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Group Theory

Invariant measures and stiffness for non-Abelian groups of toral automorphisms ☆

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Abstract

Let Γ be a non-elementary subgroup of $\mathrm{SL}_2(\mathbb{Z})$. If μ is a probability measure on $\mathbb{T}^2$ which is Γ-invariant, then μ is a convex combination of the Haar measure and an atomic probability measure supported by rational points. The same conclusion holds under the weaker assumption that μ is ν-stationary, i.e. $\mu = \nu * \mu$, where ν is a finitely supported, probability measure on Γ whose support $\operatorname{supp}\nu$ generates Γ. The approach works more generally for $\Gamma < \mathrm{SL}_d(\mathbb{Z})$. ***To cite this article: J. Bourgain et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Mesures invariantes et rigidité pour groupes non-abeliens d'automorphismes du tore. Soit Γ un sous-groupe non-élementaire du groupe $\mathrm{SL}_2(\mathbb{Z})$. Soit μ une mesure de probabilité Γ-invariante sur le tore $\mathbb{T}^2$. On démontre que μ est une moyenne de la mesure de Haar et une probabilité discrète portée par des points rationnels. La même conclusion reste vraie sous l'hypothèse que μ est ν-stationnaire, donc $\mu = \nu * \mu$, où ν est une probabilité sur Γ à support fini et engendrant Γ. L'approche se généralise aux sous-groupes Γ de $\mathrm{SL}_d(\mathbb{Z})$. ***Pour citer cet article : J. Bourgain et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Nous considérons l'action de $\mathrm{SL}_2(\mathbb{Z})$ sur le tore $\mathbb{T}^2$. Soit Γ un sous-groupe non-élémentaire du $\mathrm{SL}_2(\mathbb{Z})$. Soit μ une mesure sur $\mathbb{T}^2$ que nous supposons Γ-invariante, ou, moins restrictivement, que μ est ν-stationnaire pour une probabilité ν sur Γ à support fini et tel que $\langle\operatorname{supp}\nu\rangle = \Gamma$. Nous démontrons que si μ n'est pas un multiple de la mesure de Haar sur $\mathbb{T}^2$, alors μ a une composante discrète. La méthode comporte plusieurs étapes et des techniques d'analyse harmonique y jouent un rôle essentiel. Supposons la transformée de Fourier $\hat{\mu}(b) \neq 0$ pour un élément $b \in \mathbb{Z}^2\backslash\{0\}$. Le point de départ consiste à étudier l'ensemble $\Lambda_c = \{n \in \mathbb{Z}^2; |\hat{\mu}(n)| > c\}$ ($c > 0$ approprié) et de

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montrer que Λ_c est « riche », en un certain sens d'entropie métrique. On utilise ici divers arguments d'amplification et un résultat d'équirépartition pour convolutions multiplicatives sur $\mathbb{R}$, qui repose sur le théorème « somme-produit » obtenu dans [3] et [4]. Ensuite on déduit de la structure de Λ_c des propriétés de « porosité » pour le support de μ et finalement une composante discrète.

1. Introduction: main theorems

In this Note we present some new dichotomies for invariant and stationary measures μ on $\mathbb{T}^2$ under the action of $\mathrm{SL}_2(\mathbb{Z})$-subgroups.

Theorem A. *If μ is invariant under the action of a non-elementary subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, then μ is a linear combination of Haar measure on $\mathbb{T}^2$ and an atomic measure supported by rational points.*

Theorem B. *The same conclusion holds if we assume μ is ν-stationary, i.e. $\mu = \nu * \mu = \sum_{g \in \Gamma} \nu(g)\, g_* \mu$, with ν a finitely supported probability measure on $\mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma = \langle \operatorname{supp} \nu \rangle$ is a non-elementary subgroup.*

Theorem C. *If for a point $\theta \in \mathbb{T}^2$ the measure $\eta_n = \nu^{(n)} * \delta_\theta$ has Fourier coefficient $|\hat{\eta}_n(b)| > \delta$ for some $b \in \mathbb{Z}^2 \backslash \{0\}$, then θ admits a rational approximation*

$$\left\| \theta - \frac{a}{q} \right\| < e^{-cn} \quad \text{for some } q \in \mathbb{Z}_+,\ |q| < \left(\frac{\|b\|}{\delta} \right)^C \tag{1}$$

with $c, C > 0$ depending on ν.

