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Numerical Analysis/Partial Differential Equations

Singular limits for the Riemann problem: general diffusion, relaxation, and boundary conditions

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Abstract

We consider self-similar approximations of non-linear hyperbolic systems in one space dimension with Riemann initial data, especially the system $\partial_t u_\varepsilon + A(u_\varepsilon)\partial_x u_\varepsilon = \varepsilon t \partial_x(B(u_\varepsilon)\partial_x u_\varepsilon)$, with $\varepsilon > 0$. We assume that the matrix $A(u)$ is strictly hyperbolic and that the diffusion matrix satisfies $|B(u) - \mathrm{Id}| \ll 1$. No genuine non-linearity assumption is required. We show the existence of a smooth, self-similar solution $u_\varepsilon = u_\varepsilon(x/t)$ which has bounded total variation, uniformly in the diffusion parameter $\varepsilon > 0$. In the limit $\varepsilon \to 0$, the functions u_ε converge towards a solution of the Riemann problem associated with the hyperbolic system. A similar result is established for the relaxation approximation $\partial_t u^\varepsilon + \partial_x v^\varepsilon = 0$, $\partial_t v^\varepsilon + a^2 B(u)\partial_x u^\varepsilon = (f(u^\varepsilon) - v^\varepsilon)/(\varepsilon t)$. We also cover the boundary-value problem in a half-space for the same regularizations. ***To cite this article: K.T. Joseph, P.G. LeFloch, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Limites singulières pour le problème de Riemann : diffusion, relaxation et conditions aux limites. Nous considérons les approximations auto-semblables d'un système hyperbolique non-linéaire à une dimension d'espace avec donnée initiale de type « problème de Riemann », en particulier le système $\partial_t u_\varepsilon + A(u_\varepsilon)\partial_x u_\varepsilon = \varepsilon t \partial_x(B(u_\varepsilon)\partial_x u_\varepsilon)$, avec $\varepsilon > 0$. Nous supposons que la matrice $A(u)$ est strictement hyperbolique et que la matrice de diffusion satisfait $|B(u) - \mathrm{Id}| \ll 1$. Aucune hypothèse de « vraie non-linéarité » n'est imposée. Nous démontrons que ce problème admet une solution régulière, auto-semblable $u_\varepsilon = u_\varepsilon(x/t)$ de variation totale uniformément bornée par rapport au paramètre de diffusion $\varepsilon > 0$. Lorsque $\varepsilon \to 0$, les fonctions u_ε convergent vers une solution du problème de Riemann associé au système hyperbolique. Nous établissons aussi un résultat analogue pour les approximations par relaxation données par $\partial_t u^\varepsilon + \partial_x v^\varepsilon = 0$, $\partial_t v^\varepsilon + a^2 B(u)\partial_x u^\varepsilon = (f(u^\varepsilon) - v^\varepsilon)/(\varepsilon t)$. Ces résultats sont finalement étendus au problème de Riemann associé à ces mêmes régularisations et posé dans un demi-espace avec condition au bord. ***Pour citer cet article : K.T. Joseph, P.G. LeFloch, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

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A la suite de [1,2,13] et de nos précédents travaux [5–8], nous nous intéressons aux régularisations auto-semblables du problème de Riemann associées à un système strictement hyperbolique non-linéaire. (Voir (1), (2) dans la version en anglais.) Nous étudions en particulier le système

$$\partial_t u_\varepsilon + A(u_\varepsilon)\partial_x u_\varepsilon = \varepsilon t \partial_x \big(B(u_\varepsilon)\partial_x u_\varepsilon\big), \quad \varepsilon > 0,$$

dans le cas où la matrice de diffusion $B = B(u)$ est suffisamment proche de la matrice identité

$$\sup_{u \in \mathcal{B}_{\delta_0}} \big|B(u) - \mathrm{Id}\big| \leqslant \eta \ll 1.$$

La méthode générale introduite ici peut s'étendre à d'autres régularisations. Rappelons que les solutions des systèmes non-conservatifs [10,11,3] et des problèmes avec conditions au bord [5] dépendent des termes de régularisation. Il est donc important d'inclure des matrices de diffusion générales $B \neq \mathrm{Id}$ dans notre analyse. En posant le problème de Riemann sur un intervalle $[-L, L]$ suffisament grand (de manière à inclure toutes les ondes), nous démontrons le résultat suivant :

