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An L^P estimate for Maxwell's equations with source term

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Abstract

In this Note we establish an interior L^p-type estimate for the solutions of Maxwell's equations with source term in a domain filled with two different materials separated by a $\mathcal{C}^2$ interface. Due to the singularity of the dielectric permittivity, the usual elliptic estimates cannot be applied directly. A special curl–div decomposition is introduced for the electric field to reduce the problem to an elliptic equation in divergence form with discontinuous coefficients. The potential theory analysis and the jump condition lead to the L^p estimates which are superior to the straightforward Nash–Moser estimates. The reduction procedure is expected to be useful for numerical simulation. Such an estimate is crucial for solving nonlinear Maxwell's equations that arise for example in the modeling of nonlinear optics. ***To cite this article: G. Bao et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Estimation L^p pour les équations de Maxwell ayant un terme source. Dans cette Note, on établit une estimation de type L^p intérieure pour les solutions des équations de Maxwell avec un terme source, dans un domaine occupé par deux matériaux séparés par une interface C^2. En raison de la singularité générée par la discontinuité de la permittivité, les estimations elliptiques usuelles ne sont plus valables. Une décomposition rot–div du champ électrique est utilisée pour réduire le problème à une équation elliptique sous forme de divergence à coefficients discontinus. La théorie du potentiel et la condition de saut permettent d'obtenir des estimations L^p meilleures que les estimations directes de Nash–Moser. Nous espérons que cette procédure de réduction peut être utile pour la simulation numérique. Une telle estimation est très importante pour la résolution des équations de Maxwell non-linéaires, que l'on rencontre dans la modélisation de l'optique nonlinéaire. ***Pour citer cet article : G. Bao et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***
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Dans cette Note, on présente un résultat de régularité de type L^p intérieure pour les solutions des équations de Maxwell en domaine fréquentiel avec un terme source dans un milieu constitué de deux matériaux homogènes

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séparés par une interface C^2. De telles estimations interviennent dans le cadre de la résolution de problèmes d'optique non-linéaire.

Cet article est motivé par de récentes études en optique non-linéaire. En particulier, une des applications importantes visées concerne l'obtention, par la génération de la deuxième harmonique (SHG), de radiations cohérentes pour des longueurs d'onde inférieures à celles des lasers disponibles.

Pour les milieux linéaires, Chen et Friedman [3] ont obtenu des résultats d'existence et d'unicité pour une permittivité constante par morceaux. Dans [4], Dobson et Friedman ont montré, par une approche intégrale, l'existence et l'unicité de la solution des équations de Maxwell dans un milieu linéaire périodique constitué de deux matériaux homogènes séparés par une interface de régularité C^2. Peu de travaux ont été consacrés à l'existence et à l'unicité de la solution des équations de Maxwell en milieux non-linéaires. On peut citer les travaux de Bao et Dobson, [1,2] ils ont récemment obtenu des résultats dans deux cas plus simples pour lesquels les équations de Maxwell peuvent se découpler.

Dans cette Note, nous nous intéressons au problème suivant : soit B une boule de $\mathbb{R}^3$ et soit S une surface C^2 définie par $z = f(x, y)$ telle que S divise B en deux parties connexes B^+ et B^-. On suppose aussi $B' \subset B$. On considère les équations de Maxwell avec un terme source :

$$\begin{aligned} &\nabla \times E = -\mathrm{i}\omega\mu H, \\ &\nabla \times H = \mathrm{i}\omega\varepsilon E + g, \end{aligned} \tag{1}$$

où ε est la permittivité définie par $\varepsilon = \varepsilon^+$ sur B^+ ; $\varepsilon = \varepsilon^-$ sur B^-, avec deux constantes différentes ε^+ et ε^-. La perméabilité μ est supposée constante dans B. Le résultat principal sur la régularité des champs peut alors être résumé comme suit :

Étant donné p tel que $1 < p < \infty$, soient $E \in L^p(B)$, $H \in W^{1,p}(B)$ et $g \in L^p(B)$ vérifiant (1). Alors

$$\|E\|_{L^p(B')} + \|H\|_{W^{1,p}(B')} \leqslant C\big(\|H\|_{L^p(B)} + \|g\|_{L^p(B)} + \|E\|_{W^{-1,p}(B)}\big),$$

où C est une constante dépendant seulement de p et de B.

La démonstration utilise la théorie du potentiel, des résultats sur la régularité elliptique et une décomposition rot–div du champ électrique.

