



## Algebraic Geometry

# Finite Abelian subgroups of the Cremona group of the plane

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### Abstract

We present in this Note some results on conjugacy classes of finite Abelian subgroups of the Cremona group of the plane. *To cite this article: J. Blanc, C. R. Acad. Sci. Paris, Ser. I 344 (2007).*

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### Résumé

**Sous-groupes abéliens finis du groupe de Cremona du plan.** On présente dans cette Note quelques résultats sur les classes de conjugaisons de sous-groupes abéliens finis du groupe de Cremona du plan. *Pour citer cet article : J. Blanc, C. R. Acad. Sci. Paris, Ser. I 344 (2007).*

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### Version française abrégée

Le groupe de Cremona du plan, noté  $Cr(\mathbb{P}^2)$ , est le groupe des transformations birationnelles du plan complexe  $\mathbb{P}^2(\mathbb{C})$ . Ce très grand groupe est un sujet de recherche depuis de longues années. L'étude des classes de conjugaisons de ses sous-groupes finis est primordiale pour la compréhension de sa structure.

On présente ici un historique des résultats marquants sur le sujet, du XIX<sup>ème</sup> siècle à nos jours, puis on donne un aperçu des résultats de [4]. Parmi ceux-ci, on donne à la Section 3 une liste précise classifiant les sous-groupes finis cycliques du groupe de Cremona. (La liste analogue pour les groupes abéliens finis non-cycliques est également présente dans [4], mais trop longue pour paraître ici.) À l'intérieur de ces listes, deux groupes à des lignes séparées sont assurés d'être dans des classes de conjugaisons distinctes, alors que les classes de conjugaisons précises pour chaque ligne (lorsque les groupes dépendent de paramètres) ne sont pas explicitement données et pas toujours connues. On fait remarquer que les courbes de points fixés par un élément du groupe, ainsi que l'action du groupe entier sur ces courbes, permettent souvent de distinguer les classes de conjugaisons ; c'est ce qui a été fait dans la majorité des cas dans [4]. Bien que la classification ne soit pas totalement précisée, on peut en déduire quelques résultats plus généraux, qui sont présentés à la Section 4 :

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- Dans le groupe de Cremona, il existe une infinité de classes de conjugaisons d'éléments d'ordre  $n$ , pour tout entier  $n$  pair, alors que ceci est faux si  $n$  est impair (Théorème 4.1).
- Les éléments du groupe de Cremona qui sont racines d'un automorphisme linéaire non-trivial sont conjuguées à un automorphisme linéaire (Théorème 4.2).
- Les groupes abéliens finis de transformations de Cremona qui fixent une courbe de genre positif sont cycliques, d'ordre 2, 3, 4, 5 ou 6. De plus, en enlevant le cas des courbes elliptiques il ne reste plus que des groupes d'ordre 2 ou 3 (Théorème 4.3).
- Les groupes cycliques finis de transformations de Cremona dont aucun des éléments non-triviaux ne fixe une courbe non-rationnelle sont conjugués à un groupe d'automorphismes linéaires (Théorème 4.4).
- Le même résultat n'est pas vrai pour les groupes abéliens finis ; il existe un unique contre-exemple (Théorème 4.5).
- La liste des classes d'isomorphismes de sous-groupes abéliens finis du groupe de Cremona est donnée au Théorème 4.6.

## 1. Introduction

The Cremona group of the plane, denoted by  $Cr(\mathbb{P}^2)$ , is the group of birational maps of the surface  $\mathbb{P}^2(\mathbb{C})$  (or  $\mathbb{C}^2$ ), or equivalently, the group of  $\mathbb{C}$ -automorphisms of the field  $\mathbb{C}(X, Y)$ .

This very large group has been a subject of research for many years. Some presentations of the group by generators and relations are available (see [8] and [10]), but these do not provide substantial insight into the algebraic properties of the group. For example, given an abstract group, it is not possible to say whether it is isomorphic to a subgroup of the Cremona group. Moreover, the results of [8] and [10] do not allow one to decide whether the Cremona group is isomorphic to a linear group. The study of the finite subgroups of the Cremona group is therefore fundamental to the understanding of this group.