Théorème 0.1. *Considérons le système hyperbolique non-linéaire* (1) *et sa régularisation parabolique* (3). *Il existe alors des constantes* (*suffisamment petites*) $\delta_1, \eta > 0$ *et une constante* (*suffisamment grande*) $C_0 > 0$ *telles que pour toute données initiales* $u_l, u_r \in \mathcal{B}_{\delta_1}$ (*la boule de* $\mathbb{R}^N$ *de centre* 0 *et de rayon* δ_1) *le problème de Riemann avec diffusion auto-semblable* (2)–(3) *admette une solution régulière* $u^\varepsilon = u^\varepsilon(x/t) \in \mathcal{B}_{\delta_0}$ *définie pour tout* $y = x/t \in [-L, L]$ *et de variation totale uniformément bornée,*

$$TV_{-L}^{L}(u_\varepsilon) \leqslant C_0 |u_r - u_l|,$$

qui, de plus, converge fortement vers une limite $u : [-L, L] \to \mathcal{B}_{\delta_0}$, *c'est-à-dire* $u^\varepsilon \to u$ *en norme* L^1 *lorsque* $\varepsilon \to 0$. *Cette limite vérifie les propriétés suivantes. La fonction* $y \mapsto u(y)$ *est de variation totale bornée,*

$$TV_{-L}^{L}(u) \leqslant C_0 |u_r - u_l|,$$

et est constante sur des intervalles de la forme $[\bar{\Lambda}_j, \underline{\Lambda}_{j+1}]$. (*Voir la version en anglais.*) *Lorsque* (1) *est un système de lois de conservation, c'est-à-dire* $A = Df$ *pour un certain flux* $f : \mathcal{B}_{\delta_0} \to \mathbb{R}^N$, *la limite est alors une solution au sens des distributions de*

$$\partial_t u + \partial_x f(u) = 0.$$

Nous obtenons aussi une description précise des courbes d'ondes générées par cette approximation, et étudions de la même façon un modèle de relaxation ainsi que le problème aux limites posé dans un demi-espace.

1. Introduction

In this Note we present a continuation of the work by the authors [5–8] on self-similar regularizations of the Riemann problem associated with a non-linear strictly hyperbolic system in one-space dimension:

$$\partial_t u + A(u)\partial_x u = 0, \quad u = u(t, x) \in \mathcal{B}_{\delta_0},\ t > 0,\ x \in \mathbb{R}, \tag{1}$$

with piecewise constant, initial data

$$u(0, x) = u_l \quad \text{for } x < 0; \quad u_r \quad \text{for } x > 0, \tag{2}$$

where u_l, u_r are constant states in $\mathcal{B}_{\delta_0}$. Here, $\mathcal{B}_{\delta_0} \subset \mathbb{R}^N$ denotes the ball centered at the origin and with radius $\delta_0 > 0$, and, for all $u \in \mathcal{B}_{\delta_0}$, $A(u)$ is assumed to have distinct and real eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$ and basis of left- and right-eigenvectors $l_j(u), r_j(u)$ $(1 \leqslant j \leqslant N)$.

Following Dafermos [1,2] and Slemrod [13] who advocate the use of self-similar regularizations to capture the whole wave fan structure of the Riemann problem, we consider solutions constructed by self-similar vanishing diffusion associated with a *general* diffusion matrix $B = B(u)$, that is we search for solutions of

$$\partial_t u_\varepsilon + A(u_\varepsilon)\partial_x u_\varepsilon = \varepsilon t \partial_x \big(B(u_\varepsilon)\partial_x u_\varepsilon\big), \quad \varepsilon > 0. \tag{3}$$

Due to the choice of the scaling εt, this system admits solutions $u_\varepsilon = u_\varepsilon(x/t)$, and, therefore, we refer to (2), (3) as the *self-similar diffusive Riemann problem.* The matrix $B = B(u)$ is assumed to depend smoothly upon u and to remain sufficient close to the identity matrix, that is, for some given matrix norm and for $\eta > 0$ sufficiently small

$$\sup_{u\in\mathcal{B}_{\delta_0}} \left|B(u) - \mathrm{Id}\right| \leqslant \eta. \tag{4}$$

The method of analysis introduced below is not a priori restricted to (3), (4), and generalizations are discussed at the end of this Note and in [8,9].

The techniques developed so far for general hyperbolic systems (see [14,12] and [5–7]) were restricted to regularizations based on the identity diffusion matrix. The new approach introduced here allows us to cover classes of approximations based on general diffusion matrices (or relaxation terms, see below). This degree of generality is especially important for non-conservative systems [10,11] and for the boundary-value problem [5], whose solutions are known to *strongly depend* upon the specific regularization.

