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Group Theory

Rouquier blocks of the cyclotomic Hecke algebras

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Abstract

Following the definition of Rouquier for the 'families of characters' of a Weyl group and its generalization to the case of a complex reflection group W, already used in the works of Broué–Kim and Malle–Rouquier, we show that the Rouquier blocks of a cyclotomic Hecke algebra associated with W depend only on numerical data defined by the generic Hecke algebra. ***To cite this article: M. Chlouveraki, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***
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Résumé

Blocs de Rouquier des algèbres de Hecke cyclotomiques. Suivant la définition de Rouquier de « familles de caractères » d'un groupe de Weyl et sa généralisation au cas d'un groupe de réflexions complexe W, déjà utilisée dans les travaux de Broué–Kim et Malle–Rouquier, nous montrons que les blocs de Rouquier d'une algèbre de Hecke cyclotomique associée à W ne dépendent que des données numériques définies par l'algèbre de Hecke générique. ***Pour citer cet article : M. Chlouveraki, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***
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Soit μ_∞ le groupe des racines de l'unité de $\mathbb{C}$ et K un corps de nombres contenu dans $\mathbb{Q}(\mu_\infty)$. On note $\mu(K)$ le groupe des racines de l'unité de K et $\forall d > 1$, on pose $\zeta_d := \exp(2\pi \mathrm{i}/d)$. Soit V un K-espace vectoriel de dimension finie. Soit W un sous-groupe fini de $\mathrm{GL}(V)$ engendré par des (pseudo-)réflexions et agissant irréductiblement sur V et B le groupe de tresses associé à W. On note $\mathcal{A}$ l'ensemble des hyperplans de réflexions de W. Pour toute orbite $\mathcal{C}$ de W sur $\mathcal{A}$, on note $e_{\mathcal{C}}$ l'ordre commun des sous-groupes W_H, où $H \in \mathcal{C}$ et W_H est le sous-groupe formé de 1 et toutes les réflexions qui fixent H.

On choisit un ensemble d'indéterminées

$$\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leqslant j\leqslant e_{\mathcal{C}}-1)}$$

et on définit l'*algèbre de Hecke générique* $\mathcal{H}$ de W comme le quotient de l'algèbre du groupe $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]B$ par l'idéal engendré par les éléments de la forme $(\mathbf{s}-u_{\mathcal{C},0})(\mathbf{s}-u_{\mathcal{C},1})\cdots(\mathbf{s}-u_{\mathcal{C},e_{\mathcal{C}}-1})$, où $\mathcal{C}$ parcourt l'ensemble $\mathcal{A}/W$ et $\mathbf{s}$ l'ensemble de générateurs de monodromie autour des images dans V^{reg}/W des éléments de l'orbite d'hyperplans $\mathcal{C}$.

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Hypothèses 0.1. L'algèbre $\mathcal{H}$ est un $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-module libre et de rang $|W|$. De plus, elle est munie d'une forme symétrisante $t:\mathcal{H}\to\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ qui satisfait les conditions décrites dans [1], §2A, Hyp. 2.1.

Dorénavant, on suppose les Hypothèses 0.1 satisfaites. Dans ce cas, Malle a montré ([9], 5.2) que si l'ensemble d'indéterminées $\mathbf{v}=(v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leqslant j\leqslant e_{\mathcal{C}}-1)}$ est défini comme $v_{\mathcal{C},j}^{|\mu(K)|}:=\zeta_{e_{\mathcal{C}}}^{-j}u_{\mathcal{C},j}$, alors l'algèbre $K(\mathbf{v})\mathcal{H}$ est semi-simple déployée (cf. [9], 5.2). La forme symétrisante pour $\mathcal{H}$ est de la forme

$$t=\sum_{\chi_{\mathbf{v}}\in\mathrm{Irr}(K(\mathbf{v})\mathcal{H})}\frac{1}{s_\chi(\mathbf{v})}\chi_{\mathbf{v}},$$

où $s_\chi(\mathbf{v})$ est l'*élément de Schur* associé au caractère irréductible $\chi_{\mathbf{v}}$.

Une *spécialisation cyclotomique* est un morphisme de $\mathbb{Z}_K$-algèbres $\phi:\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]\to\mathbb{Z}_K[y,y^{-1}]$ de la forme $\phi:v_{\mathcal{C},j}\mapsto y^{n_{\mathcal{C},j}}$ où $n_{\mathcal{C},j}\in\mathbb{Z}$ pour tout $\mathcal{C}$ et j. L'*algèbre de Hecke cyclotomique* correspondante est la $\mathbb{Z}_K[y,y^{-1}]$-algèbre, notée $\mathcal{H}_\phi$, obtenue comme la spécialisation de $\mathcal{H}$ via le morphisme ϕ.

