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Algebraic Geometry

Eigenvalues of Frobenius acting on the ℓ-adic cohomology of complete intersections of low degree

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Abstract

We show that the eigenvalues of Frobenius acting on ℓ-adic cohomology of a complete intersection of low degree defined over the finite field $\mathbb{F}_q$ modulo the cohomology of the projective space are divisible as algebraic integers by q^κ, where the natural number κ is predicted by the theorem of Ax and Katz (Amer J. Math. 93 (1971) 485–499) on the congruence for the number of rational points. ***To cite this article: H. Esnault, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Valeurs propres de Frobenius agissant sur la cohomologie ℓ-adique d'intersections complètes de bas degré. Nous montrons que les valeurs propres de Frobenius agissant sur la cohomologie ℓ-adique d'une intersection complète de bas degré définie sur le corps fini $\mathbb{F}_q$ modulo la cohomologie de $\mathbb{P}^n$ sont divisibles en tant qu'entiers algébriques par q^κ, où κ est l'entier naturel prédit par le théorème de Ax et Katz (Amer J. Math. 93 (1971) 485–499) sur la congruence pour le nombre de points rationnels. ***Pour citer cet article : H. Esnault, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Si $X \subset \mathbb{P}^n$ est une intersection complète de degrés $d_1 \geqslant \cdots \geqslant d_r$ définie sur le corps fini $\mathbb{F}_q$, et si κ est le maximum de 0 et de la partie entière du nombre rationnel $(n - d_2 - \cdots - d_r)/d_1$, nous montrons que les valeurs propres de Frobenius agissant sur la cohomologie ℓ-adique relative $H^i(\overline{\mathbb{P}^n}, \overline{X}, \mathbb{Q}_\ell)$ sont divisibles par q^κ en tant qu'entiers algébriques (Theorem 1.1). Ici, $\overline{\mathbb{P}^n} = \mathbb{P}^n \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ et $\overline{\mathbb{F}_q}$ est la clôture algébrique de $\mathbb{F}_q$. Si X est lisse, utilisant le théorème de Ax et Katz [12], c'est une conséquence immédiate de la formule des traces de Grothendieck–Lefschetz [11] et de ce que la cohomologie primitive de $\overline{X}$ est concentrée en dimension moitié [3]. Si X est singulière, cela n'est plus le cas. Si $\mathbb{F}_q$ est remplacé par un corps k de caractéristique 0, l'assertion qui correspond dans la philosophie motivique au Theorem 1.1 est que le type de Hodge est $\geqslant \kappa$ sous les mêmes hypothèses de degré et d'intersection complète, et est prouvée dans [7] pour les hypersurfaces et [9] en général.

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Cela dit, le type de Hodge est $\geqslant\kappa$ même sans hypothèse d'intersection complète [10], mais la preuve de cette Note ne permet pas d'ôter l'hypothèse d'intersection complète.

1. Introduction

If $X \subset \mathbb{P}^n$ is a projective variety defined over the finite field $k = \mathbb{F}_q$ by equations of degrees $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r$, then the fundamental theorem of Ax and Katz [12] asserts that the number of rational points of X fulfills

$$\left|X(\mathbb{F}_q)\right| \equiv \left|\mathbb{P}^n(\mathbb{F}_q)\right| \bmod q^\kappa. \tag{1}$$

If X is smooth, this is equivalent to saying that the eigenvalues of the Frobenius action on $H^i(\overline{\mathbb{P}^n}, \overline{X}, \mathbb{Q}_\ell)$ for all i are of the shape

$$q^\kappa \cdot \text{algebraic integer}. \tag{2}$$

Here $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, $\overline{\mathbb{F}_q}$ is the algebraic closure of $\mathbb{F}_q$, and κ is the maximum of 0 and of the integral part of $(n - d_1 - \cdots - d_r)/d_1$. We will freely use the equivalent notations $H^i(\overline{\mathbb{P}^n}, \overline{X}, \mathbb{Q}_\ell) = H^i_c(\overline{U}, \mathbb{Q}_\ell) = H^i(\overline{\mathbb{P}^n}, j_!\mathbb{Q}_\ell)$, where $j : U = \mathbb{P}^n \setminus X \to \mathbb{P}^n$ is the open embedding, where the first notation refers to relative cohomology, the second one to cohomology with compact supports, the last one to cohomology of the trivial sheaf extended by 0 across the boundary.

