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Complex Analysis

A new characterization of a class of pseudoconvex domains in $\mathbb{C}^2$

Fabrizio Colombo [a], M. Elena Luna-Elizarrarás [b], Irene Sabadini [a], Michael Shapiro [b], Daniele C. Struppa [c]

[a] *Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, 20133 Milano, Italy*
[b] *Departamento de Matemáticas E.S.F.M. del I.P.N. 07338 México D.F., Mexico*
[c] *Department of Mathematics and Computer Science, Chapman University, 92866 Orange, CA, USA*

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Abstract

By using the right inverse of the Cauchy–Fueter operator we obtain an explicit integral characterization of a class of pseudoconvex domains in $\mathbb{C}^2$. ***To cite this article: F. Colombo et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Une nouvelle caractérisation d'une classe des domaines pseudoconvexes en $\mathbb{C}^2$. En utilisant l'inverse à droite de l'opérateur de Cauchy–Fueter, nous démontrons une caractérisation en forme intégrale d'une classe de domaines pseudoconvexes en $\mathbb{C}^2$. ***Pour citer cet article : F. Colombo et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

It is well known that every open set U in $\mathbb{C}$ is a domain of holomorphy, [3]. In terms of cohomology, this is equivalent to the vanishing of the first cohomology group of any open set U in the complex plane, with values in the sheaf of holomorphic functions $\mathcal{O}$, in symbols $H^1(U,\mathcal{O})=0$. This result is usually referred to as the Mittag–Leffler theorem. The situation is quite different in dimension two and higher, since the Hartogs' phenomenon shows that not every open set is a domain of holomorphy. Complex analysts have found a precise characterization of those open sets which are domains of holomorphy in several variables (or equivalently, a characterization of those open sets for which the cohomology groups vanish from the first on). These characterizations make use of the Levi form, and the domains of holomorphy are called, when the Levi condition is considered, pseudoconvex sets, [3]. In this Note we consider a bounded open set U in $\mathbb{C}^2$ with piecewise smooth boundary and we characterize those cocycles in $H^1(U,\mathcal{O})$, the first cohomology group, which are not cohomologous to zero through some explicit integral formulas. As a corollary,

E-mail addresses: fabcol@mate.polimi.it (F. Colombo), eluna@esfm.ipn.mx (M.E. Luna-Elizarrarás), sabadini@mate.polimi.it (I. Sabadini), shapiro@esfm.ipn.mx (M. Shapiro), struppa@chapman.edu (D.C. Struppa).

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we obtain an integral characterization for pseudoconvex sets in $\mathbb{C}^2$ with piecewise smooth boundary. Let T_1, T_2 be defined for a complex valued function $u = u(z_1, z_2)$ by:

$$T_1[u](w_1, w_2) := \frac{1}{2\pi^2} \int_U \frac{\bar{z}_1 - \bar{w}_1}{(|z_1 - w_1|^2 + |z_2 - w_2|^2)^2} u(z_1, z_2)\,\mathrm{d}V, \tag{1}$$

$$T_2[u](w_1, w_2) := -\frac{1}{2\pi^2} \int_U \frac{\bar{z}_2 - \bar{w}_2}{(|z_1 - w_1|^2 + |z_2 - w_2|^2)^2} \bar{u}(z_1, z_2)\,\mathrm{d}V. \tag{2}$$

We prove the following result:

Theorem 1.1. *Let U be any bounded open set in $\mathbb{C}^2$ with piecewise smooth boundary, and let $\xi = \{g_{ij}\}$ be a cocycle in $H^1(U, \mathcal{O})$. Let $\{\varphi_i\}$ be any partition of the unity associated to a covering $\{U_i\}$ of U, set $h_i := \sum_j \varphi_j g_{ji} \in C^\infty(U_i)$, and let $k_1 := \{2\frac{\partial h_i}{\partial \bar{z}_1}\}$, $k_2 := \{2\frac{\partial h_i}{\partial \bar{z}_2}\}$. Then $\xi = 0$ if and only if $T_1(k_2) + T_2(k_1) = 0$.*

As an immediate corollary of this result, we obtain:

Theorem 1.2. *Let U be a bounded open set in $\mathbb{C}^2$ with piecewise smooth boundary. Denote by $\{\varphi_i\}$ any partition of the unity associated to a covering $\{U_i\}$ of U, and let $\xi = \{g_{ij}\}$ be a cocycle in $H^1(U, \mathcal{O})$. Set $h_i = \sum_j \varphi_j g_{ji} \in C^\infty(U_i)$, and define $k_1 := \{2\frac{\partial h_i}{\partial \bar{z}_1}\}$, $k_2 := \{2\frac{\partial h_i}{\partial \bar{z}_2}\}$. Then the set U is pseudoconvex if and only if for every $\xi = \{g_{ij}\}$ in $H^1(U, \mathcal{O})$, one has $T_1(k_2) + T_2(k_1) = 0$.*

Our techniques are based essentially on quaternionic analysis and they can be used in order to extend the above results to domains with much less smoothness, as well as to obtain a large class of results for other important classes of functions in low real dimension. These results will be published elsewhere.

Notice that other characterizations of domains of holomorphy were given by K. Nôno [5] in terms of quaternionic analysis and by J. Ryan [6] in terms of complex Clifford analysis.

2. Hyperholomorphic functions of two complex variables

Let z_1, z_2 be two complex numbers with the imaginary unit denoted by $\mathbf{i}$, and let $\mathbf{j}$ be another imaginary unit such that the relations $\mathbf{j}^2 = -1$ and $\mathbf{ij} + \mathbf{ji} = 0$ hold.

In particular, $z_1\mathbf{j} = \mathbf{j}\bar{z}_1$ where $\bar{z}_1$ denotes the complex conjugate of z_1. A quaternion q is an element of the form

$$q = z_1 + z_2\mathbf{j}$$

and the space of quaternions equipped with the above mentioned product turns into a real, non-commutative, division algebra denoted by $\mathbb{H}$. The conjugate of a quaternion q is defined by $\bar{q} = \bar{z}_1 - z_2\mathbf{j}$. The way in which we have written a quaternion in terms of two complex numbers shows the very well known isomorphism $\mathbb{H} \cong \mathbb{C}^2$ as complex linear spaces although of course, $\mathbb{H}$ has an additional, very rich multiplicative structure.

For $\mathbb{H}$-valued functions defined on an open set $U \subseteq \mathbb{C}^2$, it is possible to define a family of generalizations of the notion of holomorphy, the so-called hyperholomorphy, see e.g. [4]. For our purposes, we will choose a specific operator which is a minor modification of the well known Cauchy–Fueter operator [2]. This operator will act on functions $f \in C^1(U; \mathbb{H})$ as

$$\frac{1}{2}\mathcal{D}[f](q) := \frac{\partial f}{\partial \bar{z}_1}(q) + \mathbf{j}\frac{\partial f}{\partial \bar{z}_2}(q) = \left(\frac{\partial}{\partial \bar{z}_1} + \mathbf{j}\frac{\partial}{\partial \bar{z}_2}\right) f(q). \tag{3}$$

Since f can be decomposed as $f = f_1 + f_2\mathbf{j}$ we can write

$$\frac{1}{2}\mathcal{D}[f](q) = \left(\frac{\partial f_1}{\partial \bar{z}_1} - \frac{\partial \bar{f}_2}{\partial z_2}\right)(q) + \mathbf{j}\left(\frac{\partial f_1}{\partial \bar{z}_2} + \frac{\partial \bar{f}_2}{\partial z_1}\right)(q). \tag{4}$$

Definition 2.1. A function $f \in C^1(U; \mathbb{H})$ is said to be hyperholomorphic if $\mathcal{D}[f](q) = 0$.

Hyperholomorphic functions form a sheaf which we will denote by $\mathcal{R}$.

Remark 1. It is easy to see that for a hyperholomorphic function $f = f_1 + f_2\mathbf{j}$ both f_1 and f_2 are either holomorphic or non-holomorphic, hence, holomorphic functions may be identified with either of f_1 or $f_2\mathbf{j}$, the final results for holomorphic functions are the same. In order to preserve the multiplicative structure of holomorphic functions, we will identify them with hyperholomorphic functions having $f_2 = 0$.

The operator T, the right inverse of the (modified) Cauchy–Fueter operator $\mathcal{D}$, see e.g. [7], is defined for any function $u \in C^1(U;\mathbb{H})$ where U is any bounded domain with piecewise smooth boundary to be

$$T[u](p) := \int_U \mathcal{K}(q-p)u(q)\,\mathrm{d}V_q \tag{5}$$

where $\mathcal{K}$ is the (modified) Cauchy–Fueter kernel $\mathcal{K}(q) = \frac{1}{2\pi^2}\frac{\bar{z}_1 - \bar{z}_2\mathbf{j}}{(|z_1|^2+|z_2|^2)^2}$ and $\mathrm{d}V_q$ is the volume form. Using (3), direct computations show that if $v = v_1 + v_2\mathbf{j}$ is an arbitrary continuous function then

$$T[v] = T_1[v_1] - T_2[v_2] + \big(T_1[v_2] + T_2[v_1]\big)\mathbf{j}. \tag{6}$$

We are now ready for our main result for the theory of hyperholomorphic functions (see e.g. [1]):

Theorem 2.2. *For any bounded open set $U \subseteq \mathbb{C}^2$ with piecewise smooth boundary, the first cohomology group $H^1(U,\mathcal{R})$ vanishes. More explicitly, if $\mathcal{U} = \{U_j\}$ is an open covering of U and if $\{g_{ij}\}$ are hyperholomorphic functions on $U_i \cap U_j$ such that $g_{ij} - g_{ik} + g_{jk} = 0$ on $U_i \cap U_j \cap U_k$ whenever this intersection is non-empty, then there are functions $g_j \in \mathcal{R}(U_j)$ such that $g_{ij} = g_j - g_i$.*

Proof. Let $\xi = \{g_{ij}\}$ represent a 1-cocycle in $H^1(U,\mathcal{R})$, i.e. if $\mathcal{U} = \{U_i\}$ is a covering of the open set U, the functions g_{ij} are hyperholomorphic on the intersections $U_i \cap U_j$ and satisfy

$$g_{ij} - g_{ik} + g_{jk} = 0.$$

Let $\{\varphi_i\}$ be a partition of unity associated to the covering $\mathcal{U}$. Then we can construct new C^∞ functions $h_j = \sum_i \varphi_i g_{ij}$ defined in U_j, and it is immediate to observe that for every i and j such that $U_i \cap U_j \neq \emptyset$, we have $\mathcal{D}[h_i] = \mathcal{D}[h_j]$. In fact, $\mathcal{D}[h_i - h_j] = \mathcal{D}[\sum_\ell \varphi_\ell g_{\ell i} - \sum_\ell \varphi_\ell g_{\ell j}] = \mathcal{D}[\sum_\ell \varphi_\ell g_{ji}] = \mathcal{D}[g_{ji}] = 0$. This implies that the collection $\{\mathcal{D}h_i\}$ defines a C^∞ function on all of U. Setting $k := \{\mathcal{D}[h_i]\}$ we have a C^∞ function and applying the right-inverse operator T we obtain a function $u = T[k]$. Setting

$$g_j := h_j - u$$

we get hyperholomorphic functions on U_j such that $g_{ij} = g_j - g_i$ belongs to $\mathcal{R}(U_i \cap U_j)$ and the statement follows. □

The vanishing of the first cohomology group for $\mathcal{R}$ is therefore a simple, and yet interesting, result which essentially follows from the existence of a right inverse to the (modified) Cauchy–Fueter operator. On the other hand, the vanishing of higher order cohomology groups for $\mathcal{R}$ is an immediate consequence of the fact that we are studying a one-dimensional theory, in the sense that we are looking at a single operator of a single (though quaternionic) variable. The reader is referred to [1] for the trivial cohomological arguments which are necessary.

3. Proofs of the main results

We are now ready to prove the results announced in the introduction which are based on the proof of Theorem 2.2 and the notation therein. In particular, we will denote $k = k_1 + k_2\mathbf{j}$.

Proof of Theorem 1.1. Every hyperholomorphic function f can be written as $f = f_1 + f_2\mathbf{j}$, where f_1 and f_2 are complex valued functions; moreover, if a function $f : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic, then it can be thought of as a hyperholomorphic function. Let $\xi = \{g_{ij}\}$ be a 1-cocycle in $H^1(U,\mathcal{O})$, i.e. if $\mathcal{U} = \{U_i\}$ is a covering of the open set U, the functions g_{ij} are holomorphic (and therefore hyperholomorphic) on the intersections $U_i \cap U_j$ and satisfy $g_{ij} + g_{ji} = 0$, $g_{ij} - g_{ik} + g_{jk} = 0$. Let $\{\varphi_i\}$ be a complex valued partition of unity associated to the covering $\mathcal{U}$. Then we can construct new C^∞ functions $h_j := \sum_i \varphi_i g_{ij}$. These functions are complex valued as well, and it is immediate to observe that for every i and j such that $U_i \cap U_j \neq \emptyset$, we have $\mathcal{D}[h_i] = \mathcal{D}[h_j]$. This implies that the collection $\{\mathcal{D}[h_i]\}$ defines a C^∞, $\mathbb{H}$-valued function $k = k_1 + k_2\mathbf{j}$ on all of U. If we now apply the right-inverse operator T (in the sense of quaternions) we obtain a quaternion valued function $u := T[k]$. Setting $g_j := h_j - u$ we get hyperholomorphic functions on U_j such that $g_{ij} := g_j - g_i$ belongs to $\mathcal{R}(U_i \cap U_j)$. Generally speaking, this is not enough to guarantee that $\xi = 0$, and in fact in general this is not the case. In fact, in order for ξ to vanish, we need the functions g_j to be holomorphic, and not only hyperholomorphic. Since g_{ij} is holomorphic, for this to be true, we need to impose that $u = T[k]$ be complex valued. This, in turns, translates into a condition on both T_1 and T_2: using $k = k_1 + k_2\mathbf{j}$ and (6), we have $T_1(k_2) + T_2(k_1) = 0$. Conversely, if $T_1(k_2) + T_2(k_1) = 0$ then $T(k)$ is complex valued and the hyperholomorphic functions g_j are holomorphic. □

Proof of Theorem 1.2. The result follows immediately if one notices that an open bounded set U is pseudoconvex if and only if $H^1(U, \mathcal{O}) = 0$, i.e. if and only if every 1-cocycle ξ with holomorphic coefficients vanishes, or, by virtue of the previous theorem, if and only if $T_1(k_2) + T_2(k_1) = 0$. □

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