On définit $\mathcal{R}_K(y):=\mathbb{Z}_K[y,y^{-1},(y^n-1)^{-1}_{n\geqslant 1}]$ l'*anneau de Rouquier* de K. Les *blocs de Rouquier* de $\mathcal{H}_\phi$ sont les blocs de l'algèbre $\mathcal{R}_K(y)\mathcal{H}_\phi$. Rouquier [11] a montré que si W est un groupe de Weyl et $\mathcal{H}_\phi$ est obtenue via la spécialisation cyclotomique « spetsiale », alors ses blocs de Rouquier coincident avec les « familles de caractères » selon Lusztig. Ainsi, les blocs de Rouquier jouent un rôle essentiel dans le program « Spets » (cf. [2]) dont l'ambition est de faire jouer à des groupes de réflexions complexes le rôle de groupes de Weyl de structures encore mystérieuses.

Proposition 0.1. *Supposons les Hypothèses* 0.1 *satisfaites. L'élément de Schur* $s_\chi(\mathbf{v})$ *associé au caractère* $\chi_{\mathbf{v}}$ *de* $K(\mathbf{v})\mathcal{H}$ *est un élément de* $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ *de la forme*

$$s_\chi(\mathbf{v})=\xi_\chi N_\chi\prod_{i\in I_\chi}\Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

où

- ξ_χ *est un élément de* $\mathbb{Z}_K$,
- $N_\chi=\prod_{\mathcal{C},j}v_{\mathcal{C},j}^{b_{\mathcal{C},j}}$ *est un monôme dans* $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ *avec* $\sum_{j=0}^{e_{\mathcal{C}}-1}b_{\mathcal{C},j}=0$ *pour tout* $\mathcal{C}\in\mathcal{A}/W$,
- I_χ *est un ensemble d'indices,*
- $(\Psi_{\chi,i})_{i\in I_\chi}$ *est une famille de polynômes* K*-cyclotomiques sur une indéterminée,*
- $(M_{\chi,i})_{i\in I_\chi}$ *est une famille de monômes dans* $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ *et si* $M_{\chi,i}=\prod_{\mathcal{C},j}v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$, *alors* $\mathrm{pgcd}(a_{\mathcal{C},j})=1$ *et* $\sum_{j=0}^{e_{\mathcal{C}}-1}a_{\mathcal{C},j}=0$ *pour tout* $\mathcal{C}\in\mathcal{A}/W$,
- $(n_{\chi,i})_{i\in I_\chi}$ *est une famille d'entiers positifs.*

La factorisation ci-dessus est unique dans $K[\mathbf{v},\mathbf{v}^{-1}]$ et les monômes $M_{\chi,i}$ sont uniques à inversibilité près. Soient $A:=\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$, $\mathfrak{p}$ un idéal premier de $\mathbb{Z}_K$ et $M:=M_{\chi,i}$ un monôme qui apparaît dans la factorisation de l'élément de Schur s_χ. Si $\mathfrak{q}_M:=(M-1)A+\mathfrak{p}A$, alors $\mathfrak{q}_M$ est un idéal premier de A et M s'appelle $\mathfrak{p}$-*essentiel pour* χ si $s_\chi/\xi_\chi\in\mathfrak{q}_M$. Un monôme M dans A s'appelle $\mathfrak{p}$-*essentiel* s'il existe un caractère irréductible χ tel que M est $\mathfrak{p}$-essentiel pour χ.

Soit ϕ une spécialisation cyclotomique et $\mathcal{O}:=\mathcal{R}_K(y)$. Les blocs de $\mathcal{O}\mathcal{H}_\phi$ sont des unions des blocs de $\mathcal{O}_{\mathfrak{p}}\mathcal{H}_\phi$ pour tous les idéaux premiers $\mathfrak{p}$ de $\mathbb{Z}_K$ (cf. [1], Proposition 1.13 et §2B). Soient $M_1,\ldots,M_k$ tous les monômes $\mathfrak{p}$-essentiels tels que $\phi(M_j)=1$ pour tout $j=1,\ldots,k$. On pose $\mathfrak{q}_0:=\mathfrak{p}A$, $\mathfrak{q}_j:=\mathfrak{p}A+(M_j-1)A$ pour tout $j=1,\ldots,k$ et $\mathcal{Q}:=\{\mathfrak{q}_0,\mathfrak{q}_1,\ldots,\mathfrak{q}_k\}$. Si $\mathfrak{q}\in\mathcal{Q}$ et deux caractères irréductibles χ,ψ appartiennent au même bloc de $A_{\mathfrak{q}}\mathcal{H}$, on écrit $\chi\sim_{\mathfrak{q}}\psi$.