1. Introduction

In this Note, we establish an interior L^p type estimate for the frequency domain solutions of Maxwell equations with source term. The domain is filled with two different materials separated by a C^2 interface. Such an estimate is crucial in the regularity study of nonlinear Maxwell's equations that arise for instance, in the modeling of nonlinear optics. One of the many applications of nonlinear optical phenomena is the use of second harmonic generation (SHG) in obtaining coherent radiation with a wavelength shorter than that of currently available lasers. This work is motivated by recent research on gratings (periodic structures) enhanced nonlinear optical effects.

The Maxwell equations [9,10] for the constant magnetic permeability μ take the form:

$$\begin{aligned} &\mu H_t = -\nabla \times E, \qquad \nabla \cdot H = 0, \\ &D_t = \nabla \times H, \qquad \nabla \cdot D = 0, \end{aligned}$$

along with the constitutive equation: $D = E + 4\pi P$, where E is the electric field, H is the magnetic field, D is the electric displacement, ω is the angular frequency, and P is the electric moment per unit volume that describes fully the response of the medium to the electromagnetic field. The medium is said to be linear if $P = \chi^{(1)}E$ where $\chi^{(1)}$ is the linear susceptibility tensor of the medium. The (linear) dielectric constant is defined by $\varepsilon = 1 + 4\pi\chi^{(1)}$.

In general, P is a nonlinear function of E. The simplest case of optical wave interaction in nonlinear media deals with SHG, a special case of second-order nonlinear optical effects. To this end, $P = \chi^{(1)}E + \chi^{(2)} : EE$, where $\chi^{(2)}$ is the second order nonlinear susceptibility tensor which measures the nonlinearity of the medium, and $\chi^{(2)} : EE$ denotes a vector whose i-th component is $\sum_{j,k} \chi^{(2)}_{ijk} E_j E_k$.

Suppose that a pumping beam with frequency ω_1 is incident on a nonlinear medium. Consider the two wave fields $E(x, \omega_1),\ H(x, \omega_1)$ at frequency ω_1 and $E(x, \omega_2),\ H(x, \omega_2)$, at frequency $\omega_2 = 2\omega_1$. That is,

$$E = \mathrm{e}^{\mathrm{i}\omega_1 t} E(x, \omega_1) + \mathrm{e}^{-\mathrm{i}\omega_1 t} \overline{E}(x, \omega_1) + \mathrm{e}^{2\mathrm{i}\omega_1 t} E(x, \omega_2) + \mathrm{e}^{-2\mathrm{i}\omega_1 t} \overline{E}(x, \omega_2) \cdots,$$
$$H = \mathrm{e}^{\mathrm{i}\omega_1 t} H(x, \omega_1) + \mathrm{e}^{-\mathrm{i}\omega_1 t} \overline{H}(x, \omega_1) + \mathrm{e}^{2\mathrm{i}\omega_1 t} H(x, \omega_2) + \mathrm{e}^{-2\mathrm{i}\omega_1 t} \overline{H}(x, \omega_2) \cdots.$$

By a comparison of the coefficients at different frequencies, one obtains the following coupled system from the original Maxwell equations:

$$\begin{aligned}
&\nabla \times E(x, \omega_1) = -\mathrm{i}\omega_1 \mu H(x, \omega_1),\\
&\nabla \times H(x, \omega_1) = \mathrm{i}\omega_1 \varepsilon E(x, \omega_1) + \mathrm{i}4\pi\omega_1 \chi^{(2)} \overline{E}(x, \omega_1) E(x, \omega_2),\\
&\nabla \times E(x, \omega_2) = -\mathrm{i}\omega_2 \mu H(x, \omega_2),\\
&\nabla \times H(x, \omega_2) = \mathrm{i}\omega_2 \varepsilon E(x, \omega_2) + \mathrm{i}4\pi\omega_2 \chi^{(2)} E(x, \omega_1) E(x, \omega_2).
\end{aligned}$$

In particular, at each frequency, we deduce the following system of Maxwell equations [8]:

$$\begin{cases} \nabla \times E = -\mathrm{i}\omega\mu H, \\ \nabla \times H = \mathrm{i}\omega\varepsilon E + g, \end{cases} \tag{2}$$

where g comes from the nonlinear interaction. To study the existence theory for this system, it is necessary to establish some a priori estimates.

Throughout this Note, B denotes a ball in $\mathbb{R}^3$ and S denotes a C^2 surface embedded in $\mathbb{R}^3$ given by $z = \phi(x, y)$ such that S divides B into two connected components: B^+ and B^-. Let $\bar{B}' \subset B$.

