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Partial Differential Equations

A uniqueness result for monotone elliptic problems

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Abstract

We give here a short proof of the uniqueness of entropy solutions for nonlinear monotone elliptic problems with L^1-data. ***To cite this article: M.M. Porzio, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Un résultat d'unicité pour des problèmes elliptiques monotones. Nous donnons une demonstration très simple de l'unicité des solutions entropiques de problèmes de Dirichlet dans L^1. ***Pour citer cet article : M.M. Porzio, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Introduction and result

Let us consider a class of nonlinear elliptic problems whose prototype is

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open set, not necessarily bounded, in $\mathbb{R}^N$, $N \geqslant 2$, $p \in (1, N)$ and f is a summable function.

Notice that, since the data is not in $W^{-1,p'}(\Omega)$ (the dual space of $W_0^{1,p}$) we cannot expect solutions in $W_0^{1,p}(\Omega)$. Hence the first difficulty that arises in the study of these problems is that the solutions cannot be used as test functions in the equation. Moreover already in the linear case there is a lack of uniqueness of weak (i.e., distributional) solutions as showed by the classical counterexample of Serrin (see [5] and [4]).

However, in the linear case, the proof of the existence and also of the uniqueness of solutions obtained by duality was proved in the sixties by Guido Stampacchia (see [6]). Unfortunately the proof does not work in a nonlinear setting.

The first existence and uniqueness results in the nonlinear case were obtained in the seventies by Haïm Brezis and Walter A. Strauss in the semilinear case (i.e., $p = 2$) (see [3]), where more general problems were considered.

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The case $p \neq 2$ was studied in [1]. In that paper a solution was constructed by approximation and has two properties: the truncation at levels $\pm k$, $T_k(u)$ (see (7)) belongs to $W_0^{1,p}(\Omega)$ and it satisfies the following inequalities

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla T_k[u-\phi] \leqslant \int_\Omega f(x) T_k[u-\phi],$$

for every $k \in \mathbb{R}^+$ and $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. This solution was named the entropy solution and it was proved that these further requirements on the solution guarantee the uniqueness.

The aim of this paper is to give an easier and shorter proof of this last result, i.e., of the uniqueness of entropy solutions.

We recall now, in more details, the results in [1].

Let us consider the following problem:

$$\begin{cases} -\operatorname{div}\big(a(x,\nabla u)\big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is an open set, not necessarily bounded, in $\mathbb{R}^N$, $N \geqslant 2$.

Here the function $a : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is a Caratheodory function (continuous in $\xi \in \mathbb{R}^N$ for a.e. $x \in \Omega$ and measurable in x for every ξ) and there exist $p \in (1, N)$, $\alpha > 0$ and $\beta > 0$ such that

$$a(x,\xi)\cdot\xi \geqslant \alpha|\xi|^p, \tag{3}$$

$$\big|a(x,\xi)\big| \leqslant \beta\big[|\xi|^{p-1} + h(x)\big], \quad h \in L^{p'}(\Omega), \tag{4}$$

hold for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, where we have set $p' = p/(p-1)$. We also require that a is monotone, i.e., that it results

$$\big[a(x,\xi) - a(x,\eta)\big][\xi - \eta] > 0, \tag{5}$$

for every ξ and $\eta \in \mathbb{R}^N$, $\xi \neq \eta$, and a.e. $x \in \Omega$. On the data we assume that

$$f(x) \in L^1(\Omega). \tag{6}$$

Definition 1.1. Let u be a measurable function defined on Ω which is finite almost everywhere, and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$, where T_k is the truncation operator, i.e.,

$$T_k(s) = \begin{cases} s & \text{if } |s| \leqslant k, \\ k\,\operatorname{sign}(s) & \text{if } |s| > k. \end{cases} \tag{7}$$

Then there exists (see [1], Lemma 2.1) a measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$\nabla T_k(u) = v\chi_{|u|\leqslant k} \quad \text{almost everywhere in } \Omega,\ \forall k > 0,$$

which is unique up to almost everywhere equivalence. We define the gradient ∇u of u as this function v, and we denote $\nabla u = v$.

Definition 1.2. A measurable function u is an entropy solution of (2) if it satisfies

$$\begin{cases} T_k(u) \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla T_k[u-\phi] \leqslant \int_\Omega f(x) T_k[u-\phi], \\ \forall k \in \mathbb{R}^+,\ \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{8}$$

where T_k is as in (7).