Théorème 0.2. *Supposons les Hypothèses* 0.1 *satisfaites. Deux caractères irréductibles* $\chi,\psi\in\mathrm{Irr}(W)$ *sont dans le même bloc de* $\mathcal{O}_{\mathfrak{p}}\mathcal{H}_\phi$ *si et seulement s'il existe une suite finie* $\chi_0,\chi_1,\ldots,\chi_n\in\mathrm{Irr}(W)$ *et une suite finie* $\mathfrak{q}_{j_1},\ldots,\mathfrak{q}_{j_n}\in\mathcal{Q}$ *telles que* $\chi_0=\chi$, $\chi_n=\psi$ *et pour tout* $i\,(1\leqslant i\leqslant n)$, $\chi_{i-1}\sim_{\mathfrak{q}_{j_i}}\chi_i$.

1. Introduction

Let μ_∞ be the group of all the roots of unity in $\mathbb{C}$ and K a number field contained in $\mathbb{Q}(\mu_\infty)$. We denote by $\mu(K)$ the group of all the roots of unity of K and $\forall d > 1$, we put $\zeta_d := \exp(2\pi \mathrm{i}/d)$. Let V be a finite dimensional K-vector space. Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo-)reflections and acting irreducibly on V. We denote by $\mathcal{A}$ the set of its reflecting hyperplanes. We set $V^{\mathrm{reg}} := V - \bigcup_{H\in\mathcal{A}} H$. For $z_0 \in V^{\mathrm{reg}}$, we define $B := \Pi_1(V^{\mathrm{reg}}/W, z_0)$ the braid group associated with W.

For every orbit $\mathcal{C}$ of W on $\mathcal{A}$, we set $e_\mathcal{C}$ the common order of the subgroups W_H, where H is any element of $\mathcal{C}$ and W_H the subgroup formed by 1 and all the reflections fixing the hyperplane H.

We choose a set of variables

$$\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leqslant j\leqslant e_\mathcal{C}-1)}$$

and we denote by $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ the Laurent polynomial ring in all the variables $\mathbf{u}$. We define the *generic Hecke algebra* $\mathcal{H}$ of W to be the quotient of the group algebra $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]B$ by the ideal generated by the elements of the form

$$(\mathbf{s} - u_{\mathcal{C},0})(\mathbf{s} - u_{\mathcal{C},1})\cdots(\mathbf{s} - u_{\mathcal{C},e_\mathcal{C}-1}),$$

where $\mathcal{C}$ runs over the set $\mathcal{A}/W$ and $\mathbf{s}$ over the set of monodromy generators around the images in V^{reg}/W of the elements of the hyperplane orbit $\mathcal{C}$.

We will now make some assumptions for the algebra $\mathcal{H}$. Note that they have been verified for all but a finite number of irreducible complex reflection groups ([2], remarks before 1.17, §2; [6]).

Assumptions 1.1. *The algebra $\mathcal{H}$ is a $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-module free and of rank $|W|$. Moreover, there exists a linear form $t:\mathcal{H}\to\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ with the following properties*:

(i) *t is a central function (i.e. $t(hh') = t(h'h)$ for all $h, h' \in \mathcal{H}$) and a symmetrizing form for $\mathcal{H}$, i.e. the form $\hat{t}:\mathcal{H}\to\mathrm{Hom}(\mathcal{H},\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}])$, $h\mapsto(x\mapsto t(hx))$ is an isomorphism of $\mathcal{H}$-bimodules*;
(ii) *t satisfies the assumptions described in* [1], § 2A, *Hyp.* 2.1.