Let $\zeta(U, t)$ be the zeta function of U defined by its logarithmic derivative

$$\frac{\zeta'(U,t)}{\zeta(U,t)} = \sum_{s\geqslant 1} \left|U(\mathbb{F}_{q^s})\right| t^{s-1}. \tag{3}$$

By the theorem of Dwork [8], we know that $\zeta(U, t)$ is a rational function

$$\zeta(X, t) \in \mathbb{Q}(t), \tag{4}$$

while Grothendieck–Lefschetz trace formula [11] gives a cohomological formula

$$\zeta(U, t) = \prod_{i=0}^{2\,\dim(\mathrm{U})} \det(1 - F_i t)^{(-1)^{i+1}}, \tag{5}$$

where F_i is the geometric Frobenius acting on the compactly supported ℓ-adic cohomology $H^i_c(\overline{U}, \mathbb{Q}_\ell)$. For X smooth, the Weil conjectures [6] assert that the eigenvalues in any complex embedding $\mathbb{Q}_\ell \subset \mathbb{C}$ of the geometric Frobenius acting on $H^i(\overline{X}, \mathbb{Q}_\ell)$ have absolute values $q^{i/2}$. The conclusion is that there is no possible cancellation between the numerator and the denominator of (5). Thus (1) for all finite field extensions $\mathbb{F}_{q^s} \supset \mathbb{F}_q$ is equivalent to (2). If we actually assume that X is not only smooth but also a complete intersection, then we know that only $H^{n-r}_c(\overline{X}, \mathbb{Q}_\ell)$ is not equal to $H^{n-r}(\overline{\mathbb{P}^n}, \mathbb{Q}_\ell)$ [4]. Thus in this case, we do not need the Weil conjectures to conclude that there is no cancellation.

On the other hand, if we replace k by a field of characteristic 0, keeping the same degree assumption, then we know that the Hodge type of X, that is the largest integer μ such that the Hodge filtration in de Rham cohomology satisfies

$$F^\mu H^i_{c,DR}(U) = H^i_{c,DR}(U) \tag{6}$$

is $\geqslant\kappa$ (see [3] in the smooth case, [7] for hypersurfaces, [9] for complete intersections and [10] for the general case). The purpose of this Note is to show

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a complete intersection defined over the finite field $k = \mathbb{F}_q$ by equations of degrees $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_r$. Then the eigenvalues of the geometric Frobenius acting on ℓ-adic cohomology $H^i(\overline{\mathbb{P}^n}, \overline{X}, \mathbb{Q}_\ell)$ are of the shape $q^\kappa \cdot$* algebraic integer *for all i, where κ is the maximum of 0 and of the integral part of $\frac{n-d_2-\cdots-d_r}{d_1}$.*

The proof reduces the assertion to Deligne's integrality statement [4] which is true for all varieties, without supplementary assumptions. However, the reduction depends heavily on the complete intersection assumption.

2. The proof of Theorem 1.1

If X is a complete intersection of dimension $n-r$, Artin's vanishing theorem [1], Theorem 3.1, implies

$$H^i\big(\overline{X},\mathbb{Q}_\ell\big)/H^i\big(\overline{\mathbb{P}^n},\mathbb{Q}_\ell\big)=H_c^{i+1}\big(\overline{U},\mathbb{Q}_\ell\big)=0\quad\text{for } i<\dim X=n-r. \tag{7}$$

By the Lefschetz trace formula (5), we see, as already observed by Wan (see introduction of [2]), that the theorem of Ax–Katz (1) together with the divisibility assertion for F_i for all i but one, implies the divisibility assertion for the last F_i. Thus we just have to show divisibility for $i>n-r$, or equivalently divisibility of the F_i acting on $H_c^i(\overline{U},\mathbb{Q}_\ell)=H^i(\overline{\mathbb{P}^n},\overline{X},\mathbb{Q}_\ell)$ for $i>n-r+1$. We observe that divisibility of the eigenvalues of Frobenius is compatible with base field extension, and consequently we are allowed to enlarge by necessity our ground field $\mathbb{F}_q$ to $\mathbb{F}_{q^s}$. Let $A\cong\mathbb{P}^{n-1}$ be a linear hyperplane in general position. One has an exact sequence of

$$\cdots\to H_A^i\big(\overline{\mathbb{P}^n},\overline{X},\mathbb{Q}_\ell\big)\to H^i\big(\overline{\mathbb{P}^n},\overline{X},\mathbb{Q}_\ell\big)\to H^i\big(\overline{\mathbb{P}^n\setminus A},\overline{X\setminus X\cap A},\mathbb{Q}_\ell\big)\to\cdots \tag{8}$$

(see, e.g., [13], III, Proposition 1.25 applied to $F=j_!\mathbb{Q}_\ell$, for $j:\mathbb{P}^n\setminus X\to\mathbb{P}^n$, and $Z=A$, $U=\mathbb{P}^n\setminus A$) which is compatible with the Frobenius action. For $i>n-r+1$, Artin's vanishing theorem [1] again implies $H^i(\overline{\mathbb{P}^n\setminus A},\overline{X\setminus X\cap A},\mathbb{Q}_\ell)=0$. Thus we are reduced to proving the assertion for the Frobenius action on $H_A^i(\overline{\mathbb{P}^n},\overline{X},\mathbb{Q}_\ell)$ for $i>n-r+1$. One has