2. Main results

By the property of propagation at finite speed, a self-similar solution $u = u(y)$ of the Riemann problem is constant outside a sufficiently large, compact interval $[-L, L]$, i.e.: $u(y) = u_l$ for $y < -L$ and $u(y) = u_r$ for $y > L$. As is customary, we assume that δ_0 is sufficiently small so that the wave speeds $\lambda_j(u)$ remain close to the constant speeds $\lambda_j(0)$ and are uniformly separated in the sense that

$$\underline{\Lambda}_j \leqslant \lambda_j(u) \leqslant \bar{\Lambda}_j, \quad u \in \mathcal{B}_{\delta_0},$$

for some constants $-L < \underline{\Lambda}_1 < \bar{\Lambda}_1 < \underline{\Lambda}_2 < \cdots < \underline{\Lambda}_N < \bar{\Lambda}_N < L$.

We establish the following theorem:

Theorem 2.1. *Consider the non-linear, strictly hyperbolic system* (1) *together with its parabolic regularization* (3), (4). *There exist* (*sufficiently small*) *constants* $\delta_1, \eta > 0$ *and a* (*sufficiently large*) *constant* $C_0 > 0$ *such that for any initial data* $u_l, u_r \in \mathcal{B}_{\delta_1}$ *the self-similar diffusive Riemann problem* (2), (3) *admits a smooth solution* $u^\varepsilon = u^\varepsilon(x/t) \in \mathcal{B}_{\delta_0}$ *defined for all* $y = x/t \in [-L, L]$*, which has uniformly bounded variation,*

$$TV_{-L}^{L}(u_\varepsilon) \leqslant C_0|u_r - u_l|,$$

and converges strongly to some limit $u : [-L, L] \to \mathcal{B}_{\delta_0}$:

$$u^\varepsilon \to u \quad \text{in the } L^1 \text{ norm, as } \varepsilon \to 0.$$

The limit function satisfies the following properties. The function $y \mapsto u(y)$ *has bounded total variation, that is,* $TV_{-L}^{L}(u) \leqslant C_0|u_r - u_l|$*, and is constant on each interval* $[\bar{\Lambda}_j, \underline{\Lambda}_{j+1}]$*. If* (1) *is a system of conservation laws, i.e.* $A = Df$ *for some flux* $f : \mathcal{B}_{\delta_0} \to \mathbb{R}^N$*, then the limit is a distributional solution of*

$$\partial_t u + \partial_x f(u) = 0. \tag{5}$$

If $(U, F) : \mathcal{B}_{\delta_1} \to \mathbb{R} \times \mathbb{R}^N$ *is an entropy/entropy flux pair associated with* (5) *and the diffusion matrix satisfies the convexity-like condition* $\nabla^2 U \cdot B \geqslant 0$*, then the solution* u *satisfies the entropy inequality*

$$\partial_t U(u) + \partial_x F(u) \leqslant 0. \tag{6}$$

We have also the following description of the wave curves:

Theorem 2.2. *With the notation and assumptions in Theorem* 2.1*, to each* j*-characteristic family and each left-hand state* u_l *one can associate a* j-**wave curve**

$$\mathcal{W}_j(u_l) := \left\{u_r = \psi_j(m; u_l) \mid m \in (\underline{m}_j, \overline{m}_j)\right\},$$

issuing from u_l*, which, by definition, is made of all right-hand states* u_r *attainable by a Riemann solution* $u = u(y)$*, with left-hand state* u_l*, by using only* j*-waves, that is such that*

$$u(y) = u_l \quad \text{for } y < \underline{\Lambda}_j; \quad u_r \quad \text{for } y > \bar{\Lambda}_j.$$

Moreover, the mapping $\psi_j : (\underline{m}_j, \overline{m}_j) \times \mathcal{B}_{\delta_1} \to \mathcal{B}_{\delta_0}$ *is Lipschitz continuous with respect to both arguments, and for some small constant* $c > 0$

$$\partial_m \psi_j(m; u_l) \in \mathcal{C}_j := \{ w \in \mathbb{R}^N \mid |w \cdot l_j(0)| \geqslant (1-c)|w| \}.$$

Moreover, the characteristic component $y \mapsto \alpha_j(y) := l_j(0) \cdot u'(y)$ *is a non-negative measure in the interval* $[\underline{\Lambda}_j, \bar{\Lambda}_j]$.

For the proof of these results as well as a characterization of the limit when (1) is a general non-conservative system, we refer to [8,9].