Notice that both the integrals in (8) are well defined as $T_k[u-\phi] \in W_0^{1,p}(\Omega)$.

Theorem 1.3. *Under assumptions* (3)–(6) *there exists an entropy solution u of Eq.* (2). *Moreover such a solution satisfies*

$$u \in M^{p_1}(\Omega), \qquad |\nabla u| \in M^{p_2}(\Omega) \tag{9}$$

where $p_1 = \frac{N(p-1)}{N-p}$ and $p_2 = \frac{N(p-1)}{N-1}$. In case $p > 2-(1/N)$ the solution belongs to $W^{1,q}_{\mathrm{loc}}(\Omega)$ for every $q < p_2$, and if Ω is bounded to $W_0^{1,q}(\Omega)$.

(See Theorem 6.1 in [1].) As just noticed the solution cited in Theorem 1.3 is the unique entropy solution of problem (2), i.e., the following result holds.

Theorem 1.4. *Under assumptions* (3)–(6) *if v and z are entropy solutions of* (2) *then $v = z$.*

(See Theorem 5.1 in [1].) As said before we give now a different proof of this last result.

Remark 1. Notice that here, differently from [1], to prove Theorem 1.4 we do not need the following estimate on the entropy solutions:

$$\lim_{h\to+\infty} \int\limits_{h<|u|<h+k} |\nabla u|^p = 0.$$

2. Proof of Theorem 1.4

Let u be the entropy solution satisfying the regularity properties stated in Theorem 1.3 obtained by approximation as in [1]. It is sufficient to show that every entropy solution of (2) coincides with u. To this aim we recall the construction of u. Let us consider the weak solutions $u_n \in W_0^{1,p}(\Omega)$ of the following problems

$$-\operatorname{div}\big(a(x,\nabla u_n)\big) = f_n \quad \text{in } \Omega, \tag{10}$$

where $\{f_n\}$ is a sequence of bounded functions which converges to f in $L^1(\Omega)$. We point out that $u_n \in L^\infty(\Omega)$ (see [6]). Moreover u_n and $\nabla u_n(x)$ converge, respectively, to u and $\nabla u(x)$, a.e. in Ω (see [1] and [2]).

Let z be a second entropy solution, i.e., it satisfies

$$\begin{cases} T_k(z) \in W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla z)\nabla T_k[z-\phi] \leqslant \int_\Omega f(x) T_k[z-\phi], \\ \forall k \in \mathbb{R}^+,\ \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{11}$$

Taking $\phi = u_n$ in (11) we obtain

$$\int_\Omega a(x,\nabla z)\nabla T_k[z-u_n] \leqslant \int_\Omega f(x) T_k[z-u_n]. \tag{12}$$

Moreover choosing $T_k[z-u_n] \in W_0^{1,p}(\Omega)$ as test function in (10) we have

$$-\int_\Omega a(x,\nabla u_n)\nabla T_k[z-u_n] = -\int_\Omega f_n(x)T_k[z-u_n]. \tag{13}$$

Then adding (12) and (13) we obtain

$$\int_\Omega \big[a(x,\nabla z)-a(x,\nabla u_n)\big]\nabla T_k[z-u_n] \leqslant \int_\Omega \big(f(x)-f_n(x)\big)T_k[z-u_n].$$

Notice that the integral in the left-hand side is non-negative by the assumption (5), bounded from above by a constant ck independent on n. Moreover, the integrand function $[a(x,\nabla z)-a(x,\nabla u_n)]\nabla T_k[z-u_n]$ converges a.e. in Ω to $[a(x,\nabla z)-a(x,\nabla u)]\nabla T_k[z-u]$. Hence using the Fatou lemma we can pass to the limit as $n\to+\infty$ obtaining

$$\int_\Omega \big[a(x,\nabla z)-a(x,\nabla u)\big]\nabla T_k[z-u] \leqslant 0$$

which by (5) and the arbitrary choice of k implies that $z=u$ a.e. in Ω.

Remark 2. We point out that our proof shows that there is uniqueness of entropy solutions whenever it is possible to construct an entropy solution u as a limit of a sequence of bounded function u_n (it is sufficient the a.e. convergence of u_n and ∇u_n, respectively to u and ∇u) which solve the following problem

$$\begin{cases} -\operatorname{div}\big(a(x,\nabla u_n)\big) = f_n & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where the data f_n converge in $L^1(\Omega)$ to f and assuming only that a is a monotone Caratheodory operator.

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