Theorem 2.1 (P. Deligne). *The Gysin homomorphism*

$$H^{i-2}\big(\bar{A},\overline{X\cap A},\mathbb{Q}_\ell\big)(-1)\xrightarrow{\text{Gysin}}H_A^i\big(\overline{\mathbb{P}^n},\overline{X},\mathbb{Q}_\ell\big)$$

is an isomorphism of Frobenius modules for A in a non-trivial open subset of the dual projective space $(\mathbb{P}^n)^\vee$. More generally, if $\mathcal{F}$ is a ℓ-adic sheaf on $\mathbb{P}^n$, then for A in a non-trivial open subset of the dual projective space $(\mathbb{P}^n)^\vee$, the Gysin homomorphism

$$H^{i-2}(A,i^*\mathcal{F})(-1)\to H_A^i\big(\mathbb{P}^n,\mathcal{F}\big),$$

where $i:A\to\mathbb{P}^n$ is the closed embedding, is an isomorphism.

Proof (P. Deligne). We consider the universal family

$$\iota:\mathcal{A}=\big\{(x,A)\in\mathbb{P}^n\times_k(\mathbb{P}^n)^\vee,\ x\in A\big\}\hookrightarrow\mathbb{P}^n\times_k\big(\mathbb{P}^n\big)^\vee \tag{9}$$

together with the projections

$$\mathrm{pr}_2:\mathbb{P}^n\times_k\big(\mathbb{P}^n\big)^\vee\to\big(\mathbb{P}^n\big)^\vee,\qquad \mathrm{pr}_1:\mathbb{P}^n\times_k\big(\mathbb{P}^n\big)^\vee\to\big(\mathbb{P}^n\big). \tag{10}$$

Since $\mathcal{F}$ comes from $\mathbb{P}^n$, and $\mathcal{A}\subset\mathbb{P}^n\times_k(\mathbb{P}^n)^\vee$ is a smooth hypersurface such that $\mathcal{A}\to\mathbb{P}^n$ is smooth, the classical Gysin homomorphism is an isomorphism

$$\iota^*\mathrm{pr}_1^*\mathcal{F}[-2](-1)\overset{\cong}{\to}\iota^!\mathrm{pr}_1^*\mathcal{F}. \tag{11}$$

By [5], Corollary 2.9 applied to pr_2, there is a non-empty open subset $V\subset(\mathbb{P}^n)^\vee$ such that for all $A\in V$ one has the base changes

$$i_A^*\big(\iota^*\mathrm{pr}_1^*\mathcal{F}[-2](-1)\big)=i^*\mathcal{F}[-2](-1),\qquad i_A^*\big(\iota^!\mathrm{pr}_1^*\mathcal{F}\big)=i^!\mathcal{F}, \tag{12}$$

where $i_A:\mathcal{A}\cap q^{-1}A=A\times\{A\}\hookrightarrow\mathbb{P}^n\times\{A\}$. Thus (11) and (12) imply that for such a A, $i^*\mathcal{F}[-2](-1)\overset{\cong}{\to}i^!\mathcal{F}$. This concludes the proof. □

Thus for $i > n-r+1$, and $n \geqslant 1$, (8) becomes

$$H^{i-2}\big(\bar{A}, \overline{X\cap A}, \mathbb{Q}_\ell\big)(-1) \twoheadrightarrow H^i\big(\overline{\mathbb{P}^n}, \overline{X}, \mathbb{Q}_\ell\big) \tag{13}$$

as a surjection of Frobenius modules. On the other hand, $X \cap A$ is a complete intersection in $\mathbb{P}^{n-1}$ of the same degrees $d_1, \ldots, d_r$ and we have

$$\kappa(A\cap X) = \max\left(0, \left[\frac{n-1-d_2-\cdots-d_r}{d_1}\right]\right) \geqslant \max\big(0, \kappa(X)-1\big), \tag{14}$$

with $\kappa(X) = \max(0, [\frac{n-d_2-\cdots-d_r}{d_1}])$. Arguing by induction on n, we know that the eigenvalues of Frobenius acting on $H^i(\bar{A}, \overline{X\cap A}, \mathbb{Q}_\ell)$ are divisible by $q^{\kappa(X\cap A)}$ for all i, thus by (13) and (14), those on $H^i(\overline{\mathbb{P}^n}, \overline{X}, \mathbb{Q}_\ell)$ by at least $q^{\kappa(X)}$ for all $i > n-r+1$, thus on all i by Ax–Katz' theorem as recalled at the beginning of the section. It remains to start the induction. Note that the formulation allows indeed $X = \emptyset$, that is $r > n$. And for $n = 0$, one has $\kappa = 0$ and the theorem is a (very) easy case of Deligne's integrality theorem [4], Corollary 5.5.3, saying that the eigenvalues of Frobenius acting on compactly supported cohomology are algebraic integers.

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