The study of finite subgroups of the Cremona group was begun over one hundred years ago and it is still a modern subject of research. We present in this Note an overview of the classification made in [4] of Abelian finite subgroups (Section 3), and give also some results that follow from this classification (Section 4). Some of these results will be proved separately in future articles. We give also an historical overview of the research in finite subgroups of the Cremona group of the plane (Section 2).

## 2. A long history of results

Let us give a historical review of some main results on the classification of conjugacy classes of finite subgroups of the Cremona group:

- In 1895, S. Kantor [12] and A. Wiman [14] gave a description of finite subgroups. The list is exhaustive, but not precise in two respects:
  - Given some abstract finite group, it is not possible using their list to say whether this group is isomorphic to a subgroup of the Cremona group.
  - The possible conjugation between the groups of the list is not considered.
- A great deal of work was done by the Russian school since the 1960's. Our main interest is in the classification of minimal  $G$ -surfaces into automorphisms of Del Pezzo surfaces and of conic bundles. This result is due to Yu. Manin [13] in the Abelian case and V.A. Iskovskikh [9] in the general case. The description of decompositions of birational maps into elementary links [11] is also a useful tool.
- The modern approach of the study of finite subgroups of the Cremona group has started with the work of L. Bayle and A. Beauville on involutions [1]. They used [13] to classify the subgroups of order 2 of the Cremona group. This is the first example of a precise description of conjugacy classes. It is shown that the non-rational curves fixed by the groups determine the conjugacy classes.
- The techniques of [1] were generalised by T. de Fernex [6] to cyclic subgroups of prime order. The list is as precise as one can wish, except for two classes of groups of order 5, for which the question of their conjugacy is not answered.

The complete classification was obtained in [3], by A. Beauville and the author. For example, the following result was proved: a cyclic group of prime order is conjugate to a linear automorphism of the plane if and only if it does not fix a non-rational curve.

- A. Beauville has further classified the  $p$ -elementary maximal subgroups up to conjugation [2]. He obtained for example the following results:
  - No group  $(\mathbb{Z}/p\mathbb{Z})^3$  belongs to the Cremona group if  $p$  is a prime  $\neq 2, 3$ .
  - There exist infinitely many conjugacy classes of subgroups isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^4$ .
 Note that the conjugacy classes of subgroups isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^4$  of the de Jonquières group are well described ([2], Proposition 2.6). However it is not clear whether two groups non-conjugate in the de Jonquières group are conjugate in the Cremona group.
- More recently, I.V. Dolgachev and V.A. Iskovskikh [7] had updated the list of S. Kantor and A. Wiman, using the modern theory of  $G$ -surfaces, the theory of elementary links, and the conjugacy classes of Weyl groups. This manuscript in preparation contains many new results and is currently the most precise classification of conjugacy classes of finite subgroups. However, some questions remain open.

### 3. The lists

We present in Theorem 3.1 below the list of finite cyclic subgroups of the Cremona group, up to conjugation. Every two lines of the list (named as in [4] with some symbol [#.#]) represent disjoint conjugacy classes of finite cyclic subgroups. However, we do not make precise the conjugacy class within the same line, when this depends on parameters. Remark that the curves fixed by one element of the group, and the action of the whole group on these curves, are often sufficient to distinguish the conjugacy classes. This was done in [4] in many cases, but some remain unsolved. Note that the result analogous to Theorem 3.1 for Abelian non-cyclic groups is given in [4], but is too long to be written here.

Before to state the result, let us recall that birational maps of rational surfaces represent conjugacy classes of elements of the Cremona group.

**Theorem 3.1** (*Classification of finite cyclic subgroups*). *Let  $G \subset Cr(\mathbb{P}^2)$  be a finite cyclic subgroup of order  $n$ . It is conjugate, in the Cremona group, to the group generated by  $\alpha$ , where  $\alpha$  is one (and only one) of the following: (we use the notations  $\zeta_m = e^{2i\pi/m}$ ,  $\omega = \zeta_3$ , and note  $[\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4]$  for the restriction of the automorphism  $(w : x : y : z) \mapsto (\alpha_1 w : \alpha_2 x : \alpha_3 y : \alpha_4 z)$  of the weighted projective space  $\mathbb{P}(n_1, n_2, n_3, n_4)$ ):*