We will here sketch the proof of Theorem 2.1. To handle general diffusion matrix $B(u)$, the following *generalized eigenvalue problem* is introduced:

$$\begin{aligned}
&(-y + A(u))\hat{r}_j(u,y) = \mu_j(u,y) B(u) \hat{r}_j(u,y),\\
&\hat{l}_j(u,y) \cdot (-y + A(u)) = \mu_j(u,y) \hat{l}_j(u,y) \cdot B(u).
\end{aligned}$$

In view of (4), one has $\hat{r}_j(u,y) = r_j(u) + \mathrm{O}(\eta)$ and $\hat{l}_j(u,y) = l_j(u) + \mathrm{O}(\eta)$. The proof relies on a suitable *asymptotic expansion* of the solution $u_\varepsilon = u_\varepsilon(x/t)$, of the form

$$u'_\varepsilon = \sum_j a_j^\varepsilon \hat{r}_j(u_\varepsilon, \cdot) \quad \text{with } a_j^\varepsilon := \hat{l}_j(u_\varepsilon, \cdot) \cdot u'_\varepsilon.$$

Omitting ε, we deduce that the components a_j satisfy a *coupled system of N differential equations*:

$$a'_i - \frac{\mu_i(u,\cdot)}{\varepsilon} a_i + \sum_j \pi_{ij}(u,\cdot) a_j = Q_i(u;\cdot) := \sum_{j,k} \kappa_{ijk}(u,\cdot) a_j a_k,$$

where

$$\begin{aligned}
&\pi_{ij}(u,\cdot) := \hat{l}_i(u,\cdot) \cdot B(u) \partial_y \hat{r}_j(u,\cdot),\\
&\kappa_{ijk}(u,\cdot) := -\hat{l}_i(u,\cdot) \cdot D_u(B\hat{r}_k)(u,\cdot)\hat{r}_j(u,\cdot).
\end{aligned}$$

The system under study has the form

$$a'_i - \frac{\mu_i(u,\cdot)}{\varepsilon} a_i + \mathrm{O}(\eta) \sum_j |a_j| = \mathrm{O}(1) \sum_{j,k} |a_j||a_k|.$$

In a central part of our argument we study the homogeneous system

$$\varphi'_i - \frac{\mu_i(u,\cdot)}{\varepsilon} \varphi_i + \sum_j \pi_{ij}(u,\cdot) \varphi_j = 0, \quad \varphi = (\varphi_1, \ldots, \varphi_N), \tag{7}$$

and establish that it has solutions φ_j, referred to as the *linearized i-wave measures* associated with the function u, which are 'close' (in a sense to be specified) to the following normalized solutions of the corresponding uncoupled system (obtain by setting $\eta = 0$):

$$\varphi_i^\star := \frac{\mathrm{e}^{-g_i/\varepsilon}}{I_i}, \quad I_i := \int_{-L}^{L} \mathrm{e}^{-g_i/\varepsilon} \, \mathrm{d}y, \quad g_i(y) := -\int_{\rho_i}^{y} \mu_i(u(x), x) \, \mathrm{d}x.$$

Here, the constants ρ_i are determined so that the functions g_i are non-negative.

Theorem 2.3. *The system* (7) *admits a solution* φ *such that for all* $i = 1, \ldots, N$ *and* $y \in [-L, L]$

$$(1 - \mathrm{O}(\eta))\varphi_i^\star(y) - \varepsilon \mathrm{O}(\eta) \sum_j \varphi_j^\star(y) \leqslant \varphi_i(y) \leqslant (1 + \mathrm{O}(\eta))\varphi_i^\star(y) + \varepsilon \mathrm{O}(\eta) \sum_j \varphi_j^\star(y).$$

In contrast with the functions $\varphi_i^\star$, the functions φ_i need not be positive. Next, to control the total variation of the solutions of (3), we derive Glimm-like estimates on the *wave interaction coefficients*

$$F_{ijk}^\star(y) := \varphi_i^\star(y) \int_{c_i}^{y} \frac{\varphi_j^\star \varphi_k^\star}{\varphi_i^\star} \, dx$$

for some constants $c_i \in [\underline{\Lambda}_i, \bar{\Lambda}_i]$. We gain useful information on the possible growth of the total variation of solutions. Roughly speaking, the coefficient F_{ijk} bounds the contribution to the i-th family due to interactions between waves of the j-th and k-th characteristic families.

