35]$. The local minima are located at $(1.24, 4)$, $(24, 1.24)$, and $(12.35, 12.35)$ on the bissectrix.

4. Application to mixed potential-Hamiltonian systems, the Lotka and the Liénard equations

Chemical dynamical systems frequently offer equations susceptible to be PH-decomposed [16]. A historical example is the Lotka system [13] built to model a bi-reactant oscillatory reaction: $\mathrm{d}x/\mathrm{d}t = -xy - Kx - Ly$, $\mathrm{d}y/\mathrm{d}t = xy + Ly$. The decomposition is obtained by choosing: $P(x, y) = \frac{(K+y)x^2}{2} - \frac{Ly^2}{2} - \frac{x^3}{6}$ and $H(x, y) = -\frac{yx^2}{2} - \frac{Ly^2}{2} - \frac{x^3}{6}$.

The Lorenz pendulum [2]: $\mathrm{d}x/\mathrm{d}t = y$, $\mathrm{d}y/\mathrm{d}t = -ax - x^3$ is a pure Hamiltonian, with $H(x, y) = y^2/2 + ax^2/2 + x^4/4$. More generally, Liénard systems are PH-decomposable: if h and l are polynomials of order respectively m and p, and if $h^{(k)}$ denotes the kth derivative of h, we have:

$$P(x,y)=\sum_{k=1}^{n}\frac{(-1)^k f^{(2k-2)}y^{2k}}{(2k)!}+\sum_{k=1}^{m}\frac{(-1)^k h^{(2k-2)}y^{2k-1}}{(2k-1)!}+\sum_{k=1}^{p}\frac{(-1)^k l^{(2k-2)}x^{2k-2}}{(2k-2)!},$$

$$H(x,y)=\int(g+h)+\frac{y^2}{2}+\sum_{k=1}^{n}\frac{(-1)^k f^{(2k-1)}y^{2k+1}}{(2k+1)!}+\sum_{k=1}^{n}\frac{(-1)^k h^{(2k-1)}y^{2k}}{(2k)!}+\sum_{k=1}^{n}\frac{(-1)^{(k+1)} l^{(2k-1)}x^{2k-1}}{(2k-1)!}.$$

The proof is just the consequence of the following proposition, lemmas and theorem, available for any 2D ODE with sufficiently regular second member.

Proposition 1. *Let us consider $V = f(x, y)\partial./\partial x + g(x, y)\partial./\partial y$ being a polynomial vector field, then there is a (non-unique) PH-decomposition of V into two parts, a potential part, which is the gradient of a polynomial P and a conservative part, which is the Hamiltonian of a polynomial H such that: $V = -\operatorname{grad} P + \operatorname{ham} H$.*

Proof. Calculate $\operatorname{div} V = D(x, y)$ and use complex coordinates $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$. Then there is a (non-unique) polynomial P such that $\partial^2P/\partial z\partial\bar{z} = \partial^2P/\partial x^2 + \partial^2P/\partial y^2 = -D(x, y)$. The divergence of $V + \operatorname{grad} P$ equals 0. Hence $V + \operatorname{grad} P$ is a Hamiltonian polynomial vector field deriving from a Hamiltonian function H: $V + \operatorname{grad} P = (\partial H/\partial y, -\partial H/\partial x) = \operatorname{ham} H$. □

Lemma 1. *Let us consider the 2D ODE*: $\mathrm{d}x/\mathrm{d}t = 0$, $\mathrm{d}y/\mathrm{d}t = 0$ *then its PH-decomposition can be any kernel of the type* $P(x, y) = (f + \mathrm{i}g)(x + \mathrm{i}y) + (f - \mathrm{i}g)(x - \mathrm{i}y)$, $H(x, y) = -\mathrm{i}(f + \mathrm{i}g)(x + \mathrm{i}y) + \mathrm{i}(f - \mathrm{i}g)(x - \mathrm{i}y)$, *$f$ and g being differentiable functions.*

Proof. It is sufficient to check that: $-\partial P/\partial x + \partial H/\partial y = 0$ and $-\partial P/\partial y - \partial H/\partial x = 0$. □

Lemma 2. *Let us consider the 2D ODE*:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x,y) + f_2(x,y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g_1(x,y) + g_2(x,y), \tag{5}$$

where the functions f_1, f_2, g_1, g_2 are infinitely differentiable and integrable. If $f^{[n](m)}(x,y)$ represents the function obtained from f by integrating n times in x and derivating m times in y, then for any integer $k \geqslant 4$, the potential P and the Hamiltonian H defined by:

$$\begin{aligned} P = {} & \sum_{j\leqslant k,\, j=4p+1} -f_1^{[j](j-1)} - g_1^{(j-1)[j]} + \sum_{j\leqslant k,\, j=4p+2} -f_2^{(j-1)[j]} - g_2^{[j](j-1)} \\ & + \sum_{j\leqslant k,\, j=4p+3} f_1^{[j](j-1)} + g_1^{(j-1)[j]} + \sum_{j\leqslant k,\, j=4p+4} f_2^{(j-1)[j]} + g_2^{[j](j-1)}, \\ H = {} & \sum_{j\leqslant k,\, j=4p+1} -g_2^{[j](j-1)} + f_2^{(j-1)[j]} + \sum_{j\leqslant k,\, j=4p+2} f_1^{[j](j-1)} - g_1^{(j-1)[j]} \\ & + \sum_{j\leqslant k,\, j=4p+3} g_2^{[j](j-1)} - f_2^{(j-1)[j]} + \sum_{j\leqslant k,\, j=4p+4} -f_1^{[j](j-1)} + g_1^{(j-1)[j]} \end{aligned}$$

constitute the PH-decomposition of the following ODE's:

- *for* $k = 4p+1$, $\mathrm{d}x/\mathrm{d}t = f_1 + f_2 + g_1^{(k)[k]} - g_2^{[k](k)}$, $\mathrm{d}y/\mathrm{d}t = g_1 + g_2 - f_2^{(k)[k]} + f_1^{[k](k)}$,
- *for* $k = 4p+2$, $\mathrm{d}x/\mathrm{d}t = f_1 + f_2 + f_2^{(k)[k]} + f_1^{[k](k)}$, $\mathrm{d}y/\mathrm{d}t = g_1 + g_2 + g_1^{(k)[k]} + g_2^{[k](k)}$,
- *for* $k = 4p+3$, $\mathrm{d}x/\mathrm{d}t = f_1 + f_2 - g_1^{(k)[k]} + g_2^{[k](k)}$, $\mathrm{d}y/\mathrm{d}t = g_1 + g_2 + f_2^{(k)[k]} - f_1^{[k](k)}$,
- *for* $k = 4p+4$, $\mathrm{d}x/\mathrm{d}t = f_1 + f_2 - f_2^{(k)[k]} - f_1^{[k](k)}$, $\mathrm{d}y/\mathrm{d}t = g_1 + g_2 - g_1^{(k)[k]} - g_2^{[k](k)}$.

If f_1, f_2, g_1, g_2 are polynomials of order less than n, then all these ODE's are equal to (5), *since $k > n$.*

Proof. Let check that: $\mathrm{d}x/\mathrm{d}t = -\partial P/\partial x + \partial H/\partial y$, $\mathrm{d}y/\mathrm{d}t = -\partial P/\partial y - \partial H/\partial x$ and $f_1^{[k](k)} = f_2^{(k)[k]} = g_1^{(k)[k]} = g_2^{[k](k)} = 0$, $k > n$. □

Theorem 1. *Let us consider the 2D ODE*: $\mathrm{d}x/\mathrm{d}t = f(x,y)$, $\mathrm{d}y/\mathrm{d}t = g(x,y)$, *where f and g are polynomials*: $f(x,y) = \sum_{i\geqslant j} a_{ij}x^iy^j + \sum_{i<j} c_{ij}x^iy^j$ *and* $g(x,y) = \sum_{i\geqslant j} b_{ij}x^iy^j + \sum_{i<j} d_{ij}x^iy^j$, *then the general PH-decomposition is given by choosing*:

$$\begin{aligned} P = {} & \sum_{j\leqslant k,\, j=4p+1} -f_1^{[j](j-1)} - g_1^{(j-1)[j]} + \sum_{j\leqslant k,\, j=4p+2} -f_2^{(j-1)[j]} - g_2^{[j](j-1)} \\ & + \sum_{j\leqslant k,\, j=4p+3} f_1^{[j](j-1)} + g_1^{(j-1)[j]} + \sum_{j\leqslant k,\, j=4p+4} f_2^{(j-1)[j]} + g_2^{[j](j-1)} \\ & + (f + \mathrm{i}g)(x + \mathrm{i}y) + f(x - \mathrm{i}g)(x - \mathrm{i}y), \\ H = {} & \sum_{j\leqslant k,\, j=4p+1} -g_2^{[j](j-1)} + f_2^{(j-1)[j]} + \sum_{j\leqslant k,\, j=4p+2} f_1^{[j](j-1)} - g_1^{(j-1)[j]} + \sum_{j\leqslant k,\, j=4p+3} g_2^{[j](j-1)} - f_2^{(j-1)[j]} \\ & + \sum_{j\leqslant k,\, j=4p+4} -f_1^{[j](j-1)} + g_1^{(j-1)[j]} - \mathrm{i}(f + \mathrm{i}g)(x + \mathrm{i}y) + \mathrm{i}(f - \mathrm{i}g)(x - \mathrm{i}y), \end{aligned}$$

where f, g are any differentiable functions and where: $f_1 = \sum_{i>j} a_{ij}x^iy^j + \sum_{i=j} a_{ij}x^iy^j/2$, $f_2 = \sum_{i<j} c_{ij}x^iy^j + \sum_{i=j} a_{ij}x^iy^j/2$, $g_1 = \sum_{i>j} b_{ij}x^iy^j + \sum_{i=j} b_{ij}x^iy^j/2$, $g_2 = \sum_{i<j} d_{ij}x^iy^j + \sum_{i=j} b_{ij}x^iy^j/2$.