We know that the form t is unique ([2], 2.1). From now on let us suppose that Assumptions 1.1 are satisfied. Then we have the following result by G. Malle ([9], 5.2)

Theorem 1.2. *Let $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leqslant j\leqslant e_\mathcal{C}-1)}$ be a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_\mathcal{C}$ variables such that, for every $\mathcal{C}$, j, we have $v_{\mathcal{C},j}^{|\mu(K)|} = \zeta_{e_\mathcal{C}}^{-j} u_{\mathcal{C},j}$. Then the $K(\mathbf{v})$-algebra $K(\mathbf{v})\mathcal{H}$ is split semisimple.*

From 'Tits' deformation theorem' (see for example [2], 7.2) we know that the specialization $v_{\mathcal{C},j}\mapsto 1$ induces a bijection $\chi\mapsto\chi_\mathbf{v}$ from the set $\mathrm{Irr}(W)$ of absolutely irreducible characters of W to the set $\mathrm{Irr}(K(\mathbf{v})\mathcal{H})$ of absolutely irreducible characters of $K(\mathbf{v})\mathcal{H}$.

Now let y be a variable. We set $x := y^{|\mu(K)|}$.

Definition 1.3. A cyclotomic specialization is a $\mathbb{Z}_K$-algebra morphism $\phi:\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]\to\mathbb{Z}_K[y,y^{-1}]$ with the following properties:

- $\phi: v_{\mathcal{C},j}\mapsto y^{n_{\mathcal{C},j}}$ where $n_{\mathcal{C},j}\in\mathbb{Z}$ for all $\mathcal{C}$ and j;
- for all $\mathcal{C}\in\mathcal{A}/W$, and if z is another variable, the element of $\mathbb{Z}_K[y,y^{-1},z]$ defined by $\Gamma_\mathcal{C}(y,z) := \prod_{j=0}^{e_\mathcal{C}-1}(z - \zeta_{e_\mathcal{C}}^j y^{n_{\mathcal{C},j}})$ is invariant by the action of $\mathrm{Gal}(K(y)|K(x))$.

If ϕ is a cyclotomic specialization like above, the corresponding *cyclotomic Hecke algebra* is the $\mathbb{Z}_K[y,y^{-1}]$-algebra, denoted by $\mathcal{H}_\phi$, which is obtained as the specialization of the $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$-algebra $\mathcal{H}$ via the morphism ϕ. It also has a symmetrizing form t_ϕ defined by the specialization of the canonical form t. We notice that $\mathcal{H}_\phi$ is an image of the group algebra $\mathbb{Z}_K[y,y^{-1}]B$, where B is the braid group, and that for $y=1$ this algebra specializes to the group algebra $\mathbb{Z}_K[W]$. Thus the specialization $v_{\mathcal{C},j}\mapsto 1$ defines the following bijections

$$\begin{array}{ccccc} \mathrm{Irr}(W) & \leftrightarrow & \mathrm{Irr}\big(K(y)\mathcal{H}_\phi\big) & \leftrightarrow & \mathrm{Irr}\big(K(\mathbf{v})\mathcal{H}\big), \\ \chi & \mapsto & \chi_\phi & \mapsto & \chi_{\mathbf{v}}. \end{array} \tag{1}$$

Definition 1.4. We call Rouquier ring of K and denote by $\mathcal{R}_K(y)$ the $\mathbb{Z}_K$-subalgebra of $K(y)$

$$\mathcal{R}_K(y) := \mathbb{Z}_K\big[y, y^{-1}, \big(y^n-1\big)^{-1}_{n\geqslant 1}\big].$$

The *Rouquier blocks* of $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi$. It has been shown by R. Rouquier [11], that if W is a Weyl group and $\mathcal{H}_\phi$ is obtained via the cyclotomic specialization ('spetsial')

$$\phi: v_{\mathcal{C},0} \mapsto y \quad \text{and} \quad v_{\mathcal{C},j} \mapsto 1 \quad \text{for } j \neq 0,$$

then its Rouquier blocks coincide with the 'families of characters' defined by Lusztig. Thus, the Rouquier blocks play an essential role in the program 'Spets' [2] whose ambition is to give to complex reflection groups the role of Weyl groups of as yet mysterious structures.

2. Schur elements and blocks

Some important facts as far as the blocks of a symmetric algebra are concerned (for more details see [4], 7.2): Let H be an algebra over an integrally closed Noetherian ring R, which is free and finitely generated as R-module. Let F be a finite Galois extension of the field of fractions of R such that the algebra $FH := F \otimes_R H$ is split semisimple. Let us denote by $\mathrm{Irr}(FH)$ the set of the irreducible characters of the algebra FH. If t is a fixed symmetrizing form for H, then

$$t = \sum_{\chi \in \mathrm{Irr}(FH)} \frac{1}{s_\chi}\chi$$

where s_χ is the *Schur element* associated with the irreducible character χ. It can be shown that s_χ belongs to the integral closure of R in F for all $\chi \in \mathrm{Irr}(FH)$ (see [4], Prop. 7.3.9).

Moreover, since FH is semisimple, there exists a bijection between $\mathrm{Irr}(FH)$ and the set of the blocks (central primitive idempotents) of FH. If e_χ is the block which corresponds to the irreducible character χ, then we have

$$e_\chi = \frac{\hat{t}^{-1}(\chi)}{s_\chi}.$$

In our case, the generic Hecke algebra $\mathcal{H}$ defined over $K(\mathbf{v})$ is split semisimple. A case by case analysis of the Schur elements of complex reflection groups as appeared in [7,8] and [10] has shown the following:

Proposition 2.1. *Suppose that Assumptions* 1.1 *are satisfied. The Schur element* $s_\chi(\mathbf{v})$ *associated with the character* $\chi_{\mathbf{v}}$ *of* $K(\mathbf{v})\mathcal{H}$ *is an element of* $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ *of the form*

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

where

- ξ_χ *is an element of* $\mathbb{Z}_K$,
- $N_\chi = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{b_{\mathcal{C},j}}$ *is a monomial in* $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ *with* $\sum_{j=0}^{e_{\mathcal{C}}-1} b_{\mathcal{C},j} = 0$ *for all* $\mathcal{C} \in \mathcal{A}/W$,
- I_χ *is an index set,*
- $(\Psi_{\chi,i})_{i \in I_\chi}$ *is a family of* K*-cyclotomic polynomials in one variable,*
- $(M_{\chi,i})_{i \in I_\chi}$ *is a family of monomials in* $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ *and if* $M_{\chi,i} = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$, *then* $\gcd(a_{\mathcal{C},j}) = 1$ *and* $\sum_{j=0}^{e_{\mathcal{C}}-1} a_{\mathcal{C},j} = 0$ *for all* $\mathcal{C} \in \mathcal{A}/W$,
- $(n_{\chi,i})_{i \in I_\chi}$ *is a family of positive integers.*

The above factorization is unique in $K[\mathbf{v}, \mathbf{v}^{-1}]$. We can also easily show that the monomials $M_{\chi,i}$ are unique up to inversion.

Let $A := \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ and $M := M_{\chi,i} \in A$ a monomial appearing in the factorization of the Schur element s_χ. Set $\mathfrak{q}_M := (M-1)A + \mathfrak{p}A$. Then $\mathfrak{q}_M$ is a prime ideal of A.

Definition 2.2. The monomial M is called $\mathfrak{p}$-essential for χ if $s_\chi/\xi_\chi \in \mathfrak{q}_M$. A monomial is called $\mathfrak{p}$-essential if it is $\mathfrak{p}$-essential for some irreducible character χ.

A monomial $M \in A$ is $\mathfrak{p}$-essential for χ if and only if s_χ has a factor of the form $\Psi(M)$ with Ψ a K-cyclotomic polynomial and $\Psi(1) \in \mathfrak{p}$.

Now let $\phi : A \to \mathbb{Z}_K[y, y^{-1}]$ be a cyclotomic specialization as defined in 1.3. The algebra $K(y)\mathcal{H}_\phi$ is split semisimple. Following Proposition 2.1 and the existing bijection between the irreducible characters of $K(\mathbf{v})\mathcal{H}$ and of $K(y)\mathcal{H}_\phi$, we have:

Proposition 2.3. *The Schur element $s_{\chi_\phi}(y)$ associated with the character χ_ϕ of $K(y)\mathcal{H}_\phi$ is a Laurent polynomial in y of the form*

$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi,\phi}}$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

Definition 2.4. A prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ lying over a prime number p is ϕ-bad for W, if there exists $\chi_\phi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$ with $\psi_{\chi,\phi} \in \mathfrak{p}$.

Note that if $\mathfrak{p}$ is ϕ-bad for W, then p divides the order of the group.

If W is a Weyl group and ϕ is the 'spetsial' specialization, then the ϕ-bad prime ideals are the ideals generated by the bad prime numbers (in the 'usual' sense) for W (see [5], 5.2).

Let $\mathcal{O} := \mathcal{R}_K(y)$. If $\mathfrak{p}$ is not a ϕ-bad prime ideal for W, then the blocks of $\mathcal{O}_\mathfrak{p}\mathcal{H}_\phi$ are singletons. The blocks of $\mathcal{O}\mathcal{H}_\phi$ are unions of blocks of $\mathcal{O}_\mathfrak{p}\mathcal{H}_\phi$ for all ϕ-bad prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$ ([1], 1.10,1.13, §2B). We will now see how the blocks of $\mathcal{O}_\mathfrak{p}\mathcal{H}_\phi$, and thus the Rouquier blocks of $\mathcal{H}_\phi$, can be calculated as unions of blocks of the generic Hecke algebra $\mathcal{H}$ defined over specific localizations of the ring A:

Let $M_1, \ldots, M_k$ be the $\mathfrak{p}$-essential monomials such that $\phi(M_j) = 1$ for all $j = 1, \ldots, k$. Set $\mathfrak{q}_0 := \mathfrak{p}A$, $\mathfrak{q}_j := \mathfrak{p}A + (M_j - 1)A$ for all $j = 1, \ldots, k$ and $\mathcal{Q} := \{\mathfrak{q}_0, \mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$. If two irreducible characters χ, ψ belong to the same block of $A_\mathfrak{q}\mathcal{H}$ for some $\mathfrak{q} \in \mathcal{Q}$, we write $\chi \sim_\mathfrak{q} \psi$.

Theorem 2.5. *Suppose that Assumptions* 1.1 *are satisfied. Two irreducible characters $\chi, \psi \in \mathrm{Irr}(W)$ are in the same block of $\mathcal{O}_\mathfrak{p}\mathcal{H}_\phi$ if and only if there exists a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{q}_{j_1}, \ldots, \mathfrak{q}_{j_n} \in \mathcal{Q}$ such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$,*
- *for all $i\,(1 \leqslant i \leqslant n)$, $\chi_{i-1} \sim_{\mathfrak{q}_{j_i}} \chi_i$.*

3. Essential hyperplanes and consequences

Let $M := \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a $\mathfrak{p}$-essential monomial (recall that $\sum_{j=0}^{e_\mathcal{C}-1} a_{\mathcal{C},j} = 0$ for all $\mathcal{C} \in \mathcal{A}/W$). Then

$$\phi(M) = 1 \Leftrightarrow \sum_{\mathcal{C},j} a_{\mathcal{C},j} n_{\mathcal{C},j} = 0.$$

The hyperplane defined in $\mathbb{C}^{\sum_{\mathcal{C}\in\mathcal{A}/W} e_\mathcal{C}}$ by the relation $\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0$, where $(t_{\mathcal{C},j})$ is a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_\mathcal{C}$ variables, is called $\mathfrak{p}$-*essential hyperplane* for W.

In order to calculate the blocks of $\mathcal{O}_\mathfrak{p}\mathcal{H}_\phi$, it is enough to check to which $\mathfrak{p}$-essential hyperplanes the $n_{\mathcal{C},j}$ belong: If the $n_{\mathcal{C},j}$ belong to no $\mathfrak{p}$-essential hyperplane, then the blocks of $\mathcal{O}_\mathfrak{p}\mathcal{H}_\phi$ coincide with the blocks of $A_\mathfrak{p}\mathcal{H}$. If the $n_{\mathcal{C},j}$

belong to only one $\mathfrak{p}$-essential hyperplane associated with the $\mathfrak{p}$-essential monomial M, then the blocks of $\mathcal{O}_{\mathfrak{p}}\mathcal{H}_\phi$ coincide with the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$. If the $n_{\mathcal{C},j}$ belong to more than one $\mathfrak{p}$-essential hyperplane, then we use Theorem 2.5 in order to calculate the blocks of $\mathcal{O}_{\mathfrak{p}}\mathcal{H}_\phi$. If $n_{\mathcal{C},j} = n \in \mathbb{Z}$ for all $\mathcal{C}$, j, then $\mathcal{O}_{\mathfrak{p}}\mathcal{H}_\phi \cong \mathcal{O}_{\mathfrak{p}}[W]$ and the $n_{\mathcal{C},j}$ belong to all $\mathfrak{p}$-essential hyperplanes. Due to Theorem 2.5, we obtain the following proposition:

Proposition 3.1. *If two irreducible characters χ and ψ are in the same block of $\mathcal{O}_{\mathfrak{p}}\mathcal{H}_\phi$, then they are in the same block of $\mathcal{O}_{\mathfrak{p}}[W]$.*

Using the above results, we have been able to calculate the Rouquier blocks of all cyclotomic Hecke algebras of all exceptional complex reflection groups [3].

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