The main result of this Note may be stated as follows:

Theorem 1.1. *Let $1 < p < \infty$ and let $E \in L^p(B)$ and $H \in W^{1,p}(B)$ be a solution of* (2). *Then*

$$\|E\|_{L^p(B')} + \|H\|_{W^{1,p}(B')} \leqslant C\big(\|H\|_{L^p(B)} + \|g\|_{L^p(B)} + \|E\|_{W^{-1,p}(B)}\big),$$

where C is a constant depending only on p, B' and B.

The proof is based on a combination of potential theory, compact operator theory, elliptic regularity results and a special curl–div decomposition of the electric field.

Results on existence and uniqueness for Maxwell's equations in linear media with periodic structures were obtained by Chen and Friedman [3] by assuming a piece-wise constant dielectric coefficient. Dobson and Friedman [4] showed the existence and uniqueness of the solutions of linear Maxwell's equations in a periodic structure that separates two homogeneous materials and is piecewise C^2, by an integral-equation approach. Little is known concerning the questions of existence and uniqueness for nonlinear Maxwell equations. Bao and Dobson [1,2] have recently obtained results on existence and uniqueness in two simpler cases where the Maxwell equations can be reduced to a system of nonlinear Helmholtz equations.

2. Some auxiliary results

Let $D_- \subset \mathbb{R}^3$ be a bounded domain of class C^2. By $\Gamma := \partial D_-$, we denote its boundary and by $D_+ := \mathbb{R}^3 \setminus \overline{D}_-$ its open complement. We assume the unit normal n to the boundary to be directed into the exterior D_+.

Given a function $\varphi \in C(\Gamma)$, the function,

$$u(x) := \int_\Gamma \varphi(y) \Phi(x, y)\, \mathrm{d}\Gamma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma,$$

denotes the single-layer potential with density φ, and the function,

$$v(x) := \int_\Gamma \varphi(y) \frac{\partial \Phi(x,y)}{\partial n} \, \mathrm{d}\Gamma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma,$$

denotes the double-layer potential with density φ, where

$$\Phi(x,y) := \frac{1}{4\pi} \frac{1}{|x-y|}$$

is the fundamental solution of Laplace's equation.

For a proof of the next lemma, we refer to [6].

Lemma 2.1. *Let Γ be of class C^2 and $\varphi \in C(\Gamma)$. Then the single-layer potential u with density φ can be continuously extended throughout $\mathbb{R}^3$. On the boundary there holds*

$$u(x) = \int_\Gamma \varphi(\xi)\Phi(x,\xi) \, \mathrm{d}\Gamma(\xi), \quad x \in \Gamma.$$

The double-layer potential v with density φ can be continuously extended from D_+ to $\overline{D}_+$ and from D_- to $\overline{D}_-$ with limiting values:

$$v_\pm(x) = \int_\Gamma \varphi(\xi) \frac{\partial \Phi(x,\xi)}{\partial n(\xi)} \, \mathrm{d}\Gamma(\xi) \pm \frac{1}{2}\varphi(x), \quad x \in \Gamma,$$

where $v_\pm(x) := \lim_{\gamma \to 0^+} v(x \pm \gamma n(x))$ and where the integrals exist as improper integrals.

A proof of the following lemma may be found in [7].

Lemma 2.2. *The operator D defined by*

$$Du = \int_\Gamma \varphi(\xi) \frac{\partial \Phi(x,\xi)}{\partial n} \, \mathrm{d}\Gamma(\xi),$$

is continuous from $W^{m,p}(\Gamma)$ into $W^{m+1,p}(\Gamma)$ for all p such that $1 < p < \infty$, m positive integer and $x \in \Gamma$.

The following result is essential to prove that the electric field E has a jump across the interface S.

Lemma 2.3. *Consider the equation*:

$$\nabla \cdot (\varepsilon \nabla h - f) = 0 \quad \text{in } B.$$

Then, for $h \in W^{1,p}$ and $f \in L^p$, on the interface S, the following jump condition holds:

$$\left[\!\left[\varepsilon \frac{\partial h}{\partial n} - f \cdot n \right]\!\right] = 0,$$

where $[\![\cdot]\!]$ is defined by $[\![u]\!] = u^+ - u^-$ for any u defined on B. Note that u^+ and u^- are notations for $u_\pm(x) = \lim_{y \to x \in B^\pm} u(y)$, $x \in S$.

Next, we establish the regularity of h introduced in Lemma 2.3 with zero boundary condition. The result is useful for deriving the desired estimates on E and H.