Theorem C answers the question of equidistribution, posed by Y. Guivarc'h [9].

Theorem D. *Unless $\theta \in \mathbb{T}^2$ is rational, $\nu^{(n)} * \delta_\theta$ tend weak* to Lebesgue measure as $n \to \infty$.*

Comments. (1) The results extend to $\mathrm{SL}_d(\mathbb{Z})$, assuming that $\operatorname{supp}(\nu)$ generates a Zariski dense subgroup in $\mathrm{SL}_d(\mathbb{R})$ or, more generally, assuming that the smallest algebraic subgroup $H_\nu \subset \mathrm{SL}_d(\mathbb{R})$ supporting ν, is strongly irreducible (leaves invariant no finite union of $\mathbb{R}^d$-hyperplanes) and contains a proximal element. Under these conditions the top exponent is simple (see [8]).

(2) ν-stationary measures play an important role in the theory of boundaries of groups, and were systematically used by H. Furstenberg and others in many works. In his paper [7] H. Furstenberg explores the relationship between ν-stationary measures and Γ-invariant measures, where ν is a probability measure on Γ whose support generates Γ. For a general action of Γ on a space X there is a big difference between the two concepts: indeed, if X is compact ν-stationary measures always exist but there may well be no Γ-invariant probability measure whatsoever. In [7] Furstenberg introduces the notion of stiff actions: an action of a group Γ on a space X is said to be ν-**stiff** if every ν-stationary measure is in fact Γ-invariant, and proves stiffness for the action of $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ on $\mathbb{T}^d$ where ν is a (very) carefully chosen probability measure on $\mathrm{SL}(d, \mathbb{Z})$.

Furstenberg conjectured that this action is stiff for any ν whose support generates $\mathrm{SL}(d, \mathbb{Z})$. Theorem B and its extension to $d > 2$ establish in particular this conjecture. Moreover, in conjunction with strong approximation results such as those in [20,17], our results imply that the action is 'superstiff', in the sense that if $\langle \operatorname{supp} \nu \rangle$ is Zariski dense in $\mathrm{SL}(d, \mathbb{R})$, any ν-stationary measure on $\mathbb{T}^d$ is invariant under a finite index subgroup of $\mathrm{SL}(d, \mathbb{Z})$ (depending only on $\operatorname{supp} \nu$).

(3) Theorem A may be viewed as a non-Abelian analogue of the well-known $\times 2$, $\times 3$ invariant measure problem on the circle $\mathbb{T}$. Thus the conjecture states that if $\mu \in M(\mathbb{T})$ satisfies $\hat{\mu}(n) = \hat{\mu}(2n) = \hat{\mu}(3n)$ for all $n \in \mathbb{Z}$, then μ is a combination of Haar and discrete measures. It is known that if we assume moreover that μ has positive entropy, then μ is Haar (see [18] and [11,12,5] for the generalization to $\mathbb{Z}^d$-actions on tori). However, in the context of $\times 2$, $\times 3$ problem, or its toral analogues, statements such as Theorem D do not hold.

(4) We also recall that there are (Abelian and non-Abelian) counterparts for orbit closures. In the Abelian case, these are the dichotomy results of H. Furstenberg [6] and D. Berend [1]. The non-Abelian problem for Γ-orbits, $\Gamma \subset \mathrm{SL}_d(\mathbb{Z})$ a semigroup action on $\mathbb{T}^d$, appears for example in G.A. Margulis list of open problems [14]. Contributions

here include the work of G.A. Starkov [19] (for Γ a strongly irreducible subgroup of $SL_d(\mathbb{Z})$), R. Muchnik [15,16] (Γ a Zariski dense semigroup) and Guivarc'h–Starkov [10].

2. Idea of the proofs

Next, we give a brief overview of the proof of Theorem B. The proof of Theorem C (which implies D, B and A) uses the same ingredients – see comments at the end. There are several distinct steps in the proofs which we summarize.