- **roots of de Jonquières involutions**  
 **$n = 2m \geq 2$ ,**  
 $\alpha^m$  is a de Jonquières involution that fixes a hyperelliptic curve  $\Gamma$  of geometric genus  $k - 1 > 0$ , and one of the two following cases occurs:
  - [C.ro.m]  **$m$  is odd:**  
 $\alpha : (x_1 : x_2) \times (y_1 : y_2) \dashrightarrow (x_1 : \zeta_m x_2) \times (y_2 \prod_{i=1}^k (x_1 - b_i x_2) : y_1 \prod_{i=1}^k (x_1 - a_i x_2))$  is a birational transformation of  $\mathbb{P}^1 \times \mathbb{P}^1$ , where  $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{C}^*$  are all distinct, and the sets  $\{a_1, \dots, a_k\}$  and  $\{b_1, \dots, b_k\}$  are both invariant by multiplication by  $\zeta_m$ .
  - [C.re.m]  **$m$  is even:**  
 $\alpha$  is an automorphism of some conic bundle  $(S, \pi)$  with  $l$  singular fibres. The action of  $\alpha$  on the ramification set of  $\pi : \Gamma \rightarrow \mathbb{P}^1$  has  $r$  orbits, each consisting of  $m$  points (so  $rm = 2k$ ).  
 Furthermore, one of the following holds:
    - $l = 2k$ ,  $r$  is even,  $\alpha$  acts on  $\Gamma$  with 4 fixed points;
    - $l = 2k + 1$ ,  $r$  is odd,  $\alpha$  acts on  $\Gamma$  with 2 fixed points;
    - $l = 2k + 2$ ,  $r$  is even,  $\alpha$  fixes no point of  $\Gamma$ .
- **Linear automorphisms of  $\mathbb{P}^2$**   
 [0.n]  $\alpha : (x : y : z) \mapsto (x : y : \zeta_n z)$  is a linear automorphism of  $\mathbb{P}^2$ .
- **Automorphisms of Del Pezzo surfaces**  
 See Table 1.

Table 1

Automorphisms of Del Pezzo surfaces: here  $L_i$  (a form of degree  $i$ ) and  $\lambda, \mu \in \mathbb{C}$  are parameters such that the surfaces given are smooth