Lemma 2.4. *Consider the equation*:

$$\nabla \cdot (\varepsilon \nabla h) = f \quad \text{in } B,$$

where $h \in W^{1,p}(B)$, $f \in W^{-1,p}(B)$ *and* $h|_{\partial B} = 0$. *Then for* $x_0 \in S$, *h is given by*:

$$\langle G, f\rangle = \frac{\varepsilon^+ + \varepsilon^-}{2} h(x_0) - (\varepsilon^+ - \varepsilon^-) \int_S h(\xi) \frac{\partial G(x_0, \xi)}{\partial n} \, \mathrm{d}S(\xi),$$

where G is the Green's function for Δ *on B* (*i.e.,* $\Delta G = \delta$ *and* $G|_{\partial B} = 0$), $\langle \cdot, f\rangle$ *denotes the linear functional on the distribution space determined by f and* $W^{-1,p}$ *is the dual space of* $W^{1,q}$ *with* $1/p + 1/q = 1$.

Remark 1. If ε is constant or a $\mathcal{C}^1$ function, then Lemma 2.4 is a standard elliptic regularity result [5]. For the model problem in this Note, ε is only piecewise constant. For general $\varepsilon \in L^\infty$, the Nash–Moser technique implies only $h \in \mathcal{C}^\alpha$ for some small α and for $f = 0$. The $W^{1,p}$ regularity here implies that $h \in C^\alpha$ for any $\alpha \in (0, 1)$.

Remark 2. In the case $h \in L^p(B^+)$ or $h \in L^p(B^-)$, the standard elliptic regularity results [5] indicate the $W^{1,p}$ regularity of h away from a tubular neighborhood of the interface S due to the definition of ε. Thus it suffices to prove the regularity result near the interface S. Note that Lemma 2.4 gives an explicit formula for h on S. Using this formula, the behavior of h near the interface S can be easily obtained.

Corollary 2.5. *Assume that h satisfies the conditions in Lemma* 2.4. *Then the following estimate holds*:

$$\|h\|_{W^{1,p}(B)} \leqslant C\big(\|h\|_{L^p(B)} + \|f\|_{W^{-1,p}(B)}\big).$$

Corollary 2.6. *For any* $f \in L^p$, *there exists* $h \in W_0^{1,p}$ *that solves*:

$$\nabla \cdot (\varepsilon \nabla h) = \nabla \cdot f \quad \textit{in } B, \quad h|_{\partial B} = 0.$$

3. L^p Estimates for Maxwell's equations

In this section, we describe the basic ideas of the proof of Theorem 1.1. The $W^{1,p}$ regularity of H follows from standard elliptic regularity estimates [5]. Note that these standard elliptic estimates cannot be applied directly for E, due to the singularity of the dielectric permittivity.

We introduce the following curl–div decomposition for the electric field:

$$E(x) = \mathrm{i}\omega\mu \nabla \times \int_B G(x, \xi) H(\xi) \, \mathrm{d}\xi + F(x),$$

where $G(x, \xi)$ is the Green's function for Δ on B and F is determined from the Maxwell's equations (2) as follows. From the first equation in (2), $F = \nabla h$ for some h and from the second equation, h satisfies:

$$\nabla \cdot (\varepsilon \nabla h) = \nabla \cdot g_1,$$

where $h \in W^{1,p}(B)$ and

$$g_1(x) = \frac{\omega\mu}{\mathrm{i}} \varepsilon \nabla \times \left(\int_B G(x, \xi) H(\xi) \, \mathrm{d}\xi \right) - \frac{1}{\mathrm{i}\omega} g(x).$$

Hence, this decomposition reduces the problem to the above elliptic equation in divergence form with discontinuous coefficients. The desired estimates follow from the above lemmas.

Note that in the assumptions and the proof of Lemma 2.4 the condition $h|_{\partial B} = 0$ is assumed and used. However, in order to prove Theorem 1.1, it is essential that these results hold for arbitrary values of h on ∂B. This can be achieved by introducing the following cutoff argument: Let $\tilde{h} = \chi h$. Construct $\chi \in \mathcal{C}_0^\infty(B)$ such that

$$\chi|_{B_\delta} = \begin{cases} 1 & \text{in } B_\delta, \\ 0 & \text{outside } B_{2\delta}, \end{cases}$$

and $\partial_n \chi\ |_S = 0$. Then $\tilde{h}$ satisfies the assumptions of Lemma 2.4 and in B_δ, $\tilde{h} = h$, hence the results hold for any values of h on ∂B.

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