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Number Theory

The André–Oort conjecture for Drinfeld modular varieties

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Abstract

We state an analogue of the André–Oort conjecture for subvarieties of Drinfeld modular varieties, and prove it in two special cases. ***To cite this article: F. Breuer, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

La conjecture d'André–Oort pour les variétées modulaires de Drinfeld. Nous énoncons un analogue de la conjecture d'André–Oort pour les sous-variétées des variétées modulaires de Drinfeld, et nous le démontrons dans deux cas. ***Pour citer cet article : F. Breuer, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

The André–Oort conjecture for complex Shimura varieties states that any closed subvariety X of a Shimura variety $S/\mathbb{C}$ containing a Zariski-dense set of special points must be of Hodge type. For an overview, see [3] or [5]. Recently, Bruno Klingler, Emmanuel Ullmo and Andrei Yafaev have announced a proof of this conjecture assuming the Generalised Riemann Hypothesis.

In the current Note we will study an analogue of this conjecture for Drinfeld modular varieties, and outline a proof for two special cases.

Let $C/\mathbb{F}_q$ be a smooth projective geometrically connected algebraic curve with function field K, choose a closed point ∞ on C, and define $A = H^0(C \setminus \infty, \mathcal{O}_C)$. We denote by K_∞ the completion of K at ∞, and by $\mathbb{C}_\infty = \hat{\bar{K}}_\infty$ the completion of an algebraic closure of K_∞. We denote by $\mathbb{A}_f = \hat{A} \otimes K$ the ring of finite adèles of K.

For an open subgroup $\mathcal{K} \subset \mathrm{GL}_r(\hat{A})$, we denote by $M_A^r(\mathcal{K})$ the coarse moduli scheme for rank r Drinfeld A-modules with $\mathcal{K}$-level structure. We write $M_A^r(1)$ for $M_A^r(\mathrm{GL}_r(\hat{A}))$, the coarse moduli scheme for Drinfeld modules without level structure.

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Definition 1.1. A closed irreducible subvariety $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$ is called *special* if X is an irreducible component of the locus of Drinfeld A-modules with endomorphism ring containing a given ring. A point $x \in M^r_A(\mathcal{K})(\mathbb{C}_\infty)$ is called a *CM point* if it corresponds to a Drinfeld module φ with complex multiplication.

We see that CM points are precisely the special subvarieties of dimension zero. Our analogue of the André–Oort conjecture is the following:

Conjecture 1.2. *A closed irreducible subvariety $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$ contains a Zariski-dense set of CM points if and only if X is special.*

We will sketch the following theorems, the first being an (unconditional) analogue of a result of Bas Edixhoven and Andrei Yafaev [3, Theorem 1.2], and the second an analogue of a result of Ben Moonen [4, §5]. Our approach is an adaptation of that of Edixhoven and Yafaev.

Theorem 1.3. *Let $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$ be an irreducible algebraic subcurve. Then X contains infinitely many CM points if and only if X is special. In particular, Conjecture* 1.2 *is true if $r = 3$.*

Theorem 1.4. *Let $\mathfrak{p} \subset A$ be a non-zero prime, and $n \in \mathbb{N}$ a positive integer. Let $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$ be a closed irreducible subvariety containing a Zariski-dense set of CM points x with the following property: Denote by $\mathcal{O}_x$ the endomorphism ring of a Drinfeld module representing x. Then there is an unramified prime $\mathfrak{P}|\mathfrak{p}$ of $\mathcal{O}_x \otimes_A K$ such that the residue degree $f(\mathfrak{P}|\mathfrak{p}) = 1$, and $\mathfrak{p}^n$ does not divide the conductor of $\mathcal{O}_x$ in its integral closure. Then X is a special subvariety.*

2. Analytic theory

Denote by $\Omega^r := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \smallsetminus \{\text{linear subspaces defined over } K_\infty\}$ Drinfeld's upper half-space, on which $\mathrm{GL}_r(K_\infty)$ acts. Then we have, as rigid analytic varieties,

$$M^r_A(\mathcal{K})^{\mathrm{an}}_{\mathbb{C}_\infty} \cong \mathrm{GL}_r(K)\backslash\Omega^r \times \mathrm{GL}_r(\mathbb{A}_f)/\mathcal{K} \cong \coprod_{s\in S} \Gamma_s\backslash\Omega^r,$$

where S denotes a finite set of representatives for $\mathrm{GL}_r(K)\backslash\mathrm{GL}_r(\mathbb{A}_f)/\mathcal{K}$, and $\Gamma_s = s\mathcal{K}s^{-1} \cap \mathrm{GL}_r(K)$.

The group $\mathrm{GL}_r(\mathbb{A}_f)$ acts from the left on $\Omega \times \mathrm{GL}_r(\mathbb{A}_f)$ via $g\cdot(\omega, h) = (\omega, hg^{-1})$, and this induces the *Hecke correspondence* T_g on $M^r_A(\mathcal{K})$, which factors through $M^r_A(\mathcal{K}_g)$, where $\mathcal{K}_g = \mathcal{K} \cap g^{-1}\mathcal{K}g$. For any non-zero ideal $\mathfrak{n} \subset A$ we denote by $T_\mathfrak{n}$ and $T_{\tilde{\mathfrak{n}}}$ the Hecke correspondences associated to $\mathrm{diag}(\mathfrak{n}, 1, \ldots, 1)$ and $\mathrm{diag}(1, \mathfrak{n}, \ldots, \mathfrak{n})$, respectively, viewed as elements in $\mathrm{GL}_r(\mathbb{A}_f)$. In the moduli interpretation of $M^r_A(1)_{\mathbb{C}_\infty}$, the correspondence $T_\mathfrak{n}$ encodes all isogenies of kernel isomorphic to $A/\mathfrak{n}$, and $T_{\tilde{\mathfrak{n}}}$ encodes their dual isogenies, with kernels isomorphic to $(A/\mathfrak{n})^{r-1}$.

We can now give an equivalent definition of special subvarieties. Let $r'|r$, and let K'/K be an imaginary extension (which means that only one place of K' lies above ∞) of degree $[K' : K] = r/r'$, and denote by A' the integral closure of A in K'. Then any rank r' Drinfeld A'-module is also a rank r Drinfeld A-module, which gives an embedding of moduli spaces $M^{r'}_{A'}(1)_{\mathbb{C}_\infty} \hookrightarrow M^r_A(1)_{\mathbb{C}_\infty}$. A closed subvariety $X \subset M^r_A(1)_{\mathbb{C}_\infty}$ is special if and only if X is an irreducible component of $T_g(M^{r'}_{A'}(1)_{\mathbb{C}_\infty})$ for some $g \in \mathrm{GL}_r(\mathbb{A}_f)$ and some A' and r' as above. A closed irreducible subvariety $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$ is special if its image under the canonical projection $M^r_A(\mathcal{K})_{\mathbb{C}_\infty} \to M^r_A(1)_{\mathbb{C}_\infty}$ is special.

Our first step is to show that suitable Hecke orbits are Zariski-dense:

Proposition 2.1. *Let $\mathfrak{n} \subset A$ be a non-trivial principal ideal. Then, for any $x \in M^r_A(1)(\mathbb{C}_\infty)$, the Hecke orbit $(T_\mathfrak{n} + T_{\tilde{\mathfrak{n}}})^\infty(x)$ is Zariski-dense in the irreducible component of $M^r_A(1)_{\mathbb{C}_\infty}$ containing x.*

It follows that $M^r_A(1)(\mathbb{C}_\infty)$, and hence any special subvariety, contains a Zariski-dense set of CM points. The idea of the proof is the following. Let $Z \subset \Omega^r$ denote an irreducible component of the preimage of the Zariski-closure of the Hecke orbit $(T_\mathfrak{n} + T_{\tilde{\mathfrak{n}}})^\infty(x)$. Then one explicitly constructs a smooth point $\omega \in Z$ which is approximated by sequences of points of Z lying on lines in sufficiently many directions (in Ω viewed as a subspace of $\mathbb{A}^{r-1}(\mathbb{C}_\infty)$) to conclude that the tangent space of Z at ω must have dimension $r-1$. The result follows.

A subvariety $X \subset M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ is called *Hodge generic* if it is not contained in any proper special subvariety. The next step is to show that suitable Hecke images of irreducible Hodge generic subvarieties are again irreducible:

Proposition 2.2. *Let $X \subset M_A^r(1)_{\mathbb{C}_\infty}$ be an irreducible Hodge generic subvariety, and $\dim(X) \geqslant 1$. Then*

(i) *There exists a non-zero ideal $\mathfrak{m}_X \subset A$, such that $T_\mathfrak{n}(X)$ is irreducible for any non-zero ideal $\mathfrak{n} \subset A$ prime to $\mathfrak{m}_X$.*
(ii) *There exists an open subgroup $\mathcal{K} \subset \mathrm{GL}_r(\hat{A})$ and an irreducible component $X' \subset M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ of the preimage of X such that $T_\mathfrak{n}(X')$ is irreducible for all non-zero ideals $\mathfrak{n} \subset A$.*

Proof sketch. We first replace X by its non-singular locus. We assume for simplicity that $X^{\mathrm{an}} \subset \mathrm{GL}_r(A)\backslash\Omega^r$. Let $\Xi \subset \Omega^r$ be an irreducible component of the preimage of X^{an} and $\Delta = \mathrm{Stab}_{\mathrm{GL}_r(A)}(\Xi)$, so that $X^{\mathrm{an}} \cong \Delta\backslash\Xi$. Denote by $\hat{\Delta}$ the closure of Δ in $\mathrm{GL}_r(\hat{A})$, then one can show using [1, Theorem 1.1] that $\hat{\Delta}$ is open in $\mathrm{GL}_r(\hat{A})$. It follows that there is a non-zero ideal $\mathfrak{m}_X \subset A$ such that $\Delta \to \mathrm{GL}_r(A/\mathfrak{n})$ is surjective for all non-zero ideals $\mathfrak{n} \subset A$ prime to $\mathfrak{m}_X$.

To prove (i), let $\mathfrak{n} \subset A$ be a non-zero ideal prime to $\mathfrak{m}_X$. Let $\mathcal{K} = \mathrm{GL}_r(\hat{A})$, and define $\Gamma = \mathcal{K} \cap \mathrm{GL}_r(K)$ and $\mathcal{K}_0(\mathfrak{n}) = \mathcal{K} \cap \mathfrak{n}^{-1}\mathcal{K}\mathfrak{n}$. Denote by $\pi : M_A^r(\mathcal{K}_0(\mathfrak{n})) \to M_A^r(\mathcal{K})$ the canonical projection. Since Δ and Γ have the same image in $\mathrm{GL}_r(A/\mathfrak{n})$, it follows that Δ acts transitively on the fibres of π, hence $\pi^{-1}(X)$, and thus also $T_\mathfrak{n}(X) = \pi(\mathrm{diag}(\mathfrak{n}, 1, \ldots, 1)\pi^{-1}(X))$, is irreducible.

To prove (ii), we make similar definitions as above, but with $\mathcal{K} = \hat{\Delta}$. We choose $X' \subset M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ such that $X'^{\mathrm{an}} \cong \Delta\backslash\Xi$. Then we see that for each non-zero ideal $\mathfrak{n} \subset A$, Δ and Γ have the same image in $\mathrm{GL}_r(A/\mathfrak{n})$, and thus again $T_\mathfrak{n}(X')$ is irreducible. □

3. Arithmetic theory

Let $x \in M_A^r(\mathcal{K})(\mathbb{C}_\infty)$ be a CM point represented by a CM Drinfeld module φ with endomorphism ring $\mathrm{End}(\varphi) = \mathcal{O}$. Then $\mathcal{O}$ is an order in an imaginary extension K'/K of degree $[K' : K] = r$. Denote by A' the integral closure of A in K'. By the theory of complex multiplication, the field $K'(x)$ of definition of x over K' is the ring class field associated to the order $\mathcal{O}$, in particular $\mathrm{Gal}(K'(x)/K') \cong \mathrm{Pic}(\mathcal{O})$.

Now let $\mathfrak{p} \subset A$ be an unramified prime of residue degree 1 in K'/K, and which does not divide the conductor $\mathfrak{c}$ of $\mathcal{O}$ in A'. Then $\sigma_\mathfrak{p}(x) \in T_\mathfrak{p}(x)$, where $\sigma_\mathfrak{p} \in \mathrm{Gal}(K'(x)/K')$ is the Frobenius element associated to $\mathfrak{p}$.

Denote by $g(K')$ the genus of K', then we define the *CM height* of x (and of φ) to be

$$\mathrm{H}_{\mathrm{CM}}(x) = \mathrm{H}_{\mathrm{CM}}(\varphi) := q^{g(K')} \cdot \#(A'/\mathfrak{c})^{1/r}.$$

One can show

Proposition 3.1.

(i) *There are only finitely many CM points in $M_A^r(\mathcal{K})(\mathbb{C}_\infty)$ with CM-height bounded by a given constant.*
(ii) *For every $\varepsilon > 0$ there is a computable constant $C_\varepsilon > 0$ such that the following holds. Let φ be a Drinfeld module with complex multiplication by an order $\mathcal{O}$, as above. Then $\#\mathrm{Pic}(\mathcal{O}) > C_\varepsilon \mathrm{H}_{\mathrm{CM}}(\varphi)^{1-\varepsilon}$.*

Proof sketch of Theorem 1.3. Let $X \subset M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ be an irreducible subcurve. We may ignore the level structure, and after replacing X by a Hecke translate and A by some A' if necessary, we may assume that $X \subset M_A^r(1)_{\mathbb{C}_\infty}$ is Hodge generic.

Let F be a field of definition of X containing the Hilbert class field of K, we may assume $[F : K]$ is finite. Using explicit polynomial equations for the Hecke correspondence $T_\mathfrak{p}$ in the case $A = \mathbb{F}_q[T]$ (see [2]), one can show that the intersection degree of $X \cap T_\mathfrak{p}(X)$ is bounded by $c|\mathfrak{p}|^n$ for constants $c, n > 0$ depending only on X, where $|\mathfrak{p}| := \#(A/\mathfrak{p})$. Let $\mathfrak{m}_X \subset A$ denote the ideal given by Proposition 2.2(i).

Let $x \in X(\mathbb{C}_\infty)$ be a CM point with endomorphism ring $\mathcal{O} \subset K'$. Then if $\mathrm{H}_{\mathrm{CM}}(x)$ is sufficiently large, it follows from the Čebotarev Theorem for function fields, and Proposition 3.1, that there exists a non-zero prime $\mathfrak{p} \subset A$ with the following properties:

(i) $\mathfrak{p}$ divides neither $\mathfrak{m}_X$ nor the conductor of $\mathcal{O}$, and has residue degree one in FK'/K.
(ii) $\#\mathrm{Pic}(\mathcal{O}) > c[F : K]|\mathfrak{p}|^n$.

Denote by F_s the separable closure of K in F, and by L the Galois closure of $F_sK'(x)$ over K'. Let $\sigma \in \mathrm{Aut}(FL/FK')$ be an extension of the Frobenius element associated to a prime of L above $\mathfrak{p}$. Then σ fixes F (a field of definition of X and of $T_\mathfrak{p}(X)$) and $\sigma(x) \in T_\mathfrak{p}(x)$ by complex multiplication. Thus $X \cap T_\mathfrak{p}(X)$ contains the entire $\mathrm{Gal}(FK'(x)/FK')$-orbit of x, which by (ii) above is larger than the intersection degree. It follows that $X \subset T_\mathfrak{p}(X)$. Since $T_\mathfrak{p}(X)$ is irreducible by Proposition 2.2, we get $X = T_\mathfrak{p}(X) = T_{\mathfrak{p}^m}(X) = T_{\tilde{\mathfrak{p}}^m}(X)$, where we choose $m \in \mathbb{N}$ such that $\mathfrak{p}^m$ is principal. Thus X contains the entire Hecke orbit $(T_{\mathfrak{p}^m} + T_{\tilde{\mathfrak{p}}^m})^\infty(x)$, which is Zariski-dense in a component of $M_A^r(1)_{\mathbb{C}_\infty}$, by Proposition 2.1. The result follows. □

Proof sketch of Theorem 1.4. By a sequence of simplifications one may reduce to the case where $X \subset M_A^r(1)_{\mathbb{C}_\infty}$ is Hodge generic, and contains a Zariski-dense set Σ of CM points x with endomorphism rings $\mathcal{O}_x$ in which $\mathfrak{p}$ has residue degree one and does not divide the conductor. Next, let $\mathcal{K} \subset \mathrm{GL}_r(\hat{A})$ and $X' \subset M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ be given by Proposition 2.2(ii). We lift Σ to a Zariski-dense set $\Sigma' \subset X'(\mathbb{C}_\infty)$. As in the proof of Theorem 1.3, we now see that $\sigma(x) \in X' \cap T_\mathfrak{p}(X')$ for all $x \in \Sigma'$, where σ is again a Frobenius element associated to $\mathfrak{p}$. It follows that $X' \subset T_\mathfrak{p}(X')$, and since $T_\mathfrak{p}(X')$ is irreducible, it follows as above that X', and hence also X, is special. □

Remark. The main obstacle to proving Conjecture 1.2 for subvarieties of higher dimension is the lack of control over the ideal $\mathfrak{m}_X$ and the level structure $\mathcal{K}$ given by Proposition 2.2.

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