3. Generalizations

The results above have been also extended to relaxation approximations and boundary-value problems. In particular, we can handle relaxation approximations associated with the conservative system (5)

$$\partial_t u^\varepsilon + \partial_x v^\varepsilon = 0, \qquad \partial_t v^\varepsilon + a^2 B(u) \partial_x u^\varepsilon = \frac{1}{\varepsilon t}\big(f(u^\varepsilon) - v^\varepsilon\big), \tag{8}$$

where $u^\varepsilon = u^\varepsilon(x,t)$ and $v^\varepsilon = v^\varepsilon(x,t)$ are the unknowns, and $\varepsilon > 0$ is a relaxation parameter.

We also study (3) and (8) in the presence of a boundary, when there exists an index p such that $0 < \bar{\Lambda}_p$, and that at most one wave family is characteristic, that is, $0 \in (\underline{\Lambda}_p, \bar{\Lambda}_p)$. We consider (1) on the interval $y \in [0, L]$, and prove the existence of a solution with uniformly bounded variation. To handle the boundary layer, we modify the previous definition of the functions $\varphi_j^\star$, $j \leqslant p$, and carefully estimate the coefficients $F_{ijk}^\star(y)$ when $\varepsilon \to 0$.

In addition, following pioneering work by Fan and Slemrod [4] who studied the effect of artificial viscosity terms, we consider a system arising in liquid–vapor phase dynamics with *physical* viscosity and capillarity effects taken into account. We establish uniform total variation bounds, allowing us to deduce new existence results. Our analysis cover both the hyperbolic and the hyperbolic-elliptic regimes and apply to arbitrarily large Riemann data. The proofs rely on a new technique of reduction to two coupled scalar equations associated with the two wave fans of the system. Strong L^1 convergence to a weak solution of bounded variation is established in the hyperbolic regime, while in the hyperbolic-elliptic regime a stationary singularity near the axis separating the two wave fans, or more generally an almost-stationary oscillating wave pattern (of thickness depending upon the capillarity-viscosity ratio) are observed which prevent the solution to have globally bounded variation.

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References

[1] C.M. Dafermos, Solution of the Riemann problem for a class of hyperbolic conservation laws by the viscosity method, Arch. Rational Mech. Anal. 52 (1973) 1–9.
[2] C.M. Dafermos, Admissible wave fans in nonlinear hyperbolic systems, Arch. Rational Mech. Anal. 106 (1989) 243–260.
[3] G. Dal Maso, P.G. LeFloch, F. Murat, Definition and weak stability of nonconservative products, J. Math. Pure Appl. 74 (1995) 483–548.
[4] H.-T. Fan, M. Slemrod, Dynamic flows with liquid/vapor phase transitions, in: Handbook of Mathematical Fluid Dynamics, vol. I, North-Holland, Amsterdam, 2002, pp. 373–420.
[5] K.T. Joseph, P.G. LeFloch, Boundary layers in weak solutions to hyperbolic conservation laws, Arch. Rational Mech. Anal. 147 (1999) 47–88.
[6] K.T. Joseph, P.G. LeFloch, Boundary layers in weak solutions of hyperbolic conservation laws II. Self-similar vanishing diffusion limits, Comm. Pure Appl. Anal. 1 (2002) 51–76.
[7] K.T. Joseph, P.G. LeFloch, Boundary layers in weak solutions of hyperbolic conservation laws III. Vanishing relaxation limits, Portugal. Math. 59 (2002) 453–494.
[8] K.T. Joseph, P.G. LeFloch, Singular limits for the Riemann problem: general diffusion, relaxation, and boundary conditions, in: O. Rozanova (Ed.), Analytical Approaches to Multidimensional Balance Laws, Nova Publishers, 2005.
[9] K.T. Joseph, P.G. LeFloch, Singular limits in phase dynamics with physical viscosity and capillarity, Proc. Roy. Soc. Edinburgh Sect. A, in press.

[10] P.G. LeFloch, Shock waves for nonlinear hyperbolic systems in nonconservative form, Institute for Math. and its Appl., Minneapolis, Preprint # 593, 1989.
[11] P.G. LeFloch, Hyperbolic Systems of Conservation Laws: The Theory of Classical and Nonclassical Shock Waves, Lecture Notes in Mathematics, ETH Zuerich, Birkhäuser, 2002.
[12] P.G. LeFloch, A. Tzavaras, Existence theory for the Riemann problem for nonconservative hyperbolic systems, C. R. Acad. Sci. Paris Ser. I 323 (1996) 347–352.
[13] M. Slemrod, A limiting viscosity approach to the Riemann problem for materials exhibiting change of phase, Arch. Rational Mech. Anal. 105 (1989) 327–365.
[14] A.E. Tzavaras, Wave interactions and variation estimates for self-similar viscous limits in systems of conservation laws, Arch. Rational Mech. Anal. 135 (1996) 1–60.