Assume μ is a ν-stationary probability measure on $\mathbb{T}^2$ different from the Haar measure. Thus

$$\hat{\mu}(b) \neq 0 \quad \text{for some } b \in \mathbb{Z}^2 \backslash \{0\}$$

and hence

$$\sum_g \left|\hat{\mu}\left(g^t(b)\right)\right| \cdot \nu^{(r)}(g) \geqslant \left|\hat{\mu}(b)\right| = c \tag{2}$$

for any convolution power $\nu^{(r)}$ of ν. It is clear from (2) that μ has many large Fourier coefficients; in fact there is $\delta > 0$ such that

$$\left|\left\{n \in \mathbb{Z}^2\colon \|n\| \leqslant N \text{ and } \left|\hat{\mu}(n)\right| > \frac{1}{2}c\right\}\right| > N^\delta$$

for all sufficiently large N. However, unless δ is sufficiently close to 2, we need a more structured set of large Fourier coefficients. This is achieved in

Step 1 (amplification).

Lemma 1. *There are positive constants $\beta > 0$ and $\kappa > 0$ such that for all sufficiently large $N \in \mathbb{Z}_+$, there is a set $\mathcal{F} \subset \mathbb{Z}^2 \cap B(0, N)$ with the following properties*

(a) $|\hat{\mu}(k)| > \beta$ *for* $k \in \mathcal{F}$.
(b) $|k - k'| > N^{1-\kappa}$ *if* $k \neq k'$ *in* $\mathcal{F}$.
(c) $|\mathcal{F}| > \beta N^{2\kappa}$.

Our proof of Lemma 1 is rather involved. It is obtained by combining the following two ingredients.

Denote $\delta(\bar{x}, \bar{y})$ the angular distance on the projective space $P(\mathbb{R}^2)$. The following statement is obtained by combining Proposition 4.1 (p. 161) and Theorem 2.5 (p. 106) from [2]:

Proposition 2 *(small ball estimate). There is a uniform estimate for $\bar{x}, \bar{y} \in P(\mathbb{R}^2)$*

$$\sum_{\delta(g\bar{x},\bar{y})<\varepsilon} \nu^{(n)}(g) < C\left(\varepsilon^\alpha + \mathrm{e}^{-cn}\right)$$

for some $\alpha, c, C > 0$.

We also use the large deviation estimate for the Lyapunov exponent γ (Theorem 6.2, p. 131 in [2]), which gives:

Proposition 3. *Uniformly in x, $\|x\| = 1$:*

$$\nu^{(n)}\left\{g\colon \left|\frac{1}{n}\log\|gx\| - \gamma\right| > \frac{\gamma}{10}\right\} < C\mathrm{e}^{-cn}.$$

The combinatorial information that can be extracted from Proposition 2 on the set of large Fourier coefficients is amplified using the following general statement on mixed multiplicative and additive convolution on $\mathbb{R}$ (which may be of independent interest).

Proposition 4. *Given $\theta > 0$, $C > 1$, there are $s \in \mathbb{Z}_+$ and $C' > 1$ such that the following holds.*

Let $\delta > 0$ and η a probability measure on $[\frac{1}{2}, 1]$ satisfying

$$\max_a \eta\big(B(a,\rho)\big) < C\rho^\theta \quad \text{for } \delta < \rho < 1.$$

Consider the image measure ν of $\eta \otimes \cdots \otimes \eta$ (s^2-fold) under the map

$$(x_1, \ldots, x_{s^2}) \mapsto (x_1 \ldots x_s) + (x_{s+1} \ldots x_{2_s}) + \cdots + (x_{s^2-s+1} \ldots x_{s^2}).$$

Then

$$\max_a \nu\big(B(a,\rho)\big) < C'\rho \quad \text{for } \delta < \rho < 1$$

where here $B(a,\rho) = [a-\rho, a+\rho]$.

Proposition 4 is deduced from a set-theoretical statement, which is the 'discretized ring conjecture' (in the sense of [13]); see [3,4].

Returning to Lemma 1, there is the following implication on the support of μ.

Step 2 (porosity property).

Using elementary harmonic analysis, one shows the following general result:

Lemma 5. *Let μ be a probability measure on $\mathbb{T}^d$, $d \geqslant 1$. Fix $\kappa_1, \kappa_2 > 0$.*

Let $N \gg M$ be large integers and assume

$$\mathcal{N}\big(\big[|\hat{\mu}| > \kappa_1\big] \cap B(0,N); M\big) > \kappa_2 \left(\frac{N}{M}\right)^d$$

where for $A \subset \mathbb{Z}^d$ and $R > 1$, $\mathcal{N}(A; R)$ denotes the smallest number of balls of radius R needed to cover A.