$n$	Name of the class	Description of $\alpha$	Equation of the surface	In the space
2	[2.G]	$[-1 : 1 : 1 : 1]$	$w^2 + L_4(x, y, z)$	$\mathbb{P}(2, 1, 1, 1)$
2	[1.B]	$[-1 : 1 : 1 : 1]$	$w^2 + z^3 + zL_4(y, z) + L_6(y, z)$	$\mathbb{P}(3, 1, 1, 2)$
3	[3.3]	$[\omega : 1 : 1 : 1]$	$w^3 + L_3(x, y, z)$	$\mathbb{P}^3$
3	[1. $\rho$ ]	$[1 : 1 : 1 : \omega]$	$w^2 = z^3 + L_6(x, y)$	$\mathbb{P}(3, 1, 1, 2)$
4	[2.4]	$[1 : 1 : 1 : \mathbf{i}]$	$w^2 = L_4(x, y) + z^4$	$\mathbb{P}(2, 1, 1, 1)$
4	[1.B2.2]	$[\mathbf{i} : 1 : -1 : -1]$	$w^2 = z^3 + zL_2(x^2, y^2) + xyL'_2(x^2, y^2)$	$\mathbb{P}(3, 1, 1, 2)$
5	[1.5]	$[1 : 1 : \zeta_5 : 1]$	$w^2 = z^3 + \lambda x^4 z + x(\mu x^5 + y^5)$	$\mathbb{P}(3, 1, 1, 2)$
6	[3.6.1]	$[\omega : 1 : 1 : -1]$	$w^3 + x^3 + y^3 + xz^2 + \lambda yz^2$	$\mathbb{P}^3$
6	[3.6.2]	$[1 : -1 : \omega : \omega^2]$	$wx^2 + w^3 + y^3 + z^3 + \lambda wyz$	$\mathbb{P}^3$
6	[2.G3.1]	$[-1 : 1 : 1 : \omega]$	$w^2 = L_4(x, y) + z^3 L_1(x, y)$	$\mathbb{P}(2, 1, 1, 1)$
6	[2.G3.2]	$[-1 : 1 : \omega : \omega^2]$	$w^2 = x(x^3 + y^3 + z^3) + yzL_1(x^2, yz)$	$\mathbb{P}(2, 1, 1, 1)$
6	[2.6]	$[-1 : \omega : 1 : -1]$	$w^2 = x^3 y + y^4 + z^4 + \lambda y^2 z^2$	$\mathbb{P}(2, 1, 1, 1)$
6	[1. $\sigma\rho$ ]	$[-1 : 1 : 1 : \omega]$	$w^2 = z^3 + L_6(x, y)$	$\mathbb{P}(3, 1, 1, 2)$
6	[1. $\rho$ 2]	$[1 : 1 : -1 : \omega]$	$w^2 = z^3 + L_3(x^2, y^2)$	$\mathbb{P}(3, 1, 1, 2)$
6	[1.B3.1]	$[-1 : 1 : \omega : 1]$	$w^2 = z^3 + xL_1(x^3, y^3)z + L_2(x^3, y^3)$	$\mathbb{P}(3, 1, 1, 2)$
6	[1.B3.2]	$[-1 : 1 : \omega : \omega]$	$w^2 = z^3 + \lambda x^2 y^2 z + L_2(x^3, y^3)$	$\mathbb{P}(3, 1, 1, 2)$
6	[1.6]	$[1 : 1 : -\omega : 1]$	$w^2 = z^3 + \lambda x^4 z + \mu x^6 + y^6$	$\mathbb{P}(3, 1, 1, 2)$
8	[1.B4.2]	$[\zeta_8 : 1 : \mathbf{i} : -\mathbf{i}]$	$w^2 = \lambda x^2 y^2 z + xy(x^4 + y^4)$	$\mathbb{P}(3, 1, 1, 2)$
9	[3.9]	$[\zeta_9 : 1 : \omega : \omega^2]$	$w^3 + xz^2 + x^2 y + y^2 z$	$\mathbb{P}^3$
10	[1.B5]	$[-1 : 1 : \zeta_5 : 1]$	$w^2 = z^3 + \lambda x^4 z + x(\mu x^5 + y^5)$	$\mathbb{P}(3, 1, 1, 2)$
12	[3.12]	$[\omega : 1 : -1 : \mathbf{i}]$	$w^3 + x^3 + yz^2 + y^2 x$	$\mathbb{P}^3$
12	[2.12]	$[1 : \omega : 1 : \mathbf{i}]$	$w^2 = x^3 y + y^4 + z^4$	$\mathbb{P}(2, 1, 1, 1)$
12	[1. $\sigma\rho$ 2.2]	$[1 : 1 : -1 : -\omega]$	$w^2 = z^3 + xyL_2(x^2, y^2)$	$\mathbb{P}(3, 1, 1, 2)$
14	[2.G7]	$[-1 : \zeta_7 : (\zeta_7)^4 : (\zeta_7)^2]$	$w^2 = x^3 y + y^3 z + xz^3$	$\mathbb{P}(2, 1, 1, 1)$
15	[1. $\rho$ 5]	$[1 : 1 : \zeta_5 : \omega]$	$w^2 = z^3 + x(x^5 + y^5)$	$\mathbb{P}(3, 1, 1, 2)$
18	[2.G9]	$[-1 : (\zeta_9)^6 : 1 : \zeta_9]$	$w^2 = x^3 y + y^4 + xz^3$	$\mathbb{P}(2, 1, 1, 1)$
20	[1.B10]	$[\mathbf{i} : 1 : \zeta_{10} : -1]$	$w^2 = z^3 + x^4 z + xy^5$	$\mathbb{P}(3, 1, 1, 2)$
24	[1. $\sigma\rho$ 4]	$[\zeta_8 : 1 : \mathbf{i} : -\mathbf{i}\omega]$	$w^2 = z^3 + xy(x^4 + y^4)$	$\mathbb{P}(3, 1, 1, 2)$
30	[1. $\sigma\rho$ 5]	$[-1 : 1 : \zeta_5 : \omega]$	$w^2 = z^3 + x(x^5 + y^5)$	$\mathbb{P}(3, 1, 1, 2)$

Note that after this classification, some questions remain open and some new ones arise. For example, we discover infinitely many conjugacy classes of cyclic groups of order  $2n$ , for any integer  $n$ , but the precise conjugation between them is not established.