Proof. The existence of a PH-decomposition is ensured by the above proposition. Then to obtain P and H, it is sufficient to apply the above Lemmas 1 and 2. □

5. Conclusion

In this article, we have shown the possibility to decompose any sufficiently regular 2D-differential system into two parts: a potential one having a dissipative gradient-like behaviour responsible, in general, for the amplitude of a periodic signal generated by the system, and another one, Hamiltonian, expressing the conservative part of the flow responsible for the frequency of this periodic signal. This decomposition, called PH-decomposition if the potential P and the Hamiltonian H are polynomials, is non-unique, but we easily conceive that we can estimate the coefficients of P and H by obliging the dynamical flow to pass through certain known points on orbits of the studied system. Then we have obtained a way to algebraically estimate these orbits. The algorithm for doing this approximation (in particular in the case of limit cycles) will be given in a further Note.

References

[1] O. Cinquin, J. Demongeot, High-dimensional switches and the modeling of cellular differentiation, J. Theoret. Biol. 233 (2005) 391–411.

[2] M. Clerc, P. Coullet, E. Tirapegui, The stationary instability in quasi-reversible systems and the Lorenz pendulum, Int. J. Bifurcation and Chaos 11 (2001) 591–603.

[3] P. Collela, D.P. Trebovich, Numerical simulation of incompressible viscous flow in deforming domains, Proc. Natl. Acad. Sci. USA 96 (1999) 5378–5381.

[4] J. Demongeot, Theoretical study of attractors in enzymatic systems, in: B.E. Paulré (Ed.), System Dynamics and Analysis of Change, North-Holland, Amsterdam, 1981, pp. 102–122.

[5] J. Demongeot, Existence de solutions périodiques pour une classe de systèmes différentiels gouvernant la cinétique de chaînes enzymatiques oscillantes, in: Lecture Notes in Biomathematics, vol. 41, 1981, pp. 40–62.

[6] J. Demongeot, J. Aracena, F. Thuderoz, T.P. Baum, O. Cohen, Genetic regulation networks: circuits, regulons and attractors, C. R. Acad. Sci. Biologies 326 (2003) 171–188.

[7] J. Demongeot, R. Thomas, M. Thellier, A mathematical model for storage and recall functions in plants, C. R. Acad. Sci., Sciences de la Vie 323 (2000) 93–97.

[8] A. El Kacimi-Alaoui, G. Hector, Basic Hodge decomposition for a Riemannian foliation, Ann. Institut Fourier 36 (1986) 207–227.

[9] M. Isalan, C. Lemerle, L. Serrano, Engineering gene networks to emulate Drosophila embryonic pattern formation, PLoS Biology 3 (2005) 488–496.

[10] T. Iwaniec, C. Scott, B. Stroffolini, Nonlinear Hodge theory on manifolds with boundary, Ann. Mat. Pura Appl. 177 (1999) 37–115.

[11] A. Jolliot, A. Prochiantz, Transduction peptides: from technology to physiology, Nature Cell Biology 6 (2004) 189–196.

[12] A. Lienard, Etude des oscillations entretenues, Revue Générale de l'Electricité 23 (1928) 901–912, 946–954.

[13] A.J. Lotka, Contribution to the theory of periodic reactions, J. Phys. Chem. 14 (1910) 271–279.

[14] A. Nazarenko, Canonical realization of the Poincaré algebra for a relativistic system of charged particles plus electromagnetic field, Proc. Institute of Mathematics of NAS of Ukraine 30 (2000) 343–349.

[15] T. Pham Dinh, J. Demongeot, P. Baconnier, G. Benchetrit, Simulation of a biological oscillator: the respiratory rhythm, J. Theoret. Biol. 103 (1983) 113–132.

[16] W. Quapp, Reaction pathways and projection operators: application to string methods, J. Comput. Chem. 25 (2004) 1277–1285.

[17] F.K. Skinner, N. Kopell, E. Marder, Mechanisms for oscillation and frequency control in reciprocally inhibitory model neural networks, J. Comput. Neurosci. 1 (1994) 69–87.

[18] A.L. Taylor, G.W. Cottrell, W.B. Kristan, Analysis of oscillations in a reciprocally inhibitory network with synaptic depression, Neural Computation 14 (2002) 561–581.

[19] M. Thellier, J. Demongeot, J. Guespin, C. Ripoll, V. Norris, R. Thomas, A logical (discrete) formulation model for the storage and recall of environmental signals in plants, Plant Biol. 10 (2004) 1055–1075.

[20] B. van der Pol, J. van der Mark, The heartbeat considered as a relaxation oscillation and an electrical model of the heart, Philos. Mag. 6 (1928) 763–775.

[21] B. van der Pol, J. van der Mark, Le battement du cœur considéré comme oscillation de relaxation et un modèle électrique du cœur, Onde élec. (1928) 365–392.

[22] V. Volterra, Fluctuations dans la lutte pour la vie, leurs lois fondamentales et de réciprocité, Bull. Soc. Math. France 67 (1939) 135–151.

[23] X.J. Wang, J. Rinzel, Alternating and synchronous rhythms in reciprocally inhibitory model neurons, Neural Computation 4 (1992) 44–97.