Then there are points $x_1, \ldots, x_\beta \in \mathbb{T}^d$ such that

$$\|x_\alpha - x_{\alpha'}\| > \frac{1}{M} \quad \text{for } \alpha \neq \alpha',$$
$$\sum_\alpha \mu\left(B\left(x_\alpha, \frac{1}{N}\right)\right) > \rho(\kappa_1, \kappa_2) > 0.$$

Combined with Lemma 1 ($d = 2$ and taking $\kappa_1 = \beta = \kappa_2$, $M = N^{1-\kappa}$), we obtain therefore

Lemma 6. *For all N large enough, there are points $x_1, \ldots, x_\beta \in \mathbb{T}^2$ such that $\|x_\alpha - x_{\alpha'}\| > \frac{1}{N^{1-\kappa}}$ for $\alpha \neq \alpha'$ and*

$$\sum_\alpha \mu\left(B\left(x_\alpha, \frac{1}{N}\right)\right) > \rho.$$

Our next aim is to improve the porosity property obtained in Lemma 6 by decreasing the radius of the balls.

Step 3 (bootstrap).

Starting from the statement in Lemma 6 and using the group action, we prove

Lemma 7. *For any fixed number C_0, there is a collection of points $\{z_\alpha\} \in \mathbb{T}^2$ such that*

$$\|z_\alpha - z_{\alpha'}\| > \frac{1}{2N^{1-\kappa}} > \frac{1}{N} \quad \text{for } \alpha \neq \alpha'$$

and

$$\sum_\alpha \mu\left(B\left(z_\alpha, \frac{1}{N^{C_0}}\right)\right) > \rho(C_0) > 0.$$

The statement follows from a simple iterative construction. Under the action of $SL_2(\mathbb{Z})$-elements, the balls become elongated ellipses and intersecting different families leads to sets of smaller diameter.

Step 4 (rational approximation).

Assume

$$\mu\big(B(x,\varepsilon)\big) > \varepsilon^\tau \tag{3}$$

where $\varepsilon > 0$ is small and $\tau > 0$ a fixed exponent.

Take $n \sim (\frac{1}{\varepsilon})^{1/2}$ and make a Diophantine approximation

$$\left|x_1 - \frac{a_1}{q}\right| < \frac{1}{q\sqrt{n}}, \qquad \left|x_2 - \frac{a_2}{q}\right| < \frac{1}{q\sqrt{n}} \tag{4}$$

where $1 \leqslant q \leqslant n$ and $\gcd(a_1, a_2, q) = 1$. It follows from (3), (4) that

$$\mu\left(B\left(\frac{a}{q}, \frac{2}{q\sqrt{n}}\right)\right) > \varepsilon^\tau$$

and the ν-stationarity of μ implies for any $r \in \mathbb{Z}_+$

$$\sum_g \mu\left(B\left(\frac{g(a)}{q}, \frac{2\|g\|}{q\sqrt{n}}\right)\right) \cdot \nu^{(r)}(g) > \varepsilon^\tau. \tag{5}$$

Take $r \sim \log n$ as to ensure that $\|g\| < n^{1/3}$ if $g \in \operatorname{supp} \nu^{(r)}$. It follows then from (5) and our choice of r that

$$\varepsilon^\tau \leqslant \sum_{b\in\mathbb{Z}_q^2} \mu\left(B\left(\frac{b}{q}, \frac{1}{2q}\right)\right) \cdot \nu^{(r)}\big(\{g \mid ga \equiv b (\mathrm{mod} q)\}\big).$$

A spectral gap of the form $\|\nu^{(r)}\| \leqslant q^{-\omega_1}$, $r \geqslant \log q$, on $\ell^2(\mathbb{Z}_q^2) \ominus \mathbb{C}$ with some fixed $\omega_1 > 0$ depending only on ν, yields the estimate

$$\max_{b\in\mathbb{Z}_q^2} \nu^{(r)}\big(\{g \mid ga \equiv b (\mathrm{mod}\, q)\}\big) < q^{-\omega}, \quad q < \left(\frac{1}{\varepsilon}\right)^{\tau/\omega}. \tag{6}$$

Recalling the conclusion of Lemma 7, the exponent τ in (3) may be taken to be an arbitrary small fixed positive number. In particular, we may ensure that in (6), $q < Q(\varepsilon) < (\frac{1}{\varepsilon})^{1/20}$. Thus we proved that there is $\rho_1 > 0$ such that for all $\varepsilon > 0$ small enough

$$\mu(\mathfrak{S}_{Q(\varepsilon),\varepsilon^{1/4}}) > \rho_1 \tag{7}$$

where we denote

$$\mathfrak{S}_{Q,\varepsilon} = \bigcup_{q<Q} \bigcup_{(a,q)=1} B\left(\frac{a}{q}, \varepsilon\right). \tag{8}$$

Step 5 (conclusion).

Starting from (7) with $\varepsilon = \varepsilon_0$ small enough (depending on ρ_1), we perform again an iterative bootstrap (as in Step 3), invoking the following.

Lemma 8. *Let $\mathfrak{S}_{Q,\varepsilon}$ be as above and let $n = n(\varepsilon) \in \mathbb{Z}_+$ satisfying*

$$n < c \log \frac{1}{\varepsilon} \quad (c \text{ depending on } \nu).$$

Assume

$$\big(\nu^{(n)} * \mu\big)(\mathfrak{S}_{Q,\varepsilon}) = \sum \nu^{(n)}(g)\mu\big(g^{-1}(\mathfrak{S}_{Q,\varepsilon})\big) > \kappa.$$

Then we have $\mu(\mathfrak{S}_{Q,\varepsilon'}) > \kappa - \mathrm{e}^{-c_2 n}$ where $\varepsilon' = \mathrm{e}^{-\frac{1}{4}\gamma n}\varepsilon$.

The proof of Lemma 8 uses again Propositions 2 and 3.

Thus with $Q = Q(\varepsilon_0)$ fixed, ε is gradually decreased and in the limit we obtain

$$\mu\left(\left\{\frac{a}{q};\ 1 \leqslant q < Q(\varepsilon_0),\ 0 \leqslant a_1, a_2 < q\right\}\right) > \frac{1}{2}\rho_1 > 0.$$

This establishes Theorem B.

We conclude with some comments on the proof of Theorem C. For $m \geqslant 1$ we denote by

$$\eta_m = \nu^{(m)} * \delta_\theta \tag{9}$$

the measure on $\mathbb{T}^2$ (δ_x stands here for the Dirac measure). In these notations, the assumption of Theorem C becomes

$$\left|\hat{\eta}_n(b)\right| > \delta \quad \text{where } b \in \mathbb{Z}^2 \setminus \{0\}. \tag{10}$$

The proof of steps 1–4 is quantitative, and even though μ_m is not ν-stationary, these arguments can still be applied if one is willing to sacrifice a few powers of ν.

For example, in step 1 we may conclude from (10) that for any $k < n$ there is some N with $c_3 k < \log N < c_4 k$ and a set $\mathcal{F} \subset \mathbb{Z}^2 \cap B(0, N)$ satisfying (a)–(c) of Lemma 1 for $\mu = \mu_{n-k}$ and $\beta = (\delta/\|b\|)^C$ (where C and c_3, c_4, as well as all the other constants appearing below depend only on ν). Similarly modifying steps 2–4 we conclude that for any k' in the range $C' \log(\|b\|/\delta) < k' < n$ there are $Q, \epsilon = Q^{-20}$ with $c_3' k' < \log Q < c_4' k'$ satisfying (cf. (7)) $\eta_{n-k'}(\mathfrak{S}_{Q,\varepsilon}) > (\delta/\|b\|)^C$.

Let $n' = n - k'$ for $c_5 \log(\|b\|/\delta) < k' < n/2$, with c_5 a large constant. Since $\eta_{n'} = \nu^{(n')} * \delta_\theta$, if c_5 is sufficiently large, iteration of Lemma 8 imply that $\delta_\theta(\mathfrak{S}_{Q,\varepsilon'}) > (\delta/\|b\|)^C - \max(Q^{-c_3}, \mathrm{e}^{-c_2 n'}) > 0$ where $\varepsilon' < \mathrm{e}^{-\frac{1}{4}\gamma n'}\varepsilon < \mathrm{e}^{-\frac{1}{8}\gamma n}$, i.e. $\theta \in \mathfrak{S}_{Q,\varepsilon'}$. Since $Q < (\|b\|/\delta)^{C_0}$ for some C_0, Eq. (1) of Theorem C follows.

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