#### 4. Results that follow from the classifications

Even though the classification is not yet totally complete, it implies several results, and answers some questions that were previously open. Note that we mean ‘geometric genus’ when we speak about the genus of a curve. A curve has thus positive genus if and only if it is non-rational.

**Theorem 4.1** (Non-linear birational maps of large order).

- For any integer  $n \geq 1$ , there are infinitely many conjugacy classes of birational maps of the plane of order  $2n$ , that are non-conjugate to a linear automorphism.
- If  $n > 15$ , a birational map of order  $2n$  is an  $n$ -th root of a de Jonquières involution and preserves a pencil of rational curves.
- If a birational map is of finite odd order and is not conjugate to a linear automorphism of the plane, then its order is 3, 5, 9 or 15. In particular, any birational map of the plane of odd order  $> 15$  is conjugate to a linear automorphism of the plane.

**Remark 1.** Prior to this work, the largest known order of a non-linear birational map was 30 [7,14].

**Theorem 4.2** (*Roots of linear automorphisms*). Any birational map which is a root of a non-trivial linear automorphism of finite order of the plane is conjugate to a linear automorphism of the plane.

**Theorem 4.3** (*Groups which fix a curve of positive genus*). Let  $G$  be a finite Abelian group which fixes some curve of positive genus. Then  $G$  is cyclic, of order 2, 3, 4, 5 or 6, and all these cases occur. If the curve has genus  $> 1$ , the order is 2 or 3.

**Remark 2.** This generalises a theorem of Castelnuovo [5], which states that an element of finite order which fixes a curve of genus  $> 1$  has order 2, 3 or 4.

**Theorem 4.4** (*Cyclic groups whose non-trivial elements do not fix a curve of positive genus*). Let  $G$  be a finite cyclic subgroup of the Cremona group. The following conditions are equivalent:

- If  $g \in G$ ,  $g \neq 1$ , then  $g$  does not fix a curve of positive genus.
- The group  $G$  is birationally conjugate to a subgroup of  $\text{Aut}(\mathbb{P}^2)$ .
- The group  $G$  is birationally conjugate to a subgroup of  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ .

**Remark 3.** This generalises Corollary 1.2 of [3], which states the analogous result for cyclic subgroups of the Cremona group of prime order.

**Theorem 4.5** (*Abelian groups whose non-trivial elements do not fix a curve of positive genus*). Let  $G$  be a finite Abelian subgroup of the Cremona group. The following conditions are equivalent:

- If  $g \in G$ ,  $g \neq 1$ , then  $g$  does not fix a curve of positive genus.
- The group  $G$  is birationally conjugate to a subgroup of  $\text{Aut}(\mathbb{P}^2)$ , or to a subgroup of  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  or to the group isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ , generated by the two elements  $(x : y : z) \mapsto (yz : xy : -xz)$  and  $(x : y : z) \mapsto (yz(y - z) : xz(y + z) : xy(y + z))$ .

Moreover, this last group is conjugate neither to a subgroup of  $\text{Aut}(\mathbb{P}^2)$ , nor to a subgroup of  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ .

**Remark 4.** This generalises Theorems 4.2 and 4.4.

**Theorem 4.6** (*Isomorphism classes of finite Abelian groups*). The isomorphism classes of finite Abelian subgroups of the Cremona group are the following:  $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ , for any integers  $m, n \geq 1$ ;  $\mathbb{Z}/2n\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$ , for any integer  $n \geq 1$ ;  $(\mathbb{Z}/4\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z}$ ;  $(\mathbb{Z}/3\mathbb{Z})^3$  and  $(\mathbb{Z}/2\mathbb{Z})^4$ .

**Remark 5.** This generalises a result of [2], which gives the isomorphism classes of  $p$ -elementary subgroups of the Cremona group. The analogous list in the non-Abelian case is, as far as we know